APPROMATIONS AND FIXED POINTS FOR CONDENSING
NON-SELF-MAPS DEFINED ON A SPHERE

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Abstract. In this paper, we investigate the validity of an interesting theorem of Ky Fan [Theorem 2, Math. Z. 112 (1969), 234-240] defined on a sphere (the boundary of a closed ball) in an infinite-dimensional Banach space. We will prove that it is true for a continuous condensing map with suitable conditions posed. As applications of our theorem, some fixed point theorems of continuous condensing non-self-maps defined on a sphere are derived. Our results generalize some results of R. Nussbaum [10] and P. Massatt [8].

Most fixed point theorems in Banach spaces deal with some classes of maps defined on a compact convex (or star-shaped) subset, a closed bound convex (or star-shaped) subset, or a closed bound subset with nonempty interior. What about the domain of a function which is neither of the above cases? The simplest interesting case would be a sphere (the boundary of a closed ball). It is clear that a continuous self-map defined on a sphere may not have fixed points; for example, a rotation of a sphere (or circle) in a plane. Nussbaum [10] proved that a continuous $k$-set-contractive map from a sphere into a sphere has a fixed point, if the dimension of the Banach space is infinite. Recently, Massatt [8] generalized this result to continuous condensing maps. For definitions of $k$-set-contractive and condensing maps, see for example [9 or 7]. Generalizations of fixed point theory from self-maps to non-self-maps has been a very active topic in nonlinear functional analysis in the past two decades. Does a continuous condensing non-self-map defined on a sphere have a fixed point? Under which conditions?

On the other hand, Fan [3] proved the following interesting theorem:

Let $K$ be a nonempty compact convex subset of a normed linear space $X$. Let $f$ be a continuous map from $K$ into $X$; then there exists a point $u$ in $K$ such that $\|u - f(u)\| = d(f(u), K)$.

The author [5] proved that it is true for a continuous condensing map defined on a closed ball of a Banach space. Is this still true for a continuous condensing map defined on a sphere of an infinite-dimensional Banach space? Under which conditions?

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In this paper, we will prove that the second question is true under appropriate conditions (Theorem 1 below). As applications of our theorem, we derive fixed point theorems for continuous condensing non-self-maps defined on a sphere under suitable conditions, which answer the first question and also generalize Massatt’s result [8].

We also remark that for a continuous condensing (even more generally, 1-set-contractive) map defined on a closed bounded convex subset of a Hilbert space, the above result of Fan is still true; see the author and Yen [5–7].

Throughout this paper, we will denote

\[ S_r = \{ x \in X \mid \|x\| = r \}, \quad B_r = \{ x \in X \mid \|x\| < r \}, \]

\[ I_A(x) = \{ x + c(z - x) \mid \text{for some } z \in A, \text{ some } c > 0 \}, \]

\[ \overline{D} \] the closure of \( D \), where \( X \) is a Banach space, \( r \) a positive number, \( A \) a convex set in \( X \) and \( D \) a set in \( X \). It is clear that \( A \subset I_A(x) \).

Now, we prove our main theorems.

**Theorem 1.** Let \( S_r \) be a sphere with center at the origin and radius \( r \) in an infinite-dimensional Banach space \( X \). Let \( f \) be a continuous condensing map from \( S \) into \( X \). If

\[ \|f(x)\| \geq r \quad \text{for each } x \in S_r, \]

then there exists a point \( u \in S_r \) such that

\[ \|u - f(u)\| = d(f(u), S_r) = d(f(u), \overline{B}_r). \]

**Proof.** Define

\[ R(x) = \begin{cases} x, & \text{if } \|x\| \leq r, \\ rx/\|x\|, & \text{if } \|x\| \geq r. \end{cases} \]

From Nussbaum [9, Corollary 1], \( R \) is a continuous 1-set-contractive map from \( X \) onto \( \overline{B}_r \). Let \( F(x) = R \circ f(x) \), then \( F \) is also a continuous condensing map. Since \( \|f(x)\| \geq r \) for each \( x \in S_r \), we have

\[ \|F(x)\| = \left\| \frac{rf(x)}{\|f(x)\|} \right\| = \frac{r}{\|f(x)\|}, \]

this implies \( F(x) \in S_r \) and \( F: S_r \to S_r \). From Massatt [8], \( F \) has a fixed point in \( S_r \), say \( u \). Therefore

\[ \|u - f(u)\| = \|F(u) - f(u)\| = \|R(f(u)) - f(u)\| \]

\[ = \left\| \frac{rf(u)}{\|f(u)\|} - f(u) \right\| = \|f(u)\| - r. \]

For any \( x \in S_r \) or \( x \in \overline{B}_r \), we have

\[ \|f(u)\| - r \leq \|f(u)\| - \|x\| \leq \|f(u) - x\|. \]

Hence

\[ \|u - f(u)\| = d(f(u), S_r) = d(f(u), \overline{B}_r). \]
Theorem 2. Let $S_r, B_r, f,$ and $X$ be defined as in Theorem 1. Moreover, let $f$ satisfy any one of the following conditions:

(i) For each $x \in S_r$, with $x \neq f(x)$, there exists $y$ in $I_{B_r}(x)$ such that $$||y - f(x)|| < ||x - f(x)||.$$ 

(ii) $f$ is weakly inward (i.e., $f(x) \in I_{B_r}(x)$ for each $x \in S_r$).

(iii) $||f(x) - x||^2 \geq ||f(x)||^2 - r^2$, for each $x \in S_r$.

Then $f$ has a fixed point in $S_r$.

Proof. From Theorem 1, there exists a point $u \in S_r$ such that

\begin{equation}
||u - f(u)|| = d(f(u), S_r) = d(f(u), B_r).
\end{equation}

If $f$ satisfies (i), and $u \neq f(u)$, then there exists $y$ in $I_{B_r}(u)$ such that $||y - f(u)|| < ||u - f(u)||$. Since $y \in I_{B_r}(u)$, there exists $z \in B_r$, $c > 0$ such that $y = u + c(z - u)$. Actually, $c > 1$, otherwise $y \in B_r$ which contradicts (1). Since

\begin{equation}
z = u + \frac{1}{c}(y - u) = \left(1 - \frac{1}{c}\right)u + \frac{1}{c}y = (1 - \beta)u + \beta y,
\end{equation}

where $0 < \beta = 1/c < 1$, we have

\begin{align*}
||z - f(u)|| &\leq (1 - \beta)||u - f(u)|| + \beta||y - f(u)|| \\
&< (1 - \beta)||u - f(u)|| + \beta||u - f(u)|| \\
&= ||u - f(u)||,
\end{align*}

which contradicts (1). Therefore $u = f(u)$. It is clear that if $f$ satisfies (ii), then $f$ satisfies (i). If $f$ satisfies (iii), we will show that $u$ is a fixed point of $f$. From (iii),

\begin{equation}
||f(u) - u||^2 \geq ||f(u)||^2 - r^2.
\end{equation}

Since $||f(u)|| \geq r$, we have $f(u) \neq 0$ and $rf(u)/||f(u)|| \in S_r$. Therefore, from (1),

\begin{equation}
||u - f(u)|| \leq \left\| \frac{rf(u)}{||f(u)||} - f(u) \right\| = ||f(u)|| - r,
\end{equation}

and

\begin{equation}
||u - f(u)||^2 \leq \left(||f(u)|| - r\right)^2.
\end{equation}

Combining (2) and (3), we have $||f(u)|| \leq r$ and hence $||f(u)|| = r$. From (1), $u$ is a fixed point of $f$. \qed

Remark. Condition (i) of Theorem 2 was first stated by Browder [2]. Condition (ii) was first considered by Halpern [4], and (iii) was considered by Altman [1]. To the best of my knowledge, Theorem 2 has not been derived through other approaches. In other words, even Theorem 2 is new to the mathematical literature. We also note that Massatt's result [8] only appeared recently, in 1983. \qed
Corollary (Massatt [8]). Let $X$, $S_r$ be defined as in Theorem 1, and let $f$ be a continuous condensing map from $S_r$ into $S_r$. Then $f$ has a fixed point.

Proof. Since $f(x) \in S_r \subset \overline{B_r} \subset I_{B_r}(x)$, from Theorem 2(ii), $f$ has a fixed point. □

Finally, we give an example of a continuous condensing map $f$, with $\|f(x)\| < r$, such that the conclusion of Theorem 1 is still true, but $f$ has no fixed point in $S_r$.

Example. Let $S_r$ be a sphere in a Banach space $X$ (finite-dimensional or infinite-dimensional). Let $f: S_r \to X$ and $f(x) = x/2$ for each $x \in S_r$. Then $f$ is a contraction and is a continuous condensing map. Clearly $f$ has no fixed point in $S_r$. But for every $u \in S_r$, we have

$$\|u - f(u)\| = d(f(u), S_r).$$

In fact, $\|u - f(u)\| = \|u\|/2 = r/2$ and

$$\|x - f(u)\| = \|x - u/2\| \geq \|x\| - \|u/2\| = |r - r/2| = r/2,$$

for $x \in S_r$. Therefore $\|u - f(u)\| = d(f(u), S_r)$. Certainly, in this example,

$$d(f(u), S_r) \neq d(f(u), \overline{B_r}).$$

References


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