THE NUMBER OF INDECOMPOSABLE SEQUENCES
OVER AN ARTIN ALGEBRA OF FINITE TYPE

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Abstract. Let \( \Lambda \) be an artin algebra of finite representation type. For a finitely generated \( \Lambda \)-module \( C \), there are only finitely many f.g. modules \( A \) such that
\[
0 \to A \to B \to C \to 0
\]
is indecomposable as a short exact sequence.

Let \( \Lambda \) be an artin algebra of finite representation type and \( \text{mod} \Lambda \) the category of finitely generated (f.g.) left \( \Lambda \) modules. If \( X \) and \( C \) are in \( \text{mod} \Lambda \), we write \( \Lambda(X,C) \) for \( \text{hom}_\Lambda(X,C) \) and \( P(X,C) \) for the submodule of \( \Lambda(X,C) \) comprising those maps \( f : X \to C \) for which there exists a factorization
\[
\begin{array}{c}
\xymatrix{X \ar[r]^f & C} \\
g \ar[r] & P
\end{array}
\]
with \( P \) projective. Also, let \( \text{Tr} \) and \( D \) be the usual transpose and dual. In this setting, Theorem 5.7 in M. Auslander’s paper \([A]\) may be stated as follows.

**Theorem A.** Let \( C \) be in \( \text{mod} \Lambda \). Let \( A_1, \ldots, A_m \) be a complete list of all non-injective indecomposable modules in \( \text{mod} \Lambda \) and let \( X_i = \text{Tr}DA_i \). For each \( i \), \( \Lambda(X_i,C)/P(X_i,C) \) is an \( \text{End} X_i \)-module of finite length. Let \( S_{i_1}, \ldots, S_{i_d} \) be a complete set of nonisomorphic simple \( \text{End} X_i \)-modules, and for each \( (\text{End} X_i)^\text{op}\)-submodule \( H \) of \( \Lambda(X_i,C) \) containing \( P(X_i,C) \) let \( n_1(A_i,H), \ldots, n_d(A_i,H) \) be the uniquely determined nonnegative integers so that the \( (\text{End} X_i)^\text{op}\)-socle of \( \Lambda(X_i,C)/H \) is isomorphic to \( \prod_{j=1}^{d_i} S_{i_j}^{n_j(A_i,H)} \). Finally let \( n(A_i) = \max\{n_j(A_i,H)\} \) as \( j \) runs through \( 1,2,\ldots,d_i \) and as \( H \) runs through all \( (\text{End} X_i)^\text{op}\)-submodules of \( \Lambda(X_i,C) \) containing \( P(X_i,C) \). Then

1. \( n(A_i) \) is finite;
2. if \( k > n(A_i) \) and \( 0 \to A_i^k \xrightarrow{g} B \to C \to 0 \) is exact, then \( A_i^k \) contains a submodule \( A' \) (isomorphic to \( A_i^{k-n(A_i)} \)) such that \( g(A') \) is a summand of \( B \).

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Keeping the notation of Theorem A we have

**Theorem 1.** Fix $C$ in $\text{mod } \Lambda$. Then there are only a finite number of modules $A$ in $\text{mod } \Lambda$ for which $0 \rightarrow A \xrightarrow{g} B \rightarrow C \rightarrow 0$ is indecomposable as a short exact sequence. In fact, if $A$ has an injective summand, or if $A \simeq \coprod_{i=1}^{m} A_i^{p_i}$ with $p_i > n(A_i)$ for some $i$, the sequence decomposes.

**Proof.** If $A$ has an injective summand then clearly the sequence decomposes. Suppose $p_i > n(A_i)$, and form the pushout diagram

\[
\begin{array}{ccc}
0 & \rightarrow & A \\
\downarrow & & \downarrow \\
0 & \rightarrow & A_i^{p_i} \rightarrow D \\
& & h' \\
& & C \\
0 & \rightarrow & 0
\end{array}
\]

Because $p_i > n(A_i)$, $A_i^{p_i}$ has a submodule $A'$ for which $h'(A')$ is a summand of $D$ (so $A'$ is actually a summand of $A_i^{p_i}$) by Theorem A. Let $A''$, $A'''$ be such that $A' \oplus A'' = A$ and $A' \oplus A''' = A_i^{p_i}$. Then we have a commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & A' \oplus A'' \rightarrow B \\
\downarrow & & \downarrow \\
0 & \rightarrow & A' \oplus A''' \rightarrow \alpha(A') \oplus B' \\
& & \rightarrow C \\
& & 0
\end{array}
\]

in which $\alpha = h'|_{A'}$ is an isomorphism and $h = h'|_{A''}$. Then $A' \overset{1}{\rightarrow} A' \overset{g}{\rightarrow} g(A')$ is a monomorphism which is split by $B \overset{\mu}{\rightarrow} \alpha(A') \overset{\alpha^{-1}}{\rightarrow} A'$, so $g(A')$ is a summand of $B$. This split monomorphism and the split inclusion of $A'$ into $A$ are coherent, i.e., the diagram

\[
\begin{array}{ccc}
A' & \xrightarrow{\text{incl}} & A' \oplus A'' \\
\text{incl} & \uparrow & \text{proj} \\
A' \oplus A'' & \xrightarrow{\text{proj}} & B
\end{array}
\]

commutes both ways. Thus the exact sequence $0 \rightarrow A' \xrightarrow{g'} g(A') \rightarrow 0 \rightarrow 0$ is a summand of $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$.

If we let $R$ be a local PID which is also a $k$-algebra, we get some interesting consequences. We let $f$ and $g$ be matrices over $R$, and say that $X = F_2 \xrightarrow{f} F_1 \xrightarrow{g} F_0$ is a representation of the diagram $A_2 = \cdots \rightarrow A_1 \rightarrow A_0$ over $R$, where $F_2$, $F_1$, and $F_0$ are free $R$-modules (see, e.g., [DR]). If $f$ and $g$ are both
t \times t \text{ matrices with nonzero determinant, then the sequence } \varepsilon = 0 \to A \to B \to C \to 0 \text{ is naturally associated with } X, \text{ where } A = \text{coker}(f), \ B = \text{coker}(gf), \text{ and } C = \text{coker}(g), \text{ by the following commutative diagram:}

\[
\begin{array}{cccccc}
0 & \to & F_1 & \to & F_0 & \to & C & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A & \to & B & \to & C & \to & 0.
\end{array}
\]

Two representations \( X, X' \) are said to be isomorphic if there is a commutative diagram

\[
\begin{array}{cccccc}
F_2 & \xrightarrow{f} & F_1 & \xrightarrow{g} & F_0 \\
\downarrow & & \downarrow & & \downarrow & & \\
F'_2 & \xrightarrow{f'} & F'_1 & \xrightarrow{g'} & F'_0
\end{array}
\]

with \( \alpha, \beta, \) and \( \gamma \) isomorphisms. It is shown in [C] that representations are isomorphic if and only if the corresponding sequences are isomorphic.

If \( m \) is the maximal ideal of \( R \) and \( f \) is a \( t \times t \) matrix, we let \( \nu(f) \) be the least integer \( n \) such that \( \det(f) \in m^n \) (where \( m^0 \) is the set of units of \( R \)). In this situation Theorem 1 yields the following.

**Corollary.** Let \( g \) be a fixed \( t \times t \) matrix with nonzero determinant, and let \( \nu(g) = r \). Then for a fixed integer \( n \), there are only finitely many nonisomorphic indecomposable representations \( X = F_2 \xrightarrow{f} F_1 \xrightarrow{g} F_0 \) with \( \nu(f) \leq n \).

**Proof.** If \( 0 \to A \to B \to C \to 0 \) is the sequence associated with \( X \), then the length of an indecomposable summand of \( C \) (respectively \( A \)) is bounded by \( r \) (respectively \( n \)); so every such sequence may be considered to be a sequence of \( R/m^s \)-modules, where \( s = \max\{r, n\} \). But \( R/m^s \) is an artin algebra of finite type, so Theorem 1 may be applied.

An application of this corollary, proved in [C], is that if \( \nu(f) < t \), where \( t \) is as above, then \( X \) must decompose.

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References


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