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ON THE MAXIMUM DENSITY OF MINIMAL ASYMPTOTIC BASES

MELVYN B. NATHANSON AND ANDRÁS SÁRKÖZY

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ABSTRACT. A set A of nonnegative integers is an asymptotic basis of order h if every sufficiently large integer is the sum of h elements of A. It is proved that if A is an asymptotic basis of order h with lower asymptotic density $d_L(A) > 1/h$, then there is a set W contained in A such that W has positive asymptotic density and $A \backslash W$ is an asymptotic basis of order h. This implies that if A is a minimal asymptotic basis of order h, then $d_L(A) \leq 1/h$.

Let A be a subset of the nonnegative integers N. Denote by hA the set of all numbers n that can be written in the form $n = a_1 + \cdots + a_h$, where $a_i \in A$ for $i = 1, \ldots, h$ and the a_i are not necessarily distinct. The set A is a *basis of order* h if $hA = N$. If hA contains every sufficiently large integer, then A is an *asymptotic basis of order* h. Much of classical additive number theory is the study of special sets A that are bases or asymptotic bases of order h for some $h \geq 2$.

Minimal asymptotic bases form an important extremal class in additive number theory. The asymptotic basis A of order h is *minimal* if no proper subset of A is an asymptotic basis of order h. It follows that if A is minimal, then for every element $a \in A$ there must be infinitely many positive integers n, each of whose representations as a sum of h elements of A includes the number a as a summand. Stöhr [7] introduced the concept of minimal asymptotic basis, and Härtter [3] proved that minimal asymptotic bases of order h exist for all $h \geq 2$. Erdös and Nathanson [1] survey recent results on minimal asymptotic bases.

For any set A of integers, the *counting function* of A, denoted $A(x)$, is defined by $A(x) = \operatorname{card}(\{a \in A | 1 \leq a \leq x\})$. The *lower asymptotic density* of A, denoted $d_L(A)$, is defined by $d_L(A) = \liminf A(x)/x$. If $\alpha = \lim A(x)/x$ exists, then α is called the *asymptotic density* of A, and denoted $d(A)$.

For every $h \geq 2$, Nathanson [5, 6] has constructed minimal asymptotic bases A of order h that satisfy $A(x) \ll x^{1/h}$. Since $A(x) \gg x^{1/h}$ for every

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asymptotic basis A of order h, it follows that these examples are the "thinnest" possible minimal asymptotic bases. Erdös and Nathanson [2] have recently constructed minimal asymptotic bases A of order h with $d(A) = 1/h$. In this paper we prove that these are the "fattest" possible minimal asymptotic bases, in the sense that $d_L(A) \leq 1/h$ for every minimal asymptotic basis A of order h. The proof uses a deep result of Kneser on the lower asymptotic density of sums of sets of integers.

Let A and B be sets of nonnegative integers. We shall write $A \sim B$ if A and B coincide for all sufficiently large integers. For $g \geq 1$, let $B^{(g)}$ be the set of all integers t such that $t \equiv b \pmod{g}$ for some $b \in B$. Let $h \geq 2$. Kneser [4] proved that either $d_L(hB) \geq hd_L(B)$ or there exists an integer $g \geq 1$ such that $hB \sim hB^{(g)}$ and $d_L(hB) \geq hd_L(B) - (h-1)/g$.

Theorem 1. *Let $h \geq 2$ and let A be an asymptotic basis of order h. If $B \subseteq A$ and $d_L(B) > 1/h$, then there is a finite set $F \subseteq A \backslash B$ such that $B \cup F$ is an asymptotic basis of order h.*

Proof. Assume that B is not an asymptotic basis of order h. Then $hB \nsim N$. Since $d_L(hB) \leq 1 < hd_L(B)$, Kneser's theorem implies that there exists an integer $g \geq 1$ such that $hB^{(g)} \sim hB \nsim N$ and

$$hd_L(B) - (h-1)/g \leq d_L(hB) = d(hB^{(g)}).$$

Since $hB^{(g)}$ is a union of congruence classes modulo g, it follows from $hB^{(g)} \nsim N$ that $d(hB^{(g)}) = r/g \leq 1 - 1/g$. Let $d_L(B) = 1/h + \delta$. Then $\delta > 0$, and $1 + h\delta - (h-1)/g \leq 1 - 1/g$, hence

$$(*) \qquad 1 \leq g \leq (h-2)/(h\delta).$$

Note that $B \subseteq B^{(g)}$ and $B \subseteq A$. If $A \subseteq B^{(g)}$, then $hB \subseteq hA \subseteq hB^{(g)}$. Since A is an asymptotic basis of order h, it follows that $N \sim hA \sim hB^{(g)} \sim hB$, which contradicts the hypothesis that B is not an asymptotic basis of order h. Therefore, $A \nsubseteq B^{(g)}$ and there exists an integer $c_1 \in A$ such that $c_1 \not\equiv b \pmod{g}$ for every $b \in B$. Let $g_1 = g$.

Let $B_1 = B \cup \{c_1\}$. If B_1 is an asymptotic basis of order h, let $F = \{c_1\}$. If B_1 is not an asymptotic basis of order h, there exists an integer g_2 satisfying $(*)$ and an integer $c_2 \in A$ such that $c_2 \not\equiv b \pmod{g_2}$ for all $b \in B_1$. Let $B_2 = B_1 \cup \{c_2\} = B \cup \{c_1, c_2\}$.

Continuing inductively, we obtain a sequence of integers $g_1, g_2, \ldots$ satisfying $(*)$ and a sequence $c_1, c_2, \ldots$ of elements of A such that $c_i \not\equiv b \pmod{g_i}$ for all $b \in B_{i-1} = B \cup \{c_1, c_2, \ldots, c_{i-1}\}$. Since there are only finitely many distinct congruence classes with respect to moduli bounded above by $(h-2)/(h\delta)$, the inductive process must terminate in a finite number of steps, and we obtain a finite set $F = \{c_1, c_2, \ldots\} \subseteq A$ such that $B \cup F$ is an asymptotic basis of order h. This completes the proof.

Theorem 2. *Let $h \geq 2$, and let A be an asymptotic basis of order h with $d_L(A) = 1/h + \delta$, where $\delta > 0$. Let $0 < \tau < \delta$. Then there is a set $W \subseteq A$ with*

asymptotic density $d(W) = \tau$ *such that* $A \backslash W$ *is an asymptotic basis of order* h.

Proof. Let W' be any subset of A with $d(W') = \tau$. Let $B = A \backslash W'$. Then $d_L(B) = 1/h + \delta - \tau > 1/h$. By Theorem 1, there is a finite set $F \subseteq A \backslash B = W'$ such that $B \cup F = A \backslash (W' \backslash F)$ is an asymptotic basis of order h. Let $W = W' \backslash F$. Then $d(W) = d(W') = \tau$. This proves the theorem.

Theorem 3. *Let* A *be a minimal asymptotic basis of order* h. *Then* $d_L(A) \leq 1/h$.

Proof. If $d_L(A) > 1/h$, then by Theorem 2 there is a set $W \subseteq A$ with $d(W) > 0$ such that $A \backslash W$ is an asymptotic basis of order h, and this contradicts the minimality of A.

References

1. P. Erdös and M. B. Nathanson, *Problems and results on minimal bases in additive number theory*, in: Number Theory, New York 1985–86, Lecture Notes in Math., vol. 1240, Springer-Verlag, Heidelberg, 1987, pp. 87–96.
2. ——, and ——, *Minimal asymptotic bases with prescribed densities*, Illinois J. Math. **32** (1988), to appear.
3. E. Härtter, *Ein Beitrag zur Theorie der Minimalbasen*, J. Reine Angew. Math. **196** (1956), 170–204.
4. M. Kneser, *Abschätzung der asymptotischen Dichte von Summenmengen*, Math. Z. **58** (1953), 459–484.
5. M. B. Nathanson, *Minimal bases and maximal nonbases in additive number theory*, J. Number Theory **6** (1974), 324–333.
6. ——, *Minimal bases and powers of* 2, Acta Arith. **51** (1988), 95–102.
7. A. Stöhr, *Gelöste und ungelöste Fragen über Basen der natürlichen Zahlenreihe*. II, J. Reine Angew. Math. **194** (1955), 111–140.

Provost and Vice President for Academic Affairs, Lehman College (CUNY), Bronx, New York 10468

Department of Mathematics, Baruch College (CUNY), New York, New York 10010
Current address of A. Sárköty:

Mathematical Institute of the Hungarian Academy of Sciences, Budapest, Hungary