ON THE MAXIMUM DENSITY
OF MINIMAL ASYMPTOTIC BASES

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Abstract. A set $A$ of nonnegative integers is an asymptotic basis of order $h$ if every sufficiently large integer is the sum of $h$ elements of $A$. It is proved that if $A$ is an asymptotic basis of order $h$ with lower asymptotic density $d_L(A) > 1/h$, then there is a set $W$ contained in $A$ such that $W$ has positive asymptotic density and $A \setminus W$ is an asymptotic basis of order $h$. This implies that if $A$ is a minimal asymptotic basis of order $h$, then $d_L(A) \leq 1/h$.

Let $A$ be a subset of the nonnegative integers $\mathbb{N}$. Denote by $hA$ the set of all numbers $n$ that can be written in the form $n = a_1 + \cdots + a_h$, where $a_i \in A$ for $i = 1, \ldots, h$ and the $a_i$ are not necessarily distinct. The set $A$ is a basis of order $h$ if $hA = \mathbb{N}$. If $hA$ contains every sufficiently large integer, then $A$ is an asymptotic basis of order $h$. Much of classical additive number theory is the study of special sets $A$ that are bases or asymptotic bases of order $h$ for some $h \geq 2$.

Minimal asymptotic bases form an important extremal class in additive number theory. The asymptotic basis $A$ of order $h$ is minimal if no proper subset of $A$ is an asymptotic basis of order $h$. It follows that if $A$ is minimal, then for every element $a \in A$ there must be infinitely many positive integers $n$, each of whose representations as a sum of $h$ elements of $A$ includes the number $a$ as a summand. Stöhr [7] introduced the concept of minimal asymptotic basis, and Härtter [3] proved that minimal asymptotic bases of order $h$ exist for all $h \geq 2$. Erdös and Nathanson [1] survey recent results on minimal asymptotic bases.

For any set $A$ of integers, the counting function of $A$, denoted $A(x)$, is defined by $A(x) = \text{card}\{a \in A|1 \leq a \leq x\}$. The lower asymptotic density of $A$, denoted $d_L(A)$, is defined by $d_L(A) = \liminf_A(x)/x$. If $\alpha = \lim A(x)/x$ exists, then $\alpha$ is called the asymptotic density of $A$, and denoted $d(A)$.

For every $h \geq 2$, Nathanson [5, 6] has constructed minimal asymptotic bases $A$ of order $h$ that satisfy $A(x) \ll x^{1/h}$. Since $A(x) \gg x^{1/h}$ for every...
asymptotic basis $A$ of order $h$, it follows that these examples are the “thinnest” possible minimal asymptotic bases. Erdös and Nathanson [2] have recently constructed minimal asymptotic bases $A$ of order $h$ with $d(A) = 1/h$. In this paper we prove that these are the “fattest” possible minimal asymptotic bases, in the sense that $d_L(A) \leq 1/h$ for every minimal asymptotic basis $A$ of order $h$. The proof uses a deep result of Kneser on the lower asymptotic density of sums of sets of integers.

Let $A$ and $B$ be sets of nonnegative integers. We shall write $A \sim B$ if $A$ and $B$ coincide for all sufficiently large integers. For $g \geq 1$, let $B^{(g)}$ be the set of all integers $t$ such that $t \equiv b \pmod{g}$ for some $b \in B$. Let $h \geq 2$. Kneser [4] proved that either $d_L(hB) \geq hd_L(B)$ or there exists an integer $g \geq 1$ such that $hB \sim hB^{(g)}$ and $d_L(hB) \geq hd_L(B) - (h - 1)/g$.

**Theorem 1.** Let $h \geq 2$ and let $A$ be an asymptotic basis of order $h$. If $B \subset A$ and $d_L(B) > 1/h$, then there is a finite set $F \subset A \setminus B$ such that $B \cup F$ is an asymptotic basis of order $h$.

**Proof.** Assume that $B$ is not an asymptotic basis of order $h$. Then $hB \not\sim N$. Since $d_L(hB) \leq 1 < hd_L(B)$, Kneser’s theorem implies that there exists an integer $g \geq 1$ such that $hB \sim hB^{(g)}$ and $d_L(hB) > hd_L(B) - (h - 1)/g$.

Let $B = B \cup \{c_1\}$. If $B$ is an asymptotic basis of order $h$, let $F = \{c_1\}$. If $B$ is not an asymptotic basis of order $h$, there exists an integer $g_2$ satisfying (*) and an integer $c_2 \in A$ such that $c_2 \not\equiv b \pmod{g_2}$ for all $b \in B$. Let $B_2 = B \cup \{c_2\}$.

Continuing inductively, we obtain a sequence of integers $g_1, g_2, \ldots$ satisfying (*) and a sequence $c_1, c_2, \ldots$ of elements of $A$ such that $c_i \not\equiv b \pmod{g_i}$ for all $b \in B_{i-1} = B \cup \{c_1, c_2, \ldots, c_{i-1}\}$. Since there are only finitely many distinct congruence classes with respect to moduli bounded above by $(h - 2)/(h\delta)$, the inductive process must terminate in a finite number of steps, and we obtain a finite set $F = \{c_1, c_2, \ldots\} \subset A$ such that $B \cup F$ is an asymptotic basis of order $h$. This completes the proof.

**Theorem 2.** Let $h \geq 2$, and let $A$ be an asymptotic basis of order $h$ with $d_L(A) = 1/h + \delta$, where $\delta > 0$. Let $0 < \tau < \delta$. Then there is a set $W \subset A$ with
asymptotic density $d(W) = \tau$ such that $A \setminus W$ is an asymptotic basis of order $h$.

**Proof.** Let $W'$ be any subset of $A$ with $d(W') = \tau$. Let $B = A \setminus W'$. Then $d_L(B) = 1/h + \delta - \tau > 1/h$. By Theorem 1, there is a finite set $F \subseteq A \setminus B = W'$ such that $B \cup F = A \setminus (W' \setminus F)$ is an asymptotic basis of order $h$. Let $W = W' \setminus F$. Then $d(W) = d(W') = \tau$. This proves the theorem.

**Theorem 3.** Let $A$ be a minimal asymptotic basis of order $h$. Then $d_L(A) \leq 1/h$.

**Proof.** If $d_L(A) > 1/h$, then by Theorem 2 there is a set $W \subseteq A$ with $d(W) > 0$ such that $A \setminus W$ is an asymptotic basis of order $h$, and this contradicts the minimality of $A$.

**References**


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