GABRIEL AND KRULL DIMENSIONS OF MODULES
OVER RINGS GRADED BY FINITE GROUPS

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Abstract. Let $R$ be a ring graded by a finite group $G$ with the identity component $R_e$ and let $M$ be a left $R$-module. It is proved that $	ext{Gdim}_R M = 	ext{Gdim}_{R_e} M$, $	ext{Kdim}_R M = 	ext{Kdim}_{R_e} M$ and $\text{Ndim}_R M = \text{Ndim}_{R_e} M$, where $\text{Gdim}$, $\text{Kdim}$ and $\text{Ndim}$ denote, respectively, Gabriel, Krull and dual Krull dimensions. The proofs are based on the use of lattice theory, a method which also gives alternative proofs of known results about normalizing extensions.

Introduction

In [10] it was proved that a module over a ring $R$ graded by a finite group $G$ satisfies ACC on $R$-submodules if and only if it satisfies the condition on $R_e$-submodules, where $R_e$ is the identity component of $R$. The corresponding question for DCC is, like the analogous one concerning normalizing extensions (cf. [2, 5, 12]), more difficult and seems that rather no direct extension of arguments of [10] or those used for normalizing extensions could provide such a result. In this paper, following the experience of [5, 9], we approach the problem via lattice theory. It appears that on a level of lattices differences between ACC and DCC disappear and the idea of [10] can be employed. It allows us not only to handle the DCC case but also prove that none of Gabriel, Krull and dual Krull dimensions of an $R$-module changes when it is considered over $R$ or $R_e$.

1. Preliminaries

Throughout this paper $R$ is a ring with unity graded by a finite group $G$, i.e., $R = \bigoplus_{g \in G} R_g$ for additive subgroups $R_g$, satisfying $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. Obviously $R_e$ is a subring of $R$ and $1 \in R_e$. All considered modules will be left unital modules and $N \leq M$ (or $N \leq_S M$ if we need to specify the ring of scalars) means that $N$ is a submodule of $M$. For proper submodules we write $N < M$.

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In this paper we use some lattices and their sublattices. All those are modular with 0 and 1 (we consider only sublattices containing 0 and 1 of the original lattice). Given a module $M$ we write $L(M)$ for the lattice of submodules of $M$. We denote by $L^0$ the dual lattice to the lattice $L$, i.e., the lattice having the same elements as $L$ but carrying the reverse ordering. For all elements $a, b \in L$ with $a \leq b$, $[a, b]$ denotes the interval $\{x \in L | a \leq x \leq b\}$. If $\theta$ is a congruence on $L$ then $\theta(a)$ denotes the congruence class determined by $a$. A lattice isomorphism will be denoted by $\approx$. All undefined terms and used results on lattices can be found in [1].

An ideal $I$ of a lattice $L$ is called essential if for every nonzero ideal $J$ of $L$ we have $I \cap J \neq 0$.

**Lemma 1.** Let $\{s_g | g \in G\}$, where $G$ is a finite group, be a set of order preserving maps of a lattice $L$ into itself such that $s_e = id$ and for every $g, h \in G$ and $x \in L$, $s_g s_h(x) \leq s_{gh}(x)$. If $f : L \to L'$ is a homomorphism of $L$ onto a nonzero lattice $L'$ and $I$ is an essential ideal of $L'$, then $f(\bigcap_{g \in G} s_g^{-1} f^{-1}(I)) \neq 0$.

**Proof.** For every subset $X$ of $G$ containing $e$, let $I_X = \bigcap_{g \in X} s_g^{-1} f^{-1}(I)$. Obviously $f(I_{\{e\}}) = I$, so $f(I_{\{e\}}) \neq 0$. Let $X$ be a subset of maximal cardinality such that $e \in X$ and $f(I_X) \neq 0$. If there is $h \in G \setminus X$ then by maximality of $|X|$, $f(s_h(a)) \neq 0$ for some $a \in I_X$. Thus, since the ideal $I$ is essential, $I \cap [0, f(s_h(a))] \neq 0$. Hence there is $b \in L$ with $b \leq s_h(a)$ and $0 \neq f(b) \in I$. Since $s_g s_h^{-1}$ preserves the order of $L$, for every $g \in X$, $s_g s_h^{-1}(b) \leq s_g s_h^{-1}(a)$. Now by the assumption, $s_{gh^{-1}}(b) \leq s_{gh^{-1}} s_h(a) \leq s_g(a)$. Since $a \in I_X$ and $g \in X$, $f(s_g(a)) \in I$. Consequently, $f(s_{gh^{-1}}(b)) \in I$. Thus $b \in I_{Xh^{-1} \cup \{e\}}$ and since $f(b) \neq 0$, we have $f(I_{Xh^{-1} \cup \{e\}}) \neq 0$. This contradicts maximality of $|X|$, so $X = G$ and the lemma follows.

A sublattice $K$ of a lattice $L$ is said to be dense in $L$ if for each element $k$ in $K \setminus \{1\}$, every essential ideal $I$ of the lattice $[k, 1]$ contains an element $k_1 \in K$, $k_1 \neq k$.

The following is the key proposition of this paper. A similar proposition is true for normalizing extensions, as proved in §4.

**Proposition 1.** If $M$ is an $R$-module and $f$ is a homomorphism of $L(R, M)$ onto a nonzero lattice $L$, then $f(L(R, M))$ is a dense sublattice of $L(L^0)$. In particular, $L(R, M)(L^0(R, M))$ is a dense sublattice of $L(R, M)(L^0(R, M))$.

**Proof.** For every $N \leq_R M$ there is a natural isomorphism $i : L(R, M/N) \to [N, M] \subseteq L(R, M)$ such that $i(L(R, M/N)) = [N, M] \cap L(R, M)$. Now, $f$ maps $[N, M]$ onto $[f(N), 1]$. Thus for every $k \in f(L(R, M))$ there is an $R$-module $K$ and a homomorphism $\overline{f}$ of $L(R, K)$ onto $[k, 1] \subseteq L$ such that $\overline{f}(L(R, K)) = f(L(R, M)) \cap [k, 1]$. Hence to prove that $f(L(R, M))$ is a dense sublattice of $L$ it suffices to show that every essential ideal $I$ of $L$ contains a nonzero element of
Define for each \( g \in G \), \( s_g: L(R, M) \to L(R, M) \) by \( s_g(N) = RgN \).

Obviously for every \( g, h \in G \) and \( T \in L(R, M) \) we have \( s_gs_h(T) \leq s_{gh}(T) \) and \( s_e = id \). Hence by Lemma 1 there is \( N \leq R, M \) such that \( f(N) \neq 0 \) and \( f(RgN) \in I \) for all \( g \in G \). Obviously \( f(\sum_{g \in G} RgN) \) is a nonzero element of \( I \cap f(L(R, M)) \).

Going the same line as above in the dual situation but now applying the maps \( s^0_g: L^0(R, M) \to L^0(R, M) \) given by \( s^0_g(N) = R^{-1}g(N) = \{m \in M | R^{-1}gm \leq N\} \) one obtains that \( f(L^0(R, M)) \) is a dense sublattice of \( L^0 \).

2. KRULL DIMENSION

In this section we study the Krull dimension (Kdim) and the dual Krull dimension (Ndim) of modules over \( G \)-graded rings. Recall that for a module \( M \), \( \text{Kdim} M \) (\( \text{Ndim} M \)) is defined (cf. [3, 7]) by induction as follows:

\[
\text{Kdim} M = -1(\text{Ndim} M = -1) \text{ if and only if } M = 0; \\
\text{if } \alpha \text{ is an ordinal number and } \text{Kdim} M \neq \alpha (\text{Ndim} M \neq \alpha) \text{ then } \text{Kdim} M = \alpha (\text{Ndim} M = \alpha) \text{ provided for every descending (ascending) chain } M_1 \geq M_2 \geq \cdots (M_1 \leq M_2 \leq \cdots) \text{ of submodules of } M \text{ there is a } n \text{ such that } \text{Kdim} M_i/M_{i+1} < \alpha (\text{Ndim} M_{i+1}/M_i < \alpha) \text{ for } i \geq n.
\]

One of basic properties of these dimensions says that if \( N \leq M \) then \( \text{Kdim} M = \sup\{\text{Kdim} N, \text{Kdim} M/N\} \) if either side exists. The same is true for \( \text{Ndim} \).

**Proposition 2.** Let \( M \) be a module, \( \alpha \geq 0 \) an ordinal number and let \( \kappa_L(\eta_L) \) be the relation on \( L = L(M) \) defined by \( M_1 \kappa_L M_2 \eta_L M_2 \) if and only if \( \text{Kdim}(M_1 + M_2/M_1 \cap M_2) < \alpha(\text{Ndim}(M_1 + M_2/M_1 \cap M_2) < \alpha) \). Then

(a) \( \kappa_L \) and \( \eta_L \) are congruences on \( L \);

(b) \( \text{Kdim} M = \alpha(\text{Ndim} M = \alpha) \) if and only if the lattice \( L/\kappa_L (L/\eta_L) \) is nonzero and satisfies DCC (ACC).

**Proof.** We prove the proposition for \( \text{Kdim} \); the proof for \( \text{Ndim} \) is similar. In view of Lemma 1.3.8 of [8], to prove that \( \kappa_L \) is a congruence it suffices to check that if \( M_1 \leq M_2 \) are submodules of \( M \) such that \( \text{Kdim} M_2/M_1 < \alpha \) then for every submodule \( T \) of \( M \), \( \text{Kdim} M_2 + T/M_1 + T < \alpha \) and \( \text{Kdim} M_2 \cap T/M_1 \cap T < \alpha \). However these are immediate consequences of isomorphisms \( M_2 + T/M_1 + T \cong M_2/M_1 + T \cap M_2, M_2 \cap T/M_1 \cap T \cong M_2 + T \cap M_2/M_1 \) and the property of \( \text{Kdim} \) formulated before the proposition.

It is clear that if the lattice \( L/\kappa_L \), is nonzero and satisfies DCC then \( \text{Kdim} L = \alpha \). Conversely, let \( \kappa_L(M_1) \geq \kappa_L(M_2) \geq \cdots \) be a descending chain of \( L/\kappa_L \). Obviously if \( N_i = M_1 \cap \cdots \cap M_i \) for \( i = 1, 2, \ldots \), then \( N_1 \geq N_2 \geq \cdots \) and \( \kappa_L(N_i) = \kappa_L(M_i) \). Hence assuming \( \text{Kdim} M = \alpha \) we get that for some \( n \), \( \kappa_L(M_n) = \kappa_L(N_n) = \kappa_L(N_{n+1}) = \cdots \), so the lattice \( L/\kappa_L \) satisfies DCC.

To obtain the main result of this section we also need the following
Lemma 2. A dense sublattice $K$ of a lattice $L$ satisfies ACC if and only if the lattice $L$ satisfies ACC.

Proof. Obviously if $L$ satisfies ACC then the same is true for $K$. Suppose that $K$ satisfies ACC and $L$ does not. Let $M$ be an element of $K$ maximal with respect to the property “[m, 1] does not satisfy ACC”. Let $a_1 < a_2 < \cdots$ be a strictly ascending chain of elements of [m, 1] and let $I = \bigcup_{k \geq 1}[m, a_k]$. Of course $I$ is an ideal of [m, 1]. By Zorn’s Lemma there is an ideal $J$ of [m, 1] maximal with respect to the property $I \cap J = \{m\}$. Using modularity one can easily check that the ideal $I \vee J$ of [m, 1] generated by $I \cup J$ is essential in [m, 1]. Hence the density property implies that there is $m_1 \in K \cap (I \vee J)$, $m_1 > m$. Hence for some $a \in I$, $b \in J$ we have $m < m_1 \leq a \vee b$. By the definition of $I$, $a \leq a_i$ for some $i$. Thus $[m_1, 1]$ contains the chain $a_i \vee b \leq a_{i+1} \vee b \leq \cdots$.

The choice of $m$ implies that it stabilizes. Hence for some $k \geq 0$, $a_{i+k+1} \vee b = a_{i+k} \vee b$. Now applying the modular law one obtains $a_{i+k+1} = a_{i+k+1} \wedge (a_{i+k+1} \vee b) = a_{i+k+1} \wedge (a_{i+k} \vee b) = a_{i+k} \vee (a_{i+k+1} \wedge b)$. But $I \cap J = \{m\}$, so we have $a_{i+k+1} \wedge b = m$. Hence $a_{i+k+1} = a_{i+k}$, contradiction.

Proposition 1 and Lemma 2 give immediately

Corollary 1. If $M$ is an $R$-module then

(i) $\text{RM}$ is Noetherian if and only if $R_M$ is Noetherian;

(ii) $\text{RM}$ is Artinian if and only if $R_M$ is Artinian.

Now we prove the main result of this section.

Theorem 1. For every $R$-module $M$, $\text{Kdim}_R M = \text{Kdim}_{R_e} M$ ($\text{Ndim}_R M = \text{Ndim}_{R_e} M$) if either side exists.

Proof. We will concentrate here on $\text{Kdim}$; the proof for $\text{Ndim}$ is parallel.

The inequality $\text{Kdim}_R M \leq \text{Kdim}_{R_e} M$ is obvious. The proof of the other inequality is by induction on $\alpha = \text{Kdim}_R M$. The case $\alpha = 0$ is done in Corollary 1(ii). Now suppose that $\alpha > 0$ and the result holds for $\beta < \alpha$. The induction assumption gives $\kappa_K = \kappa_L \cap (K \times K)$, where $L = L(R_e M), K = L(R_M), \kappa_L, \kappa_K$ are defined as in Proposition 2. Hence $K/\kappa_K \approx \{\kappa_L(N) | N \in K\} \subseteq L/\kappa_L$. Applying Proposition 1 and Lemma 2 we obtain that the lattice $K/\kappa_K$ satisfies DCC if and only if the lattice $L/\kappa_L$ satisfies DCC. This and Proposition 2 end the proof.

Theorem 1, and hence also Corollary 1, was proved in [4] when the grading is assumed inner.

3. Gabriel dimension

We will use the following Lanski’s [11] characterization of the Gabriel dimension, $\text{Gdim} M$, of a module $M$. It is given inductively starting with
Gdim \( M = 0 \) if and only if \( M = 0 \). Let \( \alpha \) be a nonlimit ordinal and suppose that \( \text{Gdim} \, M = \beta \) has been defined for \( \beta < \alpha \). Call \( A \) an \( \alpha \)-simple module if for every \( 0 \neq N \leq A \), both \( \text{Gdim} \, N \neq \alpha \) and \( \text{Gdim} \, A / N < \alpha \). Then \( \text{Gdim} \, M = \alpha \) if \( \text{Gdim} \, M \neq \alpha \) and if for each \( N < M \), \( M / N \) contains a \( \beta \)-simple module for \( \beta \leq \alpha \). When \( \alpha \) is a limit ordinal, \( \text{Gdim} \, M = \alpha \) if \( \text{Gdim} \, M \neq \alpha \) and if for each \( N < M \), \( M / N \) contains a \( \beta \)-simple module for \( \beta < \alpha \).

Now we state some auxiliary facts which will be used in the proof of the main result of this section.

**Proposition 3** [11].

(a) For any \( N \leq M \),
\[
\text{Gdim} \, M = \sup \{ \text{Gdim} \, N , \text{Gdim} \, M / N \}
\]
if either side exists.

(b) If \( M_i \leq M \), \( i \in I \) then \( \text{Gdim} \sum_{i \in I} M_i = \sup \{ \text{Gdim} \, M_i | i \in I \} \) if either side exists.

**Proposition 4.** Let \( \alpha \) be an ordinal number and let \( \gamma_L \) be the relation on \( L = L(M) \) given by \( M_1 \gamma_L M_2 \) if and only if \( \text{Gdim}(M_1 + M_2 / M_1 \cap M_2) < \alpha \). Then

(i) \( \gamma_L \) is a congruence relation;

(ii) if \( \alpha \) is nonlimit and if \( N \) is an \( \alpha \)-simple submodule of \( M \) then \( \gamma_L(N) \) is an atom of \( L / \gamma_L \);

(iii) if \( \alpha \) is nonlimit and if for some \( N \leq M \), \( \gamma_L(N) \) is an atom of \( L / \gamma_L \) then there is \( T < N \) such that \( \text{Gdim} \, T < \alpha \) and \( N / T \) is \( \alpha \)-simple.

**Proof.** The proof of (i) follows easily from Proposition 3(a). The condition (ii) is a direct consequence of the definition of \( \gamma_L \). Now we prove (iii). Let \( T = \sum \{ U \leq N \mid \text{Gdim} \, U < \alpha \} \). By Proposition 3(b) \( \text{Gdim} \, T < \alpha \). Now if \( T < T' \leq N \) then Proposition 3(a) and the choice of \( T' \) guarantee that \( \text{Gdim} \, T' / T \neq \alpha \). On the other hand, since \( \gamma_L(N) \) is an atom of \( L / \gamma_L \) and \( \gamma_L(T') \neq 0 \), we have \( \gamma_L(N) = \gamma_L(T') \). Hence, by the definition of \( \gamma_L \), \( \text{Gdim} \, N / T' < \alpha \).

**Lemma 3.** If \( K = \{ 0 , 1 \} \) is a dense sublattice of a lattice \( L \) then \( 1 = a_1 \lor \cdots \lor a_n \) for some atoms \( a_1, \ldots , a_n \in L \).

**Proof.** By Lemma 2 the lattice \( L \) satisfies ACC, so it suffices to prove that for every \( x \in L \) there is \( y \in L \) such that \( x \land y = 0 \) and \( x \lor y = 1 \). Let \( J \) be an ideal of \( L \) maximal with respect to the property \( [0 , x] \cap J = \{ 0 \} \). Then the ideal \( [0 , x] \lor J \) generated by \( [0 , x] \lor J \) is essential in \( L \). Hence, since \( \{ 0 , 1 \} \) is a dense sublattice of \( L \), \( 1 \in [0, x] \lor J \). It means that \( 1 = x \lor y \) for some \( y \in J \). Obviously \( x \land y \in [0 , x] \lor J = \{ 0 \} \).

**Lemma 4.** Suppose that \( \delta > 0 \) is an ordinal number and \( M \) is an \( R \)-module such that for every \( 0 \neq N \leq_{R_e} M \) there exists \( 0 \neq N' \leq_{R_e} N , \ R_e N ' - \beta \)-simple for some \( \beta < \delta \). Then there exists \( 0 \neq T \leq_{R_e} M \) such that for every \( g \in G \), \( R_g T = 0 \) or \( R_g T \) is a \( \beta \)-simple \( R_e \)-module for some \( \beta < \delta \).

**Proof.** Let \( H \) be a subset of \( G \) containing \( e \), of maximal cardinality such that for some \( 0 \neq X \leq_{R_e} M \) and all \( h \in H \), \( R_h X = 0 \) or \( R_h X \) is \( \beta \)-simple.
$R_e$-module for some $\beta < \delta$. If there is $g \in G \setminus H$ then $R_g X \neq 0$. Hence there exists $0 \neq Y \leq_{R_e} R_g X$, $r_r Y - \beta$-simple for some $\beta < \delta$. Now for every $h \in H$, $R_{hg^{-1}} X \leq R_{hg^{-1}} R_h X \leq R_h X$. Thus for every $h \in Hg^{-1} \cup \{e\}$, $R_h Y = 0$ or $R_{g^{-1}} Y$ is a $\beta$-simple $R_e$-module for some $\beta < \delta$. This contradiction proves that $G = H$, which gives the result.

Now we are ready to prove the main result of this section.

**Theorem 2.** For every $R$-module $M$, $\text{Gdim}_R M = \text{Gdim}_{R_e} M$ if either side exists.

**Proof.** We will prove by induction on $\alpha$ that if one side is equal to $\alpha$ then so is the other. It is clear for $\alpha = 0$. Suppose now that $\alpha > 0$ and the result holds for $\beta < \alpha$. Observe that this assumption says in particular that if $\gamma_L$ and $\gamma_K$ are the congruences defined in Proposition 4 on $L = L(R, M)$ and $K = L(R, M)$ respectively then $\gamma_K = \gamma_L \cap (K \times K)$. Thus $K/\gamma_K \cong \{\gamma_L(N) \mid N \in K\} \subseteq L/\gamma_L$ and by Proposition 1 $K/\gamma_K$ can be treated as a dense sublattice of $L/\gamma_L$.

Suppose that $\text{Gdim}_R M = \alpha$. Since $\gamma_K = \gamma_L \cap (K \times K)$, $\text{Gdim}_{R_e} M \neq 0$. Thus we have to show that for every $N <_{R_e} M$, $M/N$ contains a nonzero $\beta$-simple $R_e$-module, where $\beta \leq \alpha$ if $\alpha$ is nonlimit and $\beta < \alpha$ otherwise. Let $T$ be the largest $R$-submodule of $M$ which is contained in $N$. Passing to the module $M/T$ we can assume that $T = 0$. Since $\text{Gdim}_R M = \alpha$ there is $0 \neq T' \leq_{R_e} M$, $R_{T'} - \beta$-simple for some $\beta \leq \alpha$ if $\alpha$ is nonlimit and $\beta < \alpha$ otherwise. If $\beta < \alpha$ then by the induction assumption $\text{Gdim}_{R_e} (T' / T' \cap N) < \alpha$, so $0 < \text{Gdim}_{R_e} (T' + N/N) < \alpha$ and we are done. If $\beta = \alpha$ then by Proposition 4(ii) $\gamma_L(T')$ is an atom of the sublattice $\{\gamma_L(N) \mid N \in K\}$ of $L/\gamma_L$. Thus by Proposition 1 and Lemma 3 there are atoms $\gamma_L(N_1), \ldots, \gamma_L(N_k)$ of $L/\gamma_L$ such that $\gamma_L(T') = \gamma_L(N_1) \vee \cdots \vee \gamma_L(N_k)$. It follows that the length of $[\gamma_L(N_1), \gamma_L(T' + N)]$ is finite. Hence by Proposition 4(iii) there is $\gamma_K(T')$ such that $N < \gamma_K(T') <_{R_e} T' + N$ and $R_e(\gamma_K(T'))$ is $\beta$-simple for some $\beta \leq \alpha$, as required.

Suppose that $\text{Gdim}_{R_e} M = \alpha$. By the induction assumption $\text{Gdim}_R M \neq \alpha$. Thus we have to show that for every $N <_{R_e} M$, $M/N$ contains a $\beta$-simple $R$-submodule with $\beta \leq \alpha$ if $\alpha$ is nonlimit and $\beta < \alpha$ otherwise. We can obviously assume that $N = 0$. Observe that $M$ satisfies the assumption of Lemma 4 with $\delta = \alpha$ in the limit case and $\delta = \alpha + 1$ otherwise. Hence there is $0 \neq X \leq_{R_e} M$ such that for every $g \in G$, $R_g X = 0$ or $R_g X$ is $\beta$-simple $R_e$-module for some $\beta \leq \alpha$ if $\alpha$ is nonlimit and $\beta < \alpha$ otherwise. Now $T = \bigoplus_{g \in G} R_g X$ is a nonzero $R$-submodule of $M$. If for all $g \in G$, $\text{Gdim}_{R_e}(R_g X) < \alpha$ then the induction assumption gives $\text{Gdim}_R T < \alpha$ and we are done. If for some $g \in G$, $\text{Gdim}_{R_e}(R_g X) = \alpha$ then by Proposition 4(ii) $\gamma_L(T)$ is a join of atoms of $L/\gamma_L$. Hence also the lattice $K/\gamma_K$ contains an atom $\gamma_K(T') \leq \gamma_K(T)$. Thus by Proposition 4(iii) there exists $T'' \leq_{R_e} T$ such that $\text{Gdim}_R T'' < \alpha$ and $R(T'/T'')$ is $\alpha$-simple. If $T'' \neq 0$ then it contains a
nonzero $\beta$-simple $R$-module for some $\beta < \alpha$. If $T'' = 0$ then $T'$ is $\alpha$-simple and we are done. This ends the proof.

4. Remarks

1. The methods of this paper can be almost directly applied to obtain analogous results for some other than $R_\tau \subseteq R$ ring extensions of “finite type”. Here we comment how they could be used to obtain some known results on normalizing extensions.

Recall that an extension $R \subseteq S$ of rings with the same 1 is finite normalizing if for some $s_1 = 1$, $s_2, \ldots, s_n \in S$, $S = Rs_1 + \cdots + Rs_n$ and $Rs_j = s_j R$ for $1 \leq j \leq n$.

It is known [2, 6, 12, 13], that if $R \subseteq S$ is a finite normalizing extension, $M$ is an $S$-module then $\text{Kdim}_S M = \text{Kdim}_R M$ and $\text{Gdim}_S M = \text{Gdim}_R M$. It is clear that proving the above result by the method of this paper one will have some problems with Proposition 1 and Lemma 4 only. In the present situation instead of Proposition 1 one can apply the following

**Proposition 1'**. Let $M$ be an $S$-module and $\varphi_i, \varphi_i^0 : L(RM) \to L(RM)$ be given by $\varphi_i(N) = s_i N$ and $\varphi_i^0(N) = \{m \in M \mid s_i m \in N\}$. If $f$ is a homomorphism of $L(RM)$ onto a nonzero lattice $L$ such that $f(X) = f(Y)$ implies $f(\varphi_i(X)) = f(\varphi_i(Y))$ for $1 \leq i \leq n$, then $f(L(SM))$ is a dense sublattice of $L(L^0)$.

**Proof (Sketch)**. Similarly as in the proof of Proposition 1 to get that $f(L(RM))$ is a dense sublattice of $L$ it suffices to show that every essential ideal $I$ of $L$ contains a nonzero element of $f(L(SM))$. The assumption on $f$ and $\varphi_i$ implies that for every $1 \leq i \leq n$, $g_i : L \to L$ given by $g_i(x) = f(\varphi_i(X))$ where $x = f(X)$, is a well-defined map. One easily checks that $g_i(0) = 0$, for every $x, y \in L$, $g_i(x \lor y) = g_i(x) \lor g_i(y)$ and if $x \leq y$ then $g_i([x, y]) = [g_i(x), g_i(y)]$. These imply that $g_i^{-1}(I), 1 \leq i \leq n$, are essential ideals of $L$. Hence $\bigcap_{i=1}^{n} g_i^{-1}(I) \neq 0$. Now if $0 \neq x \in \bigcap_{i=1}^{n} g_i^{-1}(I)$ then, since $g_i = \text{id}$, $y = \bigvee_{i=1}^{n} g_i(x) \neq 0$. Clearly $y \in I \cap f(L(SM))$.

Applying $\varphi_i^0$ instead of $\varphi_i$ one gets the dual result.

Instead of Lemma 4 one can apply the following.

**Lemma 4'**. Suppose that $\delta > 0$ is an ordinal number, $M$ is an $S$-module and $N$ is a $\delta$-simple $R$-submodule of $M$. Then for every $1 \leq i \leq n$, $s_i N$ is $\delta$-simple $R$-module or $\text{Gdim}_R s_i N < \delta$.

The proof follows from the fact that $L(Rs_i N) \approx L(R(N/K))$, where $K = \{x \in N \mid s_i x = 0\}$.

2. In [13] there was introduced the notion of the dual Gabriel dimension, $\text{Gcdim}_R M$, of a module $M$. The same proof as for $\text{Gdim}$ gives the inequality $\text{Gcdim}_R M \geq \text{Gcdim}_R M$ (if $\text{Gcdim}_R M$ exists !) but we have not been able to obtain the other inequality.
REFERENCES


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