POWERS' BINARY SHIFTS
ON THE HYPERFINITE FACTOR OF TYPE $\text{II}_1$

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Abstract. A unit preserving $\ast$-endomorphism $\sigma$ on the hyperfinite $\text{II}_1$ factor $R$ is called a shift if $\bigcap_{n=0}^{\infty} \sigma^n(R) = \{ \lambda 1; \lambda \in \mathbb{C} \}$. A shift $\sigma$ is called Powers’ binary shift if there is a self-adjoint unitary $u$ such that $R = \{ \sigma^n(u); n \in \mathbb{N} \cup \{0\} \}$ and $\sigma^k(u^*u) = \pm u \sigma^k(u)$ for $k \in \mathbb{N} \cup \{0\}$. Let $q(\sigma)$ be the number $\min \{ k \in \mathbb{N}; \sigma^k(R)' \cap R \neq \mathbb{C}1 \}$. It is shown that the number $q(\sigma)$ is not the complete outer conjugacy invariant for Powers’ binary shifts.

1. INTRODUCTION

Powers [5] called an identity preserving $\ast$-endomorphism $\sigma$ of the hyperfinite $\text{II}_1$-factor $R$ such that $\bigcap_{k=0}^{\infty} \sigma^k(R) = \mathbb{C}1$ a shift and defined the index of $\sigma$ by Jones index $[R: \sigma(R)]$ [4]. A shift $\sigma$ is called a Powers’ binary shift if there is a self-adjoint unitary $u_0$ in $R$ such that $R$ is generated by $\{ \sigma^n(u_0); n = 0, 1, 2, \ldots \}$ and $\sigma^n(u_0)$ and $\sigma^m(u_0)$ pairwise commute or anticommute. Here $[R: \sigma(R)] = 2$. The unitary $u_0$ is called a $\sigma$-generator of $R$ [5]. In the following we put $u_n = \sigma^n(u_0)$. Shifts $\alpha$ and $\beta$ are called conjugate if there exists an automorphism $\theta$ on $R$ such that $\beta = \theta \alpha \theta^{-1}$. And $\alpha$ and $\beta$ are called outer conjugate if there exist a unitary $u \in R$ and an automorphism $\theta$ on $R$ such that $\beta = \theta \alpha \theta^{-1} \circ A \text{Ad} u$ [5]. Powers [5] classified binary shifts completely up to conjugacy. Subsequently Price [6] characterized Powers’ binary shifts and surprisingly found a nonbinary shift on $R$ of index two. Powers [5] also considered the outer conjugacy invariant $q(\sigma) = \min \{ k \in \mathbb{N}; \sigma^k(R)' \cap R \neq \mathbb{C}1 \}$ for binary shifts $\sigma$. Then Powers [5] raised the problem of whether the numbers $q(\sigma)$ are the complete outer conjugacy invariant or not for binary shifts $\sigma$. In this paper we shall give a negative answer of this Powers’ problem. In order to do this, we shall use the relative commutant algebras $\{ \sigma^k(R)' \cap R; k = 0, 1, 2, \ldots \}$. 

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Obviously the set of the (isomorphism classes of) relative commutant algebras \( \{ \sigma^k(R)' \cap R ; k = 0, 1, 2, \ldots \} \) is an outer conjugacy invariant for binary shifts \( \sigma \).

### 2. Powers’ problem

Powers’ binary shifts are realized as follows [3]. Let \( G = \prod_{i=0}^{\infty} G_i \) be the restricted direct product of \( G_i \cong \mathbb{Z}_2 = \{0, 1\} \). A function \( a: \mathbb{Z} \rightarrow \{0, 1\} \) is called a signature sequence if \( a(0) = 0 \) and \( a(n) = a(-n) \) for any integer \( n \in \mathbb{Z} \) (cf. [5, 6, 7]). Define the canonical shift \( \sigma \) on the group \( G \) as follows: For \( x = (x(i)) \in G \), \( (\sigma(x))(i) = x(i-1) \) for \( i \geq 1 \) and \( (\sigma(x))(0) = 0 \). Let us define a multiplier \( m_a \in \mathbb{Z}^2(G, T) \) by

\[
M_a(x, y) = (-1)^{\sum_{n \neq 0} a(n)x(n)y(n)} \quad \text{for } x = (x(i)) \quad \text{and} \quad y = (y(j)) \in G.
\]

We shall define a unitary operator \( \lambda_m(x) \) on \( l^2(G) \) by

\[
(\lambda_m(x)\xi)(y) = M_a(x, x^{-1}y)\xi(x^{-1}y) \quad \text{for } x, y \in G \quad \text{and} \quad \xi \in l^2(G).
\]

Let \( R_m(G) \) be the von Neumann algebra generated by \( \{ \lambda_m(x) ; x \in G \} \). A signature sequence \( a \) is periodic if there exists an integer \( k \) such that \( a(k+n) = a(n) \) for \( n \in \mathbb{Z} \). Price [4, 5] showed that \( a \) is aperiodic (i.e., not periodic) if and only if \( R_m(G) \) is a factor (cf. also [1, 3]). In the following we shall always assume that the signature sequence \( a \) is aperiodic and identify the sequence \( (a(i) ; i \in \mathbb{N}) \) with \( (a(i) ; i \in \mathbb{Z}) \).

Since \( M_a(\sigma(x), \sigma(y)) = M_a(x, y) \), \( \sigma \) induces a shift \( \sigma \) on \( R_m(G) \) such that \( \sigma(\lambda_m(x)) = \lambda_m(\sigma(x)) \) for \( x \in G \), where we use the same notation \( \sigma \).

Put \( e_0 = (1, 0, 0, 0, \ldots) \in G \) and \( e_n = \sigma^n(e_0) \in G \). Similarly put \( u_0 = \lambda_m(e_0) \) and \( u_n = \sigma^n(u_0) \). Then \( u_nu_m = (-1)^{a(n-m)}u_mu_n \) and the hyperfinite factor of type II_1 \( R = R_m(G) \) is generated by \( \{ u_n ; n = 0, 1, 2, \ldots \} \). Thus the shift \( \sigma = \sigma_a \) on \( R_m(G) \) is a Powers’ binary shift with a signature sequence \( a \). In the following we shall realize the relative commutant algebras \( C_k(\sigma) = \sigma^k(R)' \cap R \) concretely.

**Theorem 1.** Let \( a \) be an aperiodic signature sequence. Suppose that the set \( \{ i \in \mathbb{N} ; a(i) \neq 0 \} \) is finite. Put \( d = \max\{ i \in \mathbb{N} ; a(i) \neq 0 \} \). Let \( \sigma \) be the Powers’ binary shift with a signature sequence \( a \). Let \( u_0 = \lambda_m(e_0) \) be the \( \sigma \)-generator. Put \( u_n = \sigma^n(u) \). Then \( C_k(\sigma) = \sigma^k(R)' \cap R = C1 \) if \( 0 \leq k \leq d \) and \( C_k(\sigma) = \{ u_i ; 0 \leq i \leq k \} \) if \( d + 1 \leq k \).

**Proof.** It is clear that we have the inclusion \( C_k(\sigma) \supset C \) if \( 0 \leq k \leq d \) and \( C_k(\sigma) \supset \{ u_i ; 0 \leq i \leq k \} \) if \( d + 1 \leq k \). We shall show the reverse inclusion. In the following we denote \( \lambda = \lambda_m \). Let \( x = \sum_g x_g \lambda_g \in R_m(G) \). If \( x \) is in \( C_k(\sigma) \), then

\[
\left( \sum_g x_g \lambda_g \right)_e \lambda_e = \lambda_e \left( \sum_g x_g \lambda_g \right)
\]

for \( n \geq k \).
Hence $\sum_g x_g m_a(g, e_n) \lambda_{g + e_n} = \sum_g x_g m_a(e_n, g) \lambda_{e_n + g}$. Thus $x_g (m_a(g, e_n) - m_a(e_n, g)) = 0$ for $n \geq k$. We may suppose that $x_g \neq 0$. Then $m_a(g, e_n) = m_a(e_n, g)$ for $n \geq k$. It is enough to show that $g = 0$ if $0 \leq k \leq d$ and $g(s) = 0$ for $s \geq k - d$ if $d + 1 \leq k$. Since

$$m_a(g, e_n) = (-1)^\sum_{i>j} a(i-j) g(i) e_n(j) = (-1)^\sum_{i>n} a(i-n) g(i)$$

and

$$m_a(e_n, g) = (-1)^\sum_{i>j} a(i-j) e_n(i) g(j) = (-1)^\sum_{n>j} a(n-j) g(j),$$

we have that $\sum_{i>n} a(i-n) g(i) = \sum_{n>j} a(n-j) g(j)$. By changing variables from $i, j$ to $p$, we have that

$$\sum_{p=1}^{d} g(p + n) a(p) = \sum_{p=1}^{\min(n,d)} a(p) g(n - p) \quad \text{for } n \geq k.$$

Firstly consider the case that $0 < k < d$. We shall show that $g = 0$. Suppose that $g \neq 0$. Put $m = \max\{i \in \mathbb{N} \cup \{0\}; \ g(i) \neq 0\}$ and $n = m + d$. Then we have $n \geq k$. Therefore we can apply (#) in this case, so that we have

$$\sum_{p=1}^{d} g(p + m + d) a(p) = \sum_{p=1}^{d} g(m + d - p) a(p).$$

Hence $0 = g(m) a(d)$. Since $a(d) = 1$, we have that $g(m) = 0$. This is a contradiction. Thus we have $g = 0$. Next consider the case that $d + 1 \leq k$. Assume that $g(s) = 1$ for some $s \geq k - d$. Then we shall show the contradiction. Put $m = \max\{i \in \mathbb{N} \cup \{0\}; \ g(i) \neq 0\}$. By the assumption we have that $m \geq k - d$. Put $n = m + d$. Then we have that $n \geq k$. Since we can apply (#), we have that $\sum_{p=1}^{d} g(p + m + d) a(p) = \sum_{p=1}^{d} g(m + d - p) a(p)$. Therefore $0 = g(m) a(d)$. Since $a(d) = 1$, $g(m) = 0$. This is a contradiction. Thus $g(s) = 0$ for $k - d \leq s$.

Remark 2. In [1], Bures and Yin considered independently the relative commutant algebras for group shifts abstractly and they proved the following:

Let $G$ be a discrete abelian group and $m$ a multiplier of $G$. Let $R_m(G)$ be the von Neumann algebra as well as the above case $m = m_a$. If $H$ is a subgroup of $G$, then $R_m(H)^' \cap R_m(G) = R_m(D_H)$, where $D_H$ is the subgroup \{ $g \in G; m(g, h) = m(h, g)$ for any $h \in H$\} of $G$.

Powers [5] defined the following outer conjugacy invariant $q(\sigma)$ for shifts $\sigma$:

Put $q(\sigma) = \min\{k \in \mathbb{N}; \sigma^k(R)^' \cap R \neq C1\}.$

Remark 3. Take the signature sequence $a$ such that the set \{ $i \in \mathbb{N}; a(i) \neq 0$\} is finite. Let degree $a$ be the number $\max\{i \in \mathbb{N}; a(i) \neq 0\}$. Then Theorem 1 says that $q(\sigma) = (\text{degree } a) + 1$.

In [5], Powers raised the following problem (cf. also [7]).

Powers' problem. If $\alpha$ and $\beta$ are binary shifts and $q(\alpha) = q(\beta)$ then are $\alpha$ and $\beta$ outer conjugate?

We give a negative answer to the above problem.
Corollary 4. There exist binary shifts $\alpha$ and $\beta$ such that $q(\alpha) = q(\beta)$ but $\alpha$ and $\beta$ are not outer conjugate.

Proof. Let $a$ and $b$ be signature sequences such that $a(2) = a(3) = 1$ and $a(i) = 0$ ($i \neq 2, 3$), $b(1) = b(3) = 1$ and $b(j) = 0$ ($j \neq 1, 3$). Then $C_k(\sigma_a) \cong \mathbb{C}$ for $0 \leq k \leq 3$, $C_4(\sigma_a) = \{u_0\}'' \cong \mathbb{C}^2$ and $C_5(\sigma_a) = \{u_0, u_1\}'' \cong \mathbb{C}^2$. On the other hand $C_k(\sigma_b) \cong \mathbb{C}$ for $0 \leq k \leq 3$ and $C_4(\sigma_b) = \{u_0\}'' \cong \mathbb{C}^2$ but $C_5(\sigma_b) = \{u_0, u_1\}'' \cong M_2$. Thus $q(\sigma_a) = q(\sigma_b) = 4$ but $\sigma_a$ and $\sigma_b$ are not outer conjugate. \(\Box\)

Remark 5. Let $a$ be a signature sequence such that the set $\{i \in \mathbb{N}; a(i) \neq 0\}$ is finite. Let order $a$ be the number $\min\{n \in \mathbb{N}; a(n) \neq 0\}$. Then degree $a$ and order $a$ are outer conjugacy invariant for Powers’ binary shifts $\sigma_a$ with degree $a < +\infty$. In fact $q(\sigma_a) = (\text{degree} a) + 1$ and $q(\sigma_a) = (\text{order} a) + 1 = \min\{k \in \mathbb{N}; \sigma^k(R) \cap R \text{ is not abelian}\}$. But orders and degrees are not complete outer conjugacy invariant. This is shown by the following example.

Example 6. Let $a$ and $b$ be signature sequences such that $a(1) = a(3) = 1$ and $a(i) = 0$ ($i \neq 1, 3$), $b(1) = b(2) = b(3) = 1$ and $b(j) = 0$ ($j \neq 1, 2, 3$). Then obviously degree $a = \text{degree} b$ and order $a = \text{order} b$. On the other hand, by Theorem 1, we have that $C_7(\sigma_a) \cong M_2 \otimes \mathbb{C}^4$ and $C_7(\sigma_b) \cong M_4$. Thus $C_7(\sigma_a)$ is not isomorphic to $C_7(\sigma_b)$. Hence $\sigma_a$ and $\sigma_b$ are not outer conjugate.

Remark 7. In [2], M. Choda also uses the numbers $\min\{k \in \mathbb{N}; \sigma^k(R) \cap R \neq \mathbb{C} \}$ and $\min\{k \in \mathbb{N}; \sigma^k(R) \cap R \text{ is not abelian}\}$ for projection shifts to show that there are at least a countable infinity of outer conjugacy classes among the projection shifts of $R$ with the index $\lambda \in \{4 \cos^2(\pi/n); n = 3, 4, \ldots\} \cup \{4, \infty\}$.

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REFERENCES


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