ON A DOMAIN CHARACTERIZATION
OF SCHRODINGER OPERATORS
WITH GRADIENT MAGNETIC VECTOR POTENTIALS
 AND SINGULAR POTENTIALS

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Abstract. Of concern are the minimal and maximal operators on $L^2(\mathbb{R}^n)$ associated with the differential expression

$$\tau_Q = \sum_{j=1}^{n} (i\partial/\partial x_j + q_j(x))^2 + W(x)$$

where $(q_1, \ldots, q_n) = \text{grad } Q$ for some real function $W$ on $\mathbb{R}^n$ and $W$ satisfies $c|x|^{-2} \leq W(x) \leq C|x|^{-2}$. In particular, for $Q = 0$, $\tau_Q$ reduces to the singular Schrödinger operator $-\Delta + W(x)$. Among other results, it is shown that the maximal operator (associated with the $\tau_Q$) is the closure of the minimal operator, and its domain is precisely

$$\text{Dom} \left( \sum_{j=1}^{n} (i\partial/\partial x_j + q_j(x))^2 \right) \cap \text{Dom}(W),$$

provided that $C \geq c > -n(n-4)/4$.

1. Introduction

Consider the formal differential expression

$$\tau_c = H_0 + V = -\Delta + c/|x|^2$$

acting on functions on $\mathbb{R}^n$; here $\Delta$ is the Laplacian and $c$ is a real constant. The minimal and maximal operators associated with $\tau_c$, $H_{cm}$ and $H_{cM}$, are given by $\tau_c$ acting on the domains

$$\mathcal{D}(H_{cm}) = C_0^\infty(\mathbb{R}^n \setminus \{0\}) = \{ u \in C^\infty(\mathbb{R}^n) : u \text{ has compact support in } \mathbb{R}^n \setminus \{0\} \},$$

$$\mathcal{D}(H_{cM}) = \{ u \in L^2(\mathbb{R}^n) : c|x|^{-2}u \in L^1_{\text{loc}}(\mathbb{R}^n), \tau_c u \in L^2(\mathbb{R}^n) \}.$$
These operators, viewed as operators on $L^2(\mathbb{R}^n)$, have many remarkable properties. (Cf. [7]; see also [1, 8].) In particular we have:

(i) $H_{cm}$ is semibounded if and only if $H_{cm} \geq 0$ if and only if $c \geq -[(n-2)/2]^2$.

(ii) $H_{cm}$ is essentially selfadjoint if and only if $c > -n(n-4)/4 = 1 - [(n-2)/2]^2$.

(iii) $H_{cm} = H_{cM}$ if and only if $c \geq -n(n-4)/4$.

The remarkable aspect of (i)-(iii) is that these properties depend on the values of $c$ rather than the form of the potential. This is a highly unusual occurrence in perturbation theory, and it shows that $c/|x|^2$ cannot be thought of as a small perturbation of $-\Delta$ if $c \neq 0$.

In our earlier paper [5] we extended these and other related results by taking advantage of scaling properties. More precisely, let $H_0 = -\Delta$, $V(x) = c/|x|^2$, and $\lambda > 0$. The unitary scaling operator $U(\lambda)$ is defined by

$$(U(\lambda)f)(x) = \lambda^{n/2}f(\lambda x) \quad \text{for } f \in L^2(\mathbb{R}^n), \ x \in \mathbb{R}^n.$$ 

Then

$$U(\lambda)AU(\lambda)^{-1} = \lambda^{-2}A$$

holds for both $A = H_0$ and $A = V$ (i.e., $A$ is multiplication by $V(x)$). Thus $-\Delta$ and $V(x)$ both scale like $\lambda^{-2}$. It turns out that the same is true of $\sum_{j=1}^n (i\partial/\partial x_j + \alpha x_j/|x|^2)^2$, and this fact formed part of the heuristic background for [5]. On the other hand, the vector whose $j$th component is $\alpha x_j |x|^{-2}$ (for $\alpha \in \mathbb{R}$) is the gradient of $\alpha \log |x|$. Operators of the form

$$(1) \quad \sum_{j=1}^n (i\partial/\partial x_j + q_j(x))^2 + c|x|^2,$$

where $(q, \ldots, q_n) = \text{grad } Q$, turn out to have the same properties as $H_{cm}$. Such a $(q, \ldots, q_n)$ will be termed a gradient magnetic vector potential. Properties of the operator (1) will be discussed in the sequel.

2. Background

Clearly

$$\mathcal{D}(H_{cM}) \supset \mathcal{D}(H_0) \cap \mathcal{D}(V).$$

It was recently discovered [2, 6, 12], that the converse containment holds if and only if $c > -n(n-4)/4$. (Cf. also [3, 5, 7, 9, 10].) Thus to (i)-(iii) we can add

(iv) $\mathcal{D}(H_{cM}) = \mathcal{D}(H_0) \cap \mathcal{D}(V) = \{u \in W^{2,2}(\mathbb{R}^n): |x|^{-2}u \in L^2(\mathbb{R})\}$ for

$$c > c_0(n) = \begin{cases} 
\frac{3}{4} & \text{if } n = 1 \text{ or } 3, \\
1 & \text{if } n = 2, \\
0 & \text{if } n \geq 4.
\end{cases}$$
In particular, (iii) shows that (iv) holds only if $H_{cm}$ is essentially selfadjoint, in which case $\overline{H_{cm}} = H_{cM}$ is selfadjoint.

In fact, from [12] we can rewrite the criterion for (iv) in a more general form as follows.

**Proposition 1.** Let $H_{cM}$, $H_0$, and $V$ be as above. Then the following four statements are equivalent.

(a) $\mathcal{D}(H_{cM}) = \mathcal{D}(H_0) \cap \mathcal{D}(V)$.

(b) There exist constants $a \geq 0$, $b \geq 0$ such that

$$\|Vu\| \leq a\|H_{cM}u\| + b\|u\|$$

holds for all $u \in \mathcal{D}(H_{cM})$.

(c) There exists a constant $a \geq 0$ such that

$$\|Vu\| \leq a\|H_{cM}u\|$$

holds for all $u \in \mathcal{D}(H_{cM})$.

(d) $\mathcal{D}(H_{cM}) \subset \mathcal{D}(V)$ and $\|V(H_{cM} + I)^{-1}\| < \infty$.

In [5] we extended this to

$$\tau_{ca} = H_0a + V = \sum_{j=1}^{n} (i\partial/\partial x_j + ax_j/|x|^2)^2 + c/|x|^2$$

(for $\alpha$, $c \in \mathbb{R}$) acting on functions on $\mathbb{R}^n$. As before, let $H_{cam}$ and $H_{caM}$ denote the minimal and maximal operators associated with $\tau_{ca}$ on $L^2(\mathbb{R}^n)$, i.e.,

$$\mathcal{D}(H_{cam}) = C^\infty_0(\mathbb{R}^n \setminus \{0\}),$$

$$\mathcal{D}(H_{caM}) = \{u \in L^2(\mathbb{R}^n) : c|x|^{-2}u \in L^1_{loc}(\mathbb{R}^n)' \cap \{u \in L^2(\mathbb{R}^n) : \tau_{ca}u \in L^2(\mathbb{R}^n)\}\}.$$  

Then Proposition 1 holds with $H_{cM}$ replaced by $H_{caM}$ for all $\alpha \in \mathbb{R}$.

In [5] we also pointed out the obstacle to (iv) holding for all positive $c$ in dimensions 2 and 3. Let $\mathcal{H}_0 = L^2(\mathbb{R}^n)$ and let $\mathcal{H}_1$ be the closure in $\mathcal{H}_0$ of

$$\left\{ f \in L^2(\mathbb{R}^n) \cap C(\mathbb{R}^n) : \int_{|x|=r} f(x) dS_x = 0 \text{ for each } r > 0 \right\},$$

i.e. $\mathcal{H}_1$, consists of the functions in $\mathcal{H}_0$ having spherical means zero. For $l = 0, 1, 2, \ldots$ let

$$M_l = L^2([0, \infty)) \oplus H^l(S^{n-1}),$$

so that the spherical harmonic decomposition of $L^2(\mathbb{R}^n)$ [13, p. 138ff.] becomes

$$\mathcal{H}_0 = L^2([0, \infty)) \otimes L^2(S^{n-1}) = L^2([0, \infty)) \otimes \bigoplus_{l=0}^\infty H^l(S^{n-1}) \equiv \bigoplus_{l=0}^\infty M_l,$$

and we have

$$\mathcal{H}_1 = M_0^\perp = \bigoplus_{l=1}^\infty M_l.$$
It was shown in [5] that a careful examination of the proof in [13] shows that for $c > 0$ and $j = 0, 1$, there is a constant $a(j, n)$ such that

$$\|Vu\| \leq a(j, n)\|H_{cM}u\|$$

holds for all $u \in \mathcal{D}(H_{cM}) \cap \mathcal{H}_j$ and all $c > c_j(n)$, where $c_0(n)$, is as before (in (iv)) and

$$c_1(n) = \begin{cases} \frac{3}{4} & \text{if } n = 1, \\ 0 & \text{if } n \geq 4. \end{cases}$$

Thus the obstruction to (iv) holding in dimension 2 and 3 is the subspace $M_0$ of radial functions.

3. Gradient magnetic vector potentials

One of our goals here is to generalize the above results to the minimal and maximal operators, $H_{Qm}$ and $H_{QM}$, associated with the differential expression

$$n \sum_{j=1}^{n} (i\partial/\partial x_j + q_j(x))^2 + W(x)$$

where $(q_1, \ldots, q_n) = \text{grad } Q$ for some real function $Q$ in $W^{1,1}_{\text{loc}}(\mathbb{R}^n)$ and where the real function $W \in L^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ satisfies $W(x) \geq c/|x|^2$ for all $x$ and some $c > -n(n - 4)/4$. The connection between $\tau_Q$ and the Schrödinger operator $-\Delta + W(x)$ is made clear by the following result.

**Proposition 2.** Let $A_j = i\partial/\partial x_j + q_j(x)$ for $j = 1, \ldots, n$ where $(q_1, \ldots, q_n) = \text{grad } Q$ for some real function $Q$ in $W^{1,1}_{\text{loc}}(\mathbb{R}^n)$. Let $U$ be the unitary operator of multiplication by $e^{iQ(x)}$. Then for $j \in \{1, \ldots, n\}$, $A_j = UB_jU^{-1}$ where $B_j = i\partial/\partial x_j$ acts on $\mathcal{D}(B_j) = \{u \in L^2(\mathbb{R}^n) : \text{the distributional derivative } \partial u/\partial x_j \text{ is in } L^2(\mathbb{R}^n)\}$.

**Proof.** A straightforward computation gives

$$UB_jU^{-1}u = (i\partial/\partial x_j + q_j(x))u$$

for any $u \in \mathcal{D}(A_j) = e^{iQ} \mathcal{D}(B_j)$. \ \Box

The point of the above computation is that the unitary operator $U$ is independent of $j$.

Note that for $Q \equiv 0$, the minimal operator $H_{om}$ is known to be essentially selfadjoint [11]. Also, $H_{om} = H_{QM}$. Combining these observations with the facts that $H(Q) = \sum_{j=1}^{n} A_j^2$ and $UWU^{-1} = W$ leads to the following result.

**Corollary 3.** Let $Q$ be as in Proposition 2. Then $H_{Qm}$ is essentially selfadjoint, $H_{Qm} = H_{QM}$, and $\mathcal{D}(H_{Qm}) = e^{iQ} \mathcal{D}(H_{om})$.

H. Kalf [6] conjectured that for suitable values of $c$, $\mathcal{D}(H_{om}) = \mathcal{D}(H_0) \cap \mathcal{D}(W)$. We shall establish a special case of this conjecture. Namely, we shall...
verify it for (measurable) potentials \( W \) satisfying
\[
(3) \quad c_1/|x|^2 \leq W(x) \leq c_2/|x|^2 + c_3
\]
for any constants \( c_i \) satisfying
\[
(4) \quad -n(n-4)/4 < c_1 \leq c_2, \quad c_3 \geq 0.
\]

**Theorem 4.** Let (3) and (4) hold. Let \( Q \) be as in Proposition 2. Then there exists a constant \( a \), depending only on \( c_1 \), \( c_2 \) and \( c_3 \), such that
\[
(5) \quad \|Wu\| \leq a\|H_{QM}u\| + a\|u\|
\]
holds for all \( u \in \mathcal{D}(H_{QM}) \). Thus
\[
\mathcal{D}(H_{QM}) = \mathcal{D}(H(Q)) \cap \mathcal{D}(W)
\]
and \( H_{QM} \) is selfadjoint.

**Proof.** It follows from the previous discussion that (5) holds if and only if
\[
(6) \quad \|Wu\| \leq a\|H_{oM}u\| + b\|u\|
\]
holds for all \( u \in \mathbb{D}(H_{oM}) \). Clearly (6) is equivalent to
\[
\|W(H_{oM} + I)^{-1}\| = \|W(H_0 + W + 1)^{-1}\| < \infty.
\]
Let \( V(x) = c_1/|x|^2 \). Then by Proposition 1 and (2) we have
\[
\|V(H_{c1M} + I)^{-1}\| = \|V(H_0 + V + 1)^{-1}\| < \infty
\]
if and only if \( c_1 > -n(n-4)/4 \).

Note that \( c_1 \) can be negative only in dimension five or more. Define cutoff functions
\[
U_\nu(x) = U(x) \chi_{|x| > \nu} \chi_{|x| < \nu},
\]
according as \( |x| \geq 1/\nu \) or \( |x| < 1/\nu \), for \( U = V, W \). Then \( V_\nu \) and \( W_\nu \) are in \( L^\infty(\mathbb{R}^n) \), and \( V_\nu \leq W_\nu \) holds on \( \mathbb{R}^n \). The Trotter product formula (cf. e.g. [4, 8]) implies that
\[
\exp\{-t(H_0 + W_n)} \phi \leq \exp\{-t(H_0 + V_n)} \phi
\]
for all \( 0 \leq \phi \in L^2(\mathbb{R}^n) \). Integrating this inequality gives the resolvent inequality
\[
(H_0 + W_n + 1)^{-1} \phi \leq (H_0 + V_n + 1)^{-1} \phi
\]
for all \( 0 \leq \phi \in L^2(\mathbb{R}^n) \). Letting \( n \to \infty \) gives
\[
(7) \quad (H_0 + W + 1)^{-1} \phi \leq (H_0 + V + 1)^{-1} \phi
\]
for all such \( \phi \).

From (3) we deduce, for all \( 0 \leq \phi \in L^2(\mathbb{R}^n) \),
\[
|W|(H_0 + W + 1)^{-1} \phi \leq \max(|c_1|, |c_2|)|x|^{-2}(H_0 + W + 1)^{-1} \phi
+ c_3(H_0 + W + 1)^{-1} \phi
\leq \max(|c_1|, |c_2|)|x|^{-2}(H_0 + V + 1)^{-1} \phi
+ c_3(H_0 + V + 1)^{-1} \phi
\]
by (7).
Next recall that for a bounded, positivity preserving operator $A$ we have $|A\psi| \leq A|\psi|$ a.e. for each $\psi \in L^2(\mathbb{R}^n)$. Thus we deduce
\[
\|W(H_0 + W + 1)^{-1}\| \leq \max(|c_1|, |c_2|) \|x\|^{-2}(H_0 + V + 1)^{-1}\|
\]
\[+ c_3\| (H_0 + V + 1)^{-1}\| < \infty.
\]
This completes the proof. □

We remark that when $c_1 > 0$, the approximation argument (involving $V_1, W_1$) becomes unnecessary and the above proof simplifies.

The purpose of $c_3$ was to write $W$ as $W_1 + W_2$ where
\[c_1/|x|^2 \leq W_1(x) \leq c_2/|x|^2, \quad W_2 \in L^\infty(\mathbb{R}^n).
\]
The bounded potential $W_2$ is a small perturbation of $H_0 + W_1$ from the viewpoint of selfadjointness. Various unbounded potentials could take its place.

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REFERENCES
