ON THE MAXIMUM DENSITY OF MINIMAL ASYMPTOTIC BASES

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Abstract. A set $A$ of nonnegative integers is an asymptotic basis of order $h$ if every sufficiently large integer is the sum of $h$ elements of $A$. It is proved that if $A$ is an asymptotic basis of order $h$ with lower asymptotic density $d_L(A) > 1/h$, then there is a set $W$ contained in $A$ such that $W$ has positive asymptotic density and $A \setminus W$ is an asymptotic basis of order $h$. This implies that if $A$ is a minimal asymptotic basis of order $h$, then $d_L(A) \leq 1/h$.

Let $A$ be a subset of the nonnegative integers $\mathbb{N}$. Denote by $hA$ the set of all numbers $n$ that can be written in the form $n = a_1 + \cdots + a_h$, where $a_i \in A$ for $i = 1, \ldots, h$ and the $a_i$ are not necessarily distinct. The set $A$ is a basis of order $h$ if $hA = \mathbb{N}$. If $hA$ contains every sufficiently large integer, then $A$ is an asymptotic basis of order $h$. Much of classical additive number theory is the study of special sets $A$ that are bases or asymptotic bases of order $h$ for some $h \geq 2$.

Minimal asymptotic bases form an important extremal class in additive number theory. The asymptotic basis $A$ of order $h$ is minimal if no proper subset of $A$ is an asymptotic basis of order $h$. It follows that if $A$ is minimal, then for every element $a \in A$ there must be infinitely many positive integers $n$, each of whose representations as a sum of $h$ elements of $A$ includes the number $a$ as a summand. Stöhr [7] introduced the concept of minimal asymptotic basis, and Härtter [3] proved that minimal asymptotic bases of order $h$ exist for all $h \geq 2$. Erdös and Nathanson [1] survey recent results on minimal asymptotic bases.

For any set $A$ of integers, the counting function of $A$, denoted $A(x)$, is defined by $A(x) = \text{card}\{a \in A | 1 \leq a \leq x\}$. The lower asymptotic density of $A$, denoted $d_L(A)$, is defined by $d_L(A) = \liminf \frac{A(x)}{x}$. If $\alpha = \lim \frac{A(x)}{x}$ exists, then $\alpha$ is called the asymptotic density of $A$, and denoted $d(A)$.

For every $h \geq 2$, Nathanson [5, 6] has constructed minimal asymptotic bases $A$ of order $h$ that satisfy $A(x) \ll x^{1/h}$. Since $A(x) \gg x^{1/h}$ for every

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asymptotic basis \( A \) of order \( h \), it follows that these examples are the "thinnest" possible minimal asymptotic bases. Erdős and Nathanson [2] have recently constructed minimal asymptotic bases \( A \) of order \( h \) with \( d(A) = 1/h \). In this paper we prove that these are the "fattest" possible minimal asymptotic bases, in the sense that \( d_L(A) \leq 1/h \) for every minimal asymptotic basis \( A \) of order \( h \). The proof uses a deep result of Kneser on the lower asymptotic density of sums of sets of integers.

Let \( A \) and \( B \) be sets of nonnegative integers. We shall write \( A \sim B \) if \( A \) and \( B \) coincide for all sufficiently large integers. For \( g \geq 1 \), let \( B^{(g)} \) be the set of all integers \( t \) such that \( t \equiv b \pmod{g} \) for some \( b \in B \). Let \( h \geq 2 \). Kneser [4] proved that either \( d_L(hB) \geq hd_L(B) \) or there exists an integer \( g \geq 1 \) such that \( hB \sim hB^{(g)} \) and \( d_L(hB) \geq hd_L(B) - (h - 1)/g \).

**Theorem 1.** Let \( h \geq 2 \) and let \( A \) be an asymptotic basis of order \( h \). If \( B \subseteq A \) and \( d_L(B) > 1/h \), then there is a finite set \( F \subseteq A \setminus B \) such that \( B \cup F \) is an asymptotic basis of order \( h \).

**Proof.** Assume that \( B \) is not an asymptotic basis of order \( h \). Then \( hB \not\sim N \). Since \( d_L(hB) \leq 1 < hd_L(B) \), Kneser's theorem implies that there exists an integer \( g \geq 1 \) such that \( hB \sim hB^{(g)} \) and

\[
hd_L(B) - (h - 1)/g \leq d_L(hB) = d(hB^{(g)}) .
\]

Since \( hB^{(g)} \) is a union of congruence classes modulo \( g \), it follows from \( hB^{(g)} \not\sim N \) that \( d(hB^{(g)}) = r/g \leq 1 - 1/g \). Let \( d_L(B) = 1/h + \delta \). Then \( \delta > 0 \), and

\[
1 + h\delta - (h - 1)/g \leq 1 - 1/g ,
\]

hence

\[
(*) \quad 1 \leq g \leq (h - 2)/(h\delta) .
\]

Note that \( B \subseteq B^{(g)} \) and \( B \subseteq A \). If \( A \subseteq B^{(g)} \), then \( hB \subseteq hA \subseteq hB^{(g)} \). Since \( A \) is an asymptotic basis of order \( h \), it follows that \( N \sim hA \sim hB^{(g)} \sim hB \), which contradicts the hypothesis that \( B \) is not an asymptotic basis of order \( h \). Therefore, \( A \not\subseteq B^{(g)} \) and there exists an integer \( c_i \in A \) such that \( c_i \not\equiv b \pmod{g} \) for every \( b \in B \). Let \( g_1 = g \).

Let \( B_1 = B \cup \{c_1\} \). If \( B_1 \) is an asymptotic basis of order \( h \), let \( F = \{c_1\} \). If \( B_1 \) is not an asymptotic basis of order \( h \), there exists an integer \( g_2 \) satisfying \((*)\) and an integer \( c_2 \in A \) such that \( c_2 \not\equiv b \pmod{g_2} \) for all \( b \in B_1 \). Let \( B_2 = B_1 \cup \{c_2\} = B \cup \{c_1, c_2\} \).

Continuing inductively, we obtain a sequence of integers \( g_1, g_2, \ldots \) satisfying \((*)\) and a sequence \( c_1, c_2, \ldots \) of elements of \( A \) such that \( c_i \not\equiv b \pmod{g_i} \) for all \( b \in B_{i-1} = B \cup \{c_1, c_2, \ldots, c_{i-1}\} \). Since there are only finitely many distinct congruence classes with respect to moduli bounded above by \((h - 2)/(h\delta)\), the inductive process must terminate in a finite number of steps, and we obtain a finite set \( F = \{c_1, c_2, \ldots\} \subseteq A \) such that \( B \cup F \) is an asymptotic basis of order \( h \). This completes the proof.

**Theorem 2.** Let \( h \geq 2 \), and let \( A \) be an asymptotic basis of order \( h \) with \( d_L(A) = 1/h + \delta \), where \( \delta > 0 \). Let \( 0 < \tau < \delta \). Then there is a set \( W \subseteq A \) with...
asymptotic density $d(W) = \tau$ such that $A \setminus W$ is an asymptotic basis of order $h$.

**Proof.** Let $W'$ be any subset of $A$ with $d(W') = \tau$. Let $B = A \setminus W'$. Then $d_L(B) = 1/h + \delta - \tau > 1/h$. By Theorem 1, there is a finite set $F \subseteq A \setminus B = W'$ such that $B \cup F = A \setminus (W' \setminus F)$ is an asymptotic basis of order $h$. Let $W = W' \setminus F$. Then $d(W) = d(W') = \tau$. This proves the theorem.

**Theorem 3.** Let $A$ be a minimal asymptotic basis of order $h$. Then $d_L(A) \leq 1/h$.

**Proof.** If $d_L(A) > 1/h$, then by Theorem 2 there is a set $W \subseteq A$ with $d(W) > 0$ such that $A \setminus W$ is an asymptotic basis of order $h$, and this contradicts the minimality of $A$.

**References**


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