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POWERS' BINARY SHIFTS ON THE HYPERFINITE FACTOR OF TYPE II_1

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ABSTRACT. A unit preserving *-endomorphism σ on the hyperfinite II_1 factor R is called a shift if $\bigcap_{n=0}^{\infty} \sigma^n(R) = \{\lambda 1; \lambda \in \mathbb{C}\}$. A shift σ is called Powers' binary shift if there is a self-adjoint unitary u such that $R = \{\sigma^n(u); n \in \mathbb{N} \cup \{0\}\}''$ and $\sigma^k(u)u = \pm u\sigma^k(u)$ for $k \in \mathbb{N} \cup \{0\}$. Let $q(\sigma)$ be the number $\min\{k \in \mathbb{N}; \sigma^k(R)' \cap R \neq \mathbb{C}1\}$. It is shown that the number $q(\sigma)$ is not the complete outer conjugacy invariant for Powers' binary shifts.

1. INTRODUCTION

Powers [5] called an identity preserving *-endomorphism σ of the hyperfinite II_1-factor R such that $\bigcap_{k=0}^{\infty} \sigma^k(R) = \mathbb{C}1$ a shift and defined the index of σ by Jones index $[R:\sigma(R)]$ [4]. A shift σ is called a Powers' binary shift if there is a selfadjoint unitary u_0 in R such that R is generated by $\{\sigma^n(u_0); n = 0, 1, 2, \ldots\}$ and $\sigma^n(u_0)$ and $\sigma^m(u_0)$ pairwise commute or anticommute. Here $[R:\sigma(R)] = 2$. The unitary u_0 is called a σ-generator of R [5]. In the following we put $u_n = \sigma^n(u_0)$. Shifts α and β are called conjugate if there exists an automorphism θ on R such that $\beta = \theta\alpha\theta^{-1}$. And α and β are called outer conjugate if there exist a unitary $u \in R$ and an automorphism θ on R such that $\beta = \theta\alpha\theta^{-1} \circ Adu$ [5]. Powers [5] classified binary shifts completely up to conjugacy. Subsequently Price [6] characterized Powers' binary shifts and surprisingly found a nonbinary shift on R of index two. Powers [5] also considered the outer conjugacy invariant $q(\sigma) = \min\{k \in \mathbb{N}; \sigma^k(R)' \cap R \neq \mathbb{C}1\}$ for binary shifts σ. Then Powers [5] raised the problem of whether the numbers $q(\sigma)$ are the complete outer conjugacy invariant or not for binary shifts σ. In this paper we shall give a negative answer of this Powers' problem. In order to do this, we shall use the relative commutant algebras $\{\sigma^k(R)' \cap R; k = 0, 1, 2, \ldots\}$.

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Obviously the set of the (isomorphism classes of) relative commutant algebras $\{\sigma^k(R)' \cap R; k = 0, 1, 2, \dots\}$ is an outer conjugacy invariant for binary shifts σ.

2. Powers' problem

Powers' binary shifts are realized as follows [3]. Let $G = \coprod_{i=0}^{\infty} G_i$ be the restricted direct product of $G_i \cong \mathbb{Z}_2 = \{0, 1\}$. A function $a: \mathbb{Z} \to \{0, 1\}$ is called a *signature sequence* if $a(0) = 0$ and $a(n) = a(-n)$ for any integer $n \in \mathbb{Z}$ (cf. [5, 6, 7]). Define the canonical shift σ on the group G as follows: For $x = (x(i)) \in G$, $(\sigma(x))(i) = x(i-1)$ for $i \geq 1$ and $(\sigma(x))(0) = 0$. Let us define a multiplier $m_a \in Z^2(G, T)$ by

$$m_a(x, y) = (-1)^{\sum_{i>j} a(i-j)x(i)y(j)} \quad \text{for } x = (x(i)), \quad y = (y(j)) \in G.$$

We shall define a unitary operator $\lambda_{m_a}(x)$ on $l^2(G)$ by

$$(\lambda_{m_a}(x)\xi)(y) = m_a(x, x^{-1}y)\xi(x^{-1}y) \quad \text{for } x, y \in G \quad \text{and } \xi \in l^2(G).$$

Let $R_{m_a}(G)$ be the von Neumann algebra generated by $\{\lambda_{m_a}(x); x \in G\}$. A signature sequence a is periodic if there exists an integer k such that $a(k+n) = a(n)$ for $n \in \mathbb{Z}$. Price [4, 5] showed that a is aperiodic (i.e., not periodic) if and only if $R_{m_a}(G)$ is a factor (cf. also [1, 3]). In the following we shall always assume that the signature sequence a is aperiodic and identify the sequence $(a(i); i \in \mathbb{N} \cup \{0\})$ with $(a(i); i \in \mathbb{Z})$.

Since $m_a(\sigma(x), \sigma(y)) = m_a(x, y)$, σ induces a shift σ on $R_{m_a}(G)$ such that $\sigma(\lambda_{m_a}(x)) = \lambda_{m_a}(\sigma(x))$ for $x \in G$, where we use the same notation σ. Put $e_0 = (1, 0, 0, 0, \dots) \in G$ and $e_n = \sigma^n(e_0) \in G$. Similarly put $u_0 = \lambda_{m_a}(e_0)$ and $u_n = \sigma^n(u_0)$. Then $u_n u_m = (-1)^{a(n-m)} u_m u_n$ and the hyperfinite factor of type II_1 $R = R_{m_a}(G)$ is generated by $\{u_n; n = 0, 1, 2, \dots\}$. Thus the shift $\sigma = \sigma_a$ on $R_{m_a}(G)$ is a Powers' binary shift with a signature sequence a. In the following we shall realize the relative commutant algebras $C_k(\sigma) = \sigma^k(R)' \cap R$ concretely.

Theorem 1. *Let a be an aperiodic signature sequence. Suppose that the set $\{i \in \mathbb{N}; a(i) \neq 0\}$ is finite. Put $d = \max\{i \in \mathbb{N}; a(i) \neq 0\}$. Let σ be the Powers' binary shift with a signature sequence a. Let $u_0 = \lambda_{m_a}(e_0)$ be the σ-generator. Put $u_n = \sigma^n(u)$. Then $C_k(\sigma) = \sigma^k(R)' \cap R = \mathbb{C}1$ if $0 \leq k \leq d$ and $C_k(\sigma) = \{u_i; 0 \leq i \leq k-d-1\}''$ if $d+1 \leq k$.*

Proof. It is clear that we have the inclusion $C_k(\sigma) \supset \mathbb{C}$ if $0 \leq k \leq d$ and $C_k(\sigma) \supset \{u_i; 0 \leq i \leq k-d-1\}''$ if $d+1 \leq k$. We shall show the reverse inclusion. In the following we denote $\lambda = \lambda_{m_a}$. Let $x = \sum_g x_g \lambda_g \in R_{m_a}(G)$. If x is in $C_k(\sigma)$, then

$$\left(\sum_g x_g \lambda_g\right) \lambda_{e_n} = \lambda_{e_n} \left(\sum_g x_g \lambda_g\right) \qquad \text{for } n \geq k.$$

Hence $\sum_g x_g m_a(g, e_n)\lambda_{g+e_n} = \sum_g x_g m_a(e_n, g)\lambda_{e_n+g}$. Thus $x_g(m_a(g, e_n) - m_a(e_n, g)) = 0$ for $n \geq k$. We may suppose that $x_g \neq 0$. Then $m_a(g, e_n) = m_a(e_n, g)$ for $n \geq k$. It is enough to show that $g = 0$ if $0 \leq k \leq d$ and $g(s) = 0$ for $s \geq k - d$ if $d + 1 \leq k$. Since

$$m_a(g, e_n) = (-1)^{\sum_{i>j} a(i-j)g(i)e_n(j)} = (-1)^{\sum_{i>n} a(i-n)g(i)}$$

and

$$m_a(e_n, g) = (-1)^{\sum_{i>j} a(i-j)e_n(i)g(j)} = (-1)^{\sum_{n>j} a(n-j)g(j)},$$

we have that $\sum_{i>n} a(i-n)g(i) = \sum_{n>j} a(n-j)g(j)$. By changing variables from i, j to p, we have that

$$(\#) \qquad \sum_{p=1}^{d} g(p+n)a(p) = \sum_{p=1}^{\min(n,d)} a(p)g(n-p) \quad \text{for } n \geq k.$$

Firstly consider the case that $0 \leq k \leq d$. We shall show that $g = 0$. Suppose that $g \neq 0$. Put $m = \max\{i \in \mathbb{N} \cup \{0\};\ g(i) \neq 0\}$ and $n = m + d$. Then we have $n \geq k$. Therefore we can apply (#) in this case, so that we have $\sum_{p=1}^{d} g(p+m+d)a(p) = \sum_{p=1}^{d} g(m+d-p)a(p)$.

Hence $0 = g(m)a(d)$. Since $a(d) = 1$, we have that $g(m) = 0$. This is a contradiction. Thus we have $g = 0$. Next consider the case that $d + 1 \leq k$. Assume that $g(s) = 1$ for some $s \geq k - d$. Then we shall show the contradiction. Put $m = \max\{i \in \mathbb{N} \cup \{0\}; g(i) \neq 0\}$. By the assumption we have that $m \geq k - d$. Put $n = m + d$. Then we have that $n \geq k$. Since we can apply (#), we have that $\sum_{p=1}^{d} g(p+m+d)a(p) = \sum_{p=1}^{d} g(m+d-p)a(p)$. Therefore $0 = g(m)a(d)$. Since $a(d) = 1$, $g(m) = 0$. This is a contradiction. Thus $g(s) = 0$ for $k - d \leq s$. □

Remark 2. In [1], Bures and Yin coinsidered independently the relative commutant algebras for group shifts abstractly and they proved the following:

Let G be a discrete abelian group and m a multiplier of G. Let $R_m(G)$ be the von Neumann algebra as well as the above case $m = m_a$. If H is a subgroup of G, then $R_m(H)' \cap R_m(G) = R_m(D_H)$, where D_H is the subgroup $\{g \in G; m(g, h) = m(h, g)$ for any $h \in H\}$ of G.

Powers [5] defined the following outer conjugacy invariant $q(\sigma)$ for shifts σ: Put $q(\sigma) = \min\{k \in \mathbb{N}; \sigma^k(R)' \cap R \neq \mathbb{C}1\}$.

Remark 3. Take the signature sequence a such that the set $\{i \in \mathbb{N}; a(i) \neq 0\}$ is finite. Let degree a be the number $\max\{i \in \mathbb{N}; a(i) \neq 0\}$. Then Theorem 1 says that $q(\sigma) = (\text{degree}\, a) + 1$.

In [5], Powers raised the following problem (cf. also [7]).

Powers' problem. If α and β are binary shifts and $q(\alpha) = q(\beta)$ then are α and β outer conjugate?

We give a negative answer to the above problem.

Corollary 4. *There exist binary shifts α and β such that $q(\alpha) = q(\beta)$ but α and β are not outer conjugate.*

Proof. Let a and b be signature sequences such that $a(2) = a(3) = 1$ and $a(i) = 0$ $(i \neq 2, 3)$, $b(1) = b(3) = 1$ and $b(j) = 0$ $(j \neq 1, 3)$. Then $C_k(\sigma_a) \cong \mathbb{C}$ for $0 \leq k \leq 3$, $C_4(\sigma_a) = \{u_0\}'' \cong \mathbb{C}^2$ and $C_5(\sigma_a) = \{u_0, u_1\}'' \cong \mathbb{C}^4$. On the other hand $C_k(\sigma_b) \cong \mathbb{C}$ for $0 \leq k \leq 3$ and $C_4(\sigma_b) = \{u_0\}'' \cong \mathbb{C}^2$ but $C_5(\sigma_b) = \{u_0, u_1\}'' \cong M_2$. Thus $q(\sigma_a) = q(\sigma_b) = 4$ but σ_a and σ_b are not outer conjugate. □

Remark 5. Let a be a signature sequence such that the set $\{i \in \mathbb{N}; a(i) \neq 0\}$ is finite. Let order a be the number $\min\{n \in \mathbb{N}; a(n) \neq 0\}$. Then degree a and order a are outer conjugacy invariant for Powers' binary shifts σ_a with degree $a < +\infty$. In fact $q(\sigma_a) = (\text{degree}\, a) + 1$ and $(\text{degree}\, a) + (\text{order}\, a) + 1 = \min\{k \in \mathbb{N}; \sigma^k(R)' \cap R$ is not abelian$\}$. But orders and degrees are not complete outer conjugacy invariant. This is shown by the following example.

Example 6. Let a and b be signature sequences such that $a(1) = a(3) = 1$ and $a(i) = 0$ $(i \neq 1, 3)$, $b(1) = b(2) = b(3) = 1$ and $b(j) = 0$ $(j \neq 1, 2, 3)$. Then obviously degree a = degree b and order a = order b. On the other hand, by Theorem 1, we have that $C_7(\sigma_a) \cong M_2 \otimes \mathbb{C}^4$ and $C_7(\sigma_b) \cong M_4$. Thus $C_7(\sigma_a)$ is not isomorphic to $C_7(\sigma_b)$. Hence σ_a and σ_b are not outer conjugate.

Remark 7. In [2], M. Choda also uses the numbers $\min\{k \in \mathbb{N}; \sigma^k(R)' \cap R \neq \mathbb{C}1\}$ and $\min\{k \in \mathbb{N}; \sigma^k(R)' \cap R$ is not abelian$\}$ for projection shifts to show that there are at least a countable infinity of outer conjugacy classes among the projection shifts of R with the index $\lambda \in \{4\cos^2(\pi/n); n = 3, 4, \ldots\} \cup [4, \infty)$.

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