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GABRIEL AND KRULL DIMENSIONS OF MODULES OVER RINGS GRADED BY FINITE GROUPS

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ABSTRACT. Let R be a ring graded by a finite group G with the identity component R_e and let M be a left R-module. It is proved that $\mathrm{Gdim}_R M = \mathrm{Gdim}_{R_e} M$, $\mathrm{Kdim}_R M = \mathrm{Kdim}_{R_e} M$ and $\mathrm{Ndim}_R M = \mathrm{Ndim}_{R_e} M$, where Gdim, Kdim and Ndim denote, respectively, Gabriel, Krull and dual Krull dimensions. The proofs are based on the use of lattice theory, a method which also gives alternative proofs of known results about normalizing extensions.

INTRODUCTION

In [10] it was proved that a module over a ring R graded by a finite group G satisfies ACC on R-submodules if and only if it satisfies the condition on R_e-submodules, where R_e is the identity component of R. The corresponding question for DCC is, like the analogous one concerning normalizing extensions (cf. [2, 5, 12]), more difficult and seems that rather no direct extension of arguments of [10] or those used for normalizing extensions could provide such a result. In this paper, following the experience of [5, 9], we approach the problem via lattice theory. It appears that on a level of lattices differences between ACC and DCC disappear and the idea of [10] can be employed. It allows us not only to handle the DCC case but also prove that none of Gabriel, Krull and dual Krull dimensions of an R-module changes when it is considered over R or R_e.

1. PRELIMINARIES

Throughout this paper R is a ring with unity graded by a finite group G, i.e., $R = \bigoplus_{g\in G} R_g$ for additive subgroups R_g, satisfying $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. Obviously R_e is a subring of R and $1 \in R_e$. All considered modules will be left unital modules and $N \leq M$ (or $N \leq_S M$ if we need to specify the ring of scalars) means that N is a submodule of M. For proper submodules we write $N < M$.

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In this paper we use some lattices and their sublattices. All those are modular with 0 and 1 (we consider only sublattices containing 0 and 1 of the original lattice). Given a module M we write $L(M)$ for the lattice of submodules of M. We denote by L^0 the dual lattice to the lattice L, i.e., the lattice having the same elements as L but carrying the reverse ordering. For all elements $a, b \in L$ with $a \leq b$, $[a, b]$ denotes the interval $\{x \in L \mid a \leq x \leq b\}$. If θ is a congruence on L then $\theta(a)$ denotes the congruence class determined by a. A lattice isomorphism will be denoted by $\approx$. All undefined terms and used results on lattices can be found in [1].

An ideal I of a lattice L is called essential if for every nonzero ideal J of L we have $I \cap J \neq 0$.

Lemma 1. *Let $\{s_g \mid g \in G\}$, where G is a finite group, be a set of order preserving maps of a lattice L into itself such that $s_e = \mathrm{id}$ and for every $g, h \in G$ and $x \in L$, $s_g s_h(x) \leq s_{gh}(x)$. If $f: L \to L'$ is a homomorphism of L onto a nonzero lattice L' and I is an essential ideal of L', then $f(\bigcap_{g \in G} s_g^{-1} f^{-1}(I)) \neq 0$.*

Proof. For every subset X of G containing e, let $I_X = \bigcap_{g \in X} s_g^{-1} f^{-1}(I)$. Obviously $f(I_{\{e\}}) = I$, so $f(I_{\{e\}}) \neq 0$. Let X be a subset of maximal cardinality such that $e \in X$ and $f(I_X) \neq 0$. If there is $h \in G \backslash X$ then by maximality of $|X|$, $f(s_h(a)) \neq 0$ for some $a \in I_X$. Thus, since the ideal I is essential, $I \cap [0, f(s_h(a))] \neq 0$. Hence there is $b \in L$ with $b \leq s_h(a)$ and $0 \neq f(b) \in I$. Since $s_{gh^{-1}}$ preserves the order of L, for every $g \in X$, $s_{gh^{-1}}(b) \leq s_{gh^{-1}} s_h(a)$. Now by the assumption, $s_{gh^{-1}}(b) \leq s_{gh^{-1}} s_h(a) \leq s_g(a)$. Since $a \in I_X$ and $g \in X$, $f(s_g(a)) \in I$. Consequently, $f(s_{gh^{-1}}(b)) \in I$. Thus $b \in I_{Xh^{-1} \cup \{e\}}$ and since $f(b) \neq 0$, we have $f(I_{Xh^{-1} \cup \{e\}}) \neq 0$. This contradicts maximality of $|X|$, so $X = G$ and the lemma follows.

A sublattice K of a lattice L is said to be dense in L if for each element k in $K \backslash \{1\}$, every essential ideal I of the lattice $[k, 1]$ contains an element $k_1 \in K$, $k_1 \neq k$.

The following is the key proposition of this paper. A similar proposition is true for normalizing extensions, as proved in §4.

Proposition 1. *If M is an R-module and f is a homomorphism of $L({}_{R_e}M)$ onto a nonzero lattice L, then $f(L({}_R M))$ $(f(L^0({}_R M)))$ is a dense sublattice of L (L^0). In particular, $L({}_R M)$ $(L^0({}_R M))$ is a dense sublattice of $L({}_{R_e}M)$ $(L^0({}_{R_e}M))$.*

Proof. For every $N \leq_R M$ there is a natural isomorphism $i: L({}_{R_e}(M/N)) \to [N, M] \subseteq L({}_{R_e}M)$ such that $i(L({}_R M/N)) = [N, M] \cap L({}_R M)$. Now, f maps $[N, M]$ onto $[f(N), 1]$. Thus for every $k \in f(L({}_R M))$ there is an R-module K and a homomorphism $\overline{f}$ of $L({}_{R_e}K)$ onto $[k, 1] \subset L$ such that $\overline{f}(L({}_R K)) = f(L({}_R M)) \cap [k, 1]$. Hence to prove that $f(L({}_R M))$ is a dense sublattice of L it suffices to show that every essential ideal I of L contains a nonzero element of

$f(L({}_RM))$. Define for each $g \in G$, $s_g: L({}_{R_e}M) \to L({}_{R_e}M)$ by $s_g(N) = R_gN$. Obviously for every $g, h \in G$ and $T \in L({}_{R_e}M)$ we have $s_gs_h(T) \leq s_{gh}(T)$ and $s_e = \mathrm{id}$. Hence by Lemma 1 there is $N \leq_{R_e} M$ such that $f(N) \neq 0$ and $f(R_gN) \in I$ for all $g \in G$. Obviously $f(\sum_{g\in G} R_gN)$ is a nonzero element of $I \cap f(L({}_RM))$.

Going the same line as above in the dual situation but now applying the maps $s_g^0: L^0({}_{R_e}M) \to L^0({}_{R_e}M)$ given by $s_g^0(N) = R_{g^{-1}}^{-1}(N) = \{m \in M \mid R_{g^{-1}}m \leq N\}$ one obtains that $f(L^0({}_RM))$ is a dense sublattice of L^0.

2. Krull dimension

In this section we study the Krull dimension (Kdim) and the dual Krull dimension (Ndim) of modules over G-graded rings. Recall that for a module M, $\mathrm{Kdim}\, M$ (Ndim) is defined (cf. [3, 7]) by induction as follows:

$$\mathrm{Kdim}\, M = -1 (\mathrm{Ndim}\, M = -1) \text{ if and only if } M = 0;$$

if α is an ordinal number and $\mathrm{Kdim}\, M \not< \alpha$ ($\mathrm{Ndim}\, M \not< \alpha$) then $\mathrm{Kdim}\, M = \alpha$ ($\mathrm{Ndim}\, M = \alpha$) provided for every descending (ascending) chain $M_1 \geq M_2 \geq \cdots (M_1 \leq M_2 \leq \cdots)$ of submodules of M there is an n such that $\mathrm{Kdim}\, M_i/M_{i+1} < \alpha$ ($\mathrm{Ndim}\, M_{i+1}/M_i < \alpha$) for $i \geq n$.

One of basic properties of these dimensions says that if $N \leq M$ then $\mathrm{Kdim}\, M = \sup\{\mathrm{Kdim}\, N, \mathrm{Kdim}\, M/N\}$ if either side exists. The same is true for Ndim.

Proposition 2. *Let M be a module, $\alpha \geq 0$ an ordinal number and let $\kappa_L(\eta_L)$ be the relation on $L = L(M)$ defined by $M_1\kappa_LM_2$ ($M_1\eta_LM_2$) if and only if* $\mathrm{Kdim}(M_1 + M_2/M_1 \cap M_2) < \alpha(\mathrm{Ndim}(M_1 + M_2/M_1 \cap M_2) < \alpha)$. *Then*

(a) *κ_L and η_L are congruences on L;*
(b) $\mathrm{Kdim}\, M = \alpha(\mathrm{Ndim}\, M = \alpha)$ *if and only if the lattice L/κ_L (L/η_L) is nonzero and satisfies DCC (ACC).*

Proof. We prove the proposition for Kdim; the proof for Ndim is similar. In view of Lemma 1.3.8 of [8], to prove that κ_L is a congruence it suffices to check that if $M_1 \leq M_2$ are submodules of M such that $\mathrm{Kdim}\, M_2/M_1 < \alpha$ then for every submodule T of M, $\mathrm{Kdim}\, M_2 + T/M_1 + T < \alpha$ and $\mathrm{Kdim}\, M_2 \cap T/M_1 \cap T < \alpha$. However these are immediate consequences of isomorphisms $M_2 + T/M_1 + T \simeq M_2/M_1 + T \cap M_2$, $M_2 \cap T/M_1 \cap T \simeq M_1 + T \cap M_2/M_1$ and the property of Kdim formulated before the proposition.

It is clear that if the lattice L/κ_L, is nonzero and satisfies DCC then $\mathrm{Kdim}\, L = \alpha$. Conversely, let $\kappa_L(M_1) \geq \kappa_L(M_2) \geq \cdots$ be a descending chain of L/κ_L. Obviously if $N_i = M_1 \cap \cdots \cap M_i$ for $i = 1, 2, \ldots,$ then $N_1 \geq N_2 \geq \cdots$ and $\kappa_L(N_i) = \kappa_L(M_i)$. Hence assuming $\mathrm{Kdim}\, M = \alpha$ we get that for some n, $\kappa_L(M_n) = \kappa_L(N_n) = \kappa_L(N_{n+1}) = \kappa_L(M_{n+i})$, $i = 1, 2, \ldots,$ so the lattice L/κ_L satisfies DCC.

To obtain the main result of this section we also need the following

Lemma 2. *A dense sublattice K of a lattice L satisfies ACC if and only if the lattice L satisfies ACC.*

Proof. Obviously if L satisfies ACC then the same is true for K. Suppose that K satisfies ACC and L does not. Let M be an element of K maximal with respect to the property "$[m, 1]$ does not satisfy ACC". Let $a_1 < a_2 < \cdots$ be a strictly ascending chain of elements of $[m, 1]$ and let $I = \bigcup_{k \geq 1}[m, a_k]$. Of course I is an ideal of $[m, 1]$. By Zorn's Lemma there is an ideal J of $[m, 1]$ maximal with respect to the property $I \cap J = \{m\}$. Using modularity one can easily check that the ideal $I \vee J$ of $[m, 1]$ generated by $I \cup J$ is essential in $[m, 1]$. Hence the density property implies that there is $m_1 \in K \cap (I \vee J)$, $m_1 > m$. Hence for some $a \in I$, $b \in J$ we have $m < m_1 \leq a \vee b$. By the definition of I, $a \leq a_i$ for some i. Thus $[m_1, 1]$ contains the chain

$$a_i \vee b \leq a_{i+1} \vee b \leq \cdots .$$

The choice of m implies that it stabilizes. Hence for some $k \geq 0$, $a_{i+k+1} \vee b = a_{i+k} \vee b$. Now applying the modular law one obtains $a_{i+k+1} = a_{i+k+1} \wedge (a_{i+k+1} \vee b) = a_{i+k+1} \wedge (a_{i+k} \vee b) = a_{i+k} \vee (a_{i+k+1} \wedge b)$. But $I \cap J = \{m\}$, so we have $a_{i+k+1} \wedge b = m$. Hence $a_{i+k+1} = a_{i+k}$, contradiction.

Proposition 1 and Lemma 2 give immediately

Corollary 1. *If M is an R-module then*

(i) [10] *${}_RM$ is Noetherian if and only if ${}_{R_e}M$ is Noetherian;*
(ii) *${}_RM$ is Artinian if and only if ${}_{R_e}M$ is Artinian.*

Now we prove the main result of this section.

Theorem 1. *For every R-module M,* $\operatorname{Kdim}_R M = \operatorname{Kdim}_{R_e} M$ ($\operatorname{Ndim}_R M = \operatorname{Ndim}_{R_e} M$) *if either side exists.*

Proof. We will concentrate here on Kdim; the proof for Ndim is parallel.

The inequality $\operatorname{Kdim}_R M \leq \operatorname{Kdim}_{R_e} M$ is obvious. The proof of the other inequality is by induction on $\alpha = \operatorname{Kdim}_R M$. The case $\alpha = 0$ is done in Corollary 1(ii). Now suppose that $\alpha > 0$ and the result holds for $\beta < \alpha$. The induction assumption gives $\kappa_K = \kappa_L \cap (K \times K)$, where $L = L({}_{R_e}M)$, $K = L({}_RM)$ and κ_L, κ_K are defined as in Proposition 2. Hence $K/\kappa_K \approx \{\kappa_L(N) \mid N \in K\} \subseteq L/\kappa_L$. Applying Proposition 1 and Lemma 2 we obtain that the lattice K/κ_K satisfies DCC if and only if the lattice L/κ_L satisfies DCC. This and Proposition 2 end the proof.

Theorem 1, and hence also Corollary 1, was proved in [4] when the grading is assumed inner.

3. Gabriel dimension

We will use the following Lanski's [11] characterization of the Gabriel dimension, $\operatorname{Gdim} M$, of a module M. It is given inductively starting with

$\operatorname{Gdim} M = 0$ if and only if $M = 0$. Let α be a nonlimit ordinal and suppose that $\operatorname{Gdim} M = \beta$ has been defined for $\beta < \alpha$. Call A an α-simple module if for every $0 \neq N \leq A$, both $\operatorname{Gdim} N \not< \alpha$ and $\operatorname{Gdim} A/N < \alpha$. Then $\operatorname{Gdim} M = \alpha$ if $\operatorname{Gdim} M \not< \alpha$ and if for each $N < M$, M/N contains a β-simple module for $\beta \leq \alpha$. When α is a limit ordinal, $\operatorname{Gdim} M = \alpha$ if $\operatorname{Gdim} M \not< \alpha$ and if for each $N < M$, M/N contains a β-simple module for $\beta < \alpha$.

Now we state some auxiliary facts which will be used in the proof of the main result of this section.

Proposition 3 [11]. (a) *For any* $N \leq M$,

$$\operatorname{Gdim} M = \sup\{\operatorname{Gdim} N, \operatorname{Gdim} M/N\}$$

if either side exists.

(b) *If* $M_i \leq M$, $i \in I$ *then* $\operatorname{Gdim} \sum_{i \in I} M_i = \sup\{\operatorname{Gdim} M_i \mid i \in I\}$ *if either side exists.*

Proposition 4. *Let* α *be an ordinal number and let* γ_L *be the relation on* $L = L(M)$ *given by* $M_1 \gamma_L M_2$ *if and only if* $\operatorname{Gdim}(M_1 + M_2/M_1 \cap M_2) < \alpha$. *Then*

(i) γ_L *is a congruence relation*;

(ii) *if* α *is nonlimit and if* N *is an* α*-simple submodule of* M *then* $\gamma_L(N)$ *is an atom of* L/γ_L;

(iii) *if* α *is nonlimit and if for some* $N \leq M$, $\gamma_L(N)$ *is an atom of* L/γ_L *then there is* $T < N$ *such that* $\operatorname{Gdim} T < \alpha$ *and* N/T *is* α*-simple.*

Proof. The proof of (i) follows easily from Proposition 3(a). The condition (ii) is a direct consequence of the definition of γ_L. Now we prove (iii). Let $T = \sum\{U \leq N \mid \operatorname{Gdim} U < \alpha\}$. By Proposition 3(b) $\operatorname{Gdim} T < \alpha$. Now if $T < T' \leq N$ then Proposition 3(a) and the choice of T' guarantee that $\operatorname{Gdim} T'/T \not< \alpha$. On the other hand, since $\gamma_L(N)$ is an atom of L/γ_L and $\gamma_L(T') \neq 0$, we have $\gamma_L(N) = \gamma_L(T')$. Hence, by the definition of γ_L, $\operatorname{Gdim} N/T' < \alpha$.

Lemma 3. *If* $K = \{0, 1\}$ *is a dense sublattice of a lattice* L *then* $1 = a_1 \vee \cdots \vee a_n$ *for some atoms* $a_1, \ldots, a_n \in L$.

Proof. By Lemma 2 the lattice L satisfies ACC, so it suffices to prove that for every $x \in L$ there is $y \in L$ such that $x \wedge y = 0$ and $x \vee y = 1$. Let J be an ideal of L maximal with respect to the property $[0, x] \cap J = \{0\}$. Then the ideal $[0, x] \vee J$ generated by $[0, x] \cup J$ is essential in L. Hence, since $\{0, 1\}$ is a dense sublattice of L, $1 \in [0, x] \vee J$. It means that $1 = x \vee y$ for some $y \in J$. Obviously $x \wedge y \in [0, x] \cap J = \{0\}$.

Lemma 4. *Suppose that* $\delta > 0$ *is an ordinal number and* M *is an* R*-module such that for every* $0 \neq N \leq_{R_e} M$ *there exists* $0 \neq N' \leq_{R_e} N$, ${}_{R_e}N'$ *–* β*-simple for some* $\beta < \delta$. *Then there exists* $0 \neq T \leq_{R_e} M$ *such that for every* $g \in G$, $R_g T = 0$ *or* $R_g T$ *is a* β*-simple* R_e*-module for some* $\beta < \delta$.

Proof. Let H be a subset of G containing e, of maximal cardinality such that for some $0 \neq X \leq_{R_e} M$ and all $h \in H$, $R_h X = 0$ or $R_h X$ is β-simple

R_e-module for some $\beta < \delta$. If there is $g \in G \backslash H$ then $R_g X \neq 0$. Hence there exists $0 \neq Y \leq_{R_e} R_g X$, ${}_{R_e}Y - \beta$-simple for some $\beta < \delta$. Now for every $h \in H$, $R_{hg^{-1}} Y \leq R_{hg^{-1}} R_g X \leq R_h X$. Thus for every $\overline{h} \in Hg^{-1} \cup \{e\}$, $R_{\overline{h}} Y = 0$ or $R_{\overline{h}} Y$ is a β-simple R_e-module for some $\beta < \delta$. This contradiction proves that $G = H$, which gives the result.

Now we are ready to prove the main result of this section.

Theorem 2. *For every R-module M,* $\operatorname{Gdim}_R M = \operatorname{Gdim}_{R_e} M$ *if either side exists.*

Proof. We will prove by induction on α that if one side is equal to α then so is the other. It is clear for $\alpha = 0$. Suppose now that $\alpha > 0$ and the result holds for $\beta < \alpha$. Observe that this assumption says in particular that if γ_L and γ_K are the congruences defined in Proposition 4 on $L = L({}_{R_e}M)$ and $K = L({}_R M)$ respectively then $\gamma_K = \gamma_L \cap (K \times K)$. Thus $K/\gamma_K \approx \{\gamma_L(N) \mid N \in K\} \subseteq L/\gamma_L$ and by Proposition 1 K/γ_K can be treated as a dense sublattice of L/γ_L.

Suppose that $\operatorname{Gdim}_R M = \alpha$. Since $\gamma_K = \gamma_L \cap (K \times K)$, $\operatorname{Gdim}_{R_e} M \not< \alpha$. Thus we have to show that for every $N <_{R_e} M$, M/N contains a nonzero β-simple R_e-module, where $\beta \leq \alpha$ if α is nonlimit and $\beta < \alpha$ otherwise. Let T be the largest R-submodule of M which is contained in N. Passing to the module M/T we can assume that $T = 0$. Since $\operatorname{Gdim}_R M = \alpha$ there is $0 \neq T' \leq_R M$, ${}_R T' - \beta$-simple for some $\beta \leq \alpha$ if α is nonlimit and $\beta < \alpha$ otherwise. If $\beta < \alpha$ then by the induction assumption $\operatorname{Gdim}_{R_e}(T'/T' \cap N) < \alpha$, so $0 < \operatorname{Gdim}_{R_e}(T' + N/N) < \alpha$ and we are done. If $\beta = \alpha$ then by Proposition 4(ii) $\gamma_L(T')$ is an atom of the sublattice $\{\gamma_L(N) \mid N \in K\}$ of L/γ_L. Thus by Proposition 1 and Lemma 3 there are atoms $\gamma_L(N_1), \ldots, \gamma_L(N_k)$ of L/γ_L such that $\gamma_L(T') = \gamma_L(N_1) \vee \cdots \vee \gamma_L(N_k)$. It follows that the length of $[\gamma_L(N), \gamma_L(T' + N)]$ is finite. Hence by Proposition 4(iii) there is ${}_{R_e}\overline{N}$ such that $N < \overline{N} < T' + N$ and ${}_{R_e}(\overline{N}/N)$ is β-simple for some $\beta \leq \alpha$, as required.

Suppose that $\operatorname{Gdim}_{R_e} M = \alpha$. By the induction assumption $\operatorname{Gdim}_R M \not< \alpha$. Thus we have to show that for every $N <_R M$, M/N contains a β-simple R-submodule with $\beta \leq \alpha$ if α is nonlimit and $\beta < \alpha$ otherwise. We can obviously assume that $N = 0$. Observe that M satisfies the assumption of Lemma 4 with $\delta = \alpha$ in the limit case and $\delta = \alpha + 1$ otherwise. Hence there is $0 \neq X \leq_{R_e} M$ such that for every $g \in G$, $R_g X = 0$ or $R_g X$ is β-simple R_e-module for some $\beta \leq \alpha$ if α is nonlimit and $\beta < \alpha$ otherwise. Now $T = \sum_{g \in G} R_g X$ is a nonzero R-submodule of M. If for all $g \in G$, $\operatorname{Gdim}_{R_e}(R_g X) < \alpha$ then the induction assumption gives $\operatorname{Gdim}_R T < \alpha$ and we are done. If for some $g \in G$, $\operatorname{Gdim}_{R_e}(R_g X) = \alpha$ then by Proposition 4(ii) $\gamma_L(T)$ is a join of atoms of L/γ_L. Hence also the lattice K/γ_K contains an atom $\gamma_K(T') \leq \gamma_K(T)$. Thus by Proposition 4(iii) there exists $T'' \leq_R T$ such that $\operatorname{Gdim}_R T'' < \alpha$ and ${}_R(T'/T'')$ is α-simple. If $T'' \neq 0$ then it contains a

nonzero β-simple R-module for some $\beta < \alpha$. If $T'' = 0$ then T' is α-simple and we are done. This ends the proof.

4. Remarks

1. The methods of this paper can be almost directly applied to obtain analogous results for some other than $R_e \subseteq R$ ring extensions of "finite type". Here we comment how they could be used to obtain some known results on normalizing extensions.

Recall that an extension $R \subseteq S$ of rings with the same 1 is finite normalizing if for some $s_1 = 1$, $s_2, \dots, s_n \in S$, $S = Rs_1 + \cdots + Rs_n$ and $Rs_j = s_jR$ for $1 \leq j \leq n$.

It is known [2, 6, 12, 13], that if $R \subseteq S$ is a finite normalizing extension, M is an S-module then $\operatorname{Kdim}_S M = \operatorname{Kdim}_R M$ and $\operatorname{Gdim}_S M = \operatorname{Gdim}_R M$. It is clear that proving the above result by the method of this paper one will have some problems with Proposition 1 and Lemma 4 only. In the present situation instead of Proposition 1 one can apply the following

Proposition 1′. *Let M be an S-module and φ_i, $\varphi_i^0 : L({}_RM) \to L({}_RM)$ be given by $\varphi_i(N) = s_iN$ and $\varphi_i^0(N) = \{m \in M \mid s_im \in N\}$. If f is a homomorphism of $L({}_RM)$ onto a nonzero lattice L such that $f(X) = f(Y)$ implies $f(\varphi_i(X)) = f(\varphi_i(Y))$ $(f(\varphi_i^0(X)) = f(\varphi_i^0(Y)))$ for $1 \leq i \leq n$, then $f(L({}_SM))$ $(f(L^0({}_SM)))$ is a dense sublattice of L (L^0).*

Proof (Sketch). Similarly as in the proof of Proposition 1 to get that $f(L({}_RM))$ is a dense sublattice of L it suffices to show that every essential ideal I of L contains a nonzero element of $f(L({}_SM))$. The assumption on f and φ_i implies that for every $1 \leq i \leq n$, $g_i : L \to L$ given by $g_i(x) = f(\varphi_i(X))$ where $x = f(X)$, is a well-defined map. One easily checks that $g_i(0) = 0$, for every $x, y \in L$, $g_i(x \vee y) = g_i(x) \vee g_i(y)$ and if $x \leq y$ then $g_i([x, y]) = [g_i(x), g_i(y)]$. These imply that $g_i^{-1}(I)$, $1 \leq i \leq n$, are essential ideals of L. Hence $\bigcap_{i=1}^n g_i^{-1}(I) \neq 0$. Now if $0 \neq x \in \bigcap_{i=1}^n g_i^{-1}(I)$ then, since $g_1 = \mathrm{id}$, $y = \bigvee_{i=1}^n g_i(x) \neq 0$. Clearly $y \in I \cap f(L({}_SM))$.

Applying φ_i^0 instead of φ_i one gets the dual result.

Instead of Lemma 4 one can apply the following.

Lemma 4′. *Suppose that $\delta > 0$ is an ordinal number, M is an S-module and N is a δ-simple R-submodule of M. Then for every $1 \leq i \leq n$, s_iN is δ-simple R-module or* $\operatorname{Gdim}_R s_iN < \delta$.

The proof follows from the fact that $L({}_Rs_iN) \approx L({}_R(N/K))$, where $K = \{x \in N \mid s_ix = 0\}$.

2. In [13] there was introduced the notion of the dual Gabriel dimension, $\operatorname{Gcodim} M$, of a module M. The same proof as for Gdim gives the inequality $\operatorname{Gcodim}_{R_e} M \geq \operatorname{Gcodim}_R M$ (if $\operatorname{Gcodim}_R M$ exists !) but we have not been able to obtain the other inequality.

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