Sums and Products of Hilbert Spaces

Jesús M. F. Castillo

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Abstract. Let $H$ be a Hilbert space. We prove that the locally convex sum $\bigoplus_I H$ is a subspace of the product $H^I$ if and only if $I$ is countable, $H$ is infinite dimensional, and $\text{card } J \geq 2^{\aleph_0}$.

Notations. For the general terminology on locally convex spaces we refer to [1, 2].

If $E$ is a locally convex space, $U(E)$ will denote a fundamental system of absolutely convex closed neighborhoods of 0. If $p_U$ is the associated semi-norm of $U \in U(E)$, we note $\hat{E}_U$ for the completion of the normed space $(E/\text{Ker } p_U, \|\cdot\|_U)$, where $\|\phi_U(x)\|_U = p_U(x)$, $\phi_U$ being the quotient map. The spaces $\hat{E}_U$ will be referred as the associated Banach spaces. If $U, V \in U(E)$, $V \subseteq U$, the canonical linking map $\hat{T}_{VU}$ is the extension to the completions of the operator $T_{VU} \in L(E_U, E_V)$ defined by $T_{VU} \phi_U x = \phi_U x$.

Let $T$ be an operator acting between the Banach spaces $T : X \rightarrow Y$. Let $Z$ be a Banach space. By a subfactorization of $T$ through $Z$ we mean two operators $A : X \rightarrow Z$ and $B : \overline{\text{Im } A} \rightarrow Y$ such that $T = BA$. Note that $B$ need not be defined on all of $Z$, but only on the closure of the range of $A$ in $Z$. When $B$ is defined on the whole $Z$ then we have a factorization of $T$ through $Z$.

The spaces $l_p(I)$, $0 < p \leq +\infty$, are defined to be the Banach ($p$-Banach if $0 < p < 1$) spaces

$$l_p(I) = \{(x_i)_{i \in I} \in K^I : \|\sum_{i \in I} |x_i|^p\|_p\}^{1/p} \quad < +\infty$$
if $p < +\infty$, and

$$l_{\infty}(I) = \{(x_i)_{i \in I} \in K^I : \|x_i\|_\infty = \sup_{i \in I} |x_i| < +\infty\}$$

if $p = +\infty$.

We write

$$l^+_\infty(I) = \{x \in l_\infty(I) : x_i > 0 \ \forall i \in I\}$$

and recall the well-known fact that any Hilbert space is isometric with some $l_2(I)$.

Let $X$ be a Banach space, we will also consider the vector valued sequence spaces

$$l_p(X) = \{(x_n) \in X^N : (\|x_n\|) \in l_p\}, \quad 1 \leq p < +\infty$$

and

$$c_0(X) = \{(x_n) \in X^N : (\|x_n\|) \in c_0\}$$

which in fact are Banach spaces.

**Main results**

**Theorem.** Let $H$ be an infinite-dimensional Hilbert space. Then the locally convex sum $\bigoplus_I H$ is a subspace of some product $\prod_I H$ if and only if $I$ is countable and $\operatorname{card} J \geq 2^{\aleph_0}$.

Obviously $H$ needs to be infinite-dimensional since $\varnothing$, cannot be a subspace of any product $K^I$. On the other hand $\operatorname{card} J \geq 2^{\aleph_0}$ is required since $\bigoplus_N H$ is not metrizable.

**Proposition 1.** $\bigoplus_N l_2$ is a subspace of $\prod_I l_2, \quad \operatorname{card} J \geq 2^{\aleph_0}$.

**Proof.** Since the locally convex sum topology coincides with the so-called box-topology (see [1]) on countable sums, a fundamental system of neighborhoods of 0 is given by the sets: $U(z) = \prod_N z_n B \cap \bigoplus_N l_2$, where $B$ is the unit ball of $l_2$, and $z \in c_0$. We may suppose $z_n \neq 0$ for all $n \neq N$. Its associated seminorm is: $p_z((x_n)) = \sup_n z_n^{-1} \|x_n\|_2$, and the associated Banach space is clearly seen to be the completion of $\bigoplus_N l_2$ endowed with the norm $p_z$, that is:

$$\left\{(x_n) \in l_2^N : \sup_n z_n^{-1} \|x_n\|_2 \to 0\right\}.$$

This space is isometric with $c_0(l_2)$. Under this isometry, if $k \in c_0$ and $0 < k_n \leq z_n$, then the linking map between the associated Banach spaces to $p_k$ and $p_z$ is precisely the “diagonal” operator $D_\sigma : c_0(l_2) \to c_0(l_2), \quad D_\sigma((x_n)) = (\sigma_n x_n)$, where $\sigma_n = z_n^{-1} k_n^{-1}$.

If we choose $k$ such that $\sigma$ belongs to $l_2$, then $D_\sigma$ factorizes through $l_2(l_2)$:

$$\begin{array}{cccc}
  c_0(l_2) & \xrightarrow{D_\sigma} & c_0(l_2) \\
  \downarrow & & \uparrow \text{inclusion} \\
  l_2(l_2) & & \\
\end{array}$$
\[ \|D_\sigma x\|_{l^2(I)} = \left(\sum \|\sigma_n x_n\|_2^2\right)^{1/2} = \left(\sum |\sigma_n|^2 \|x_n\|^2_2\right)^{1/2} \leq \sup_n \|x_n\|_2 \left(\sum |\sigma_n|^2\right)^{1/2} = \|x\|_{c_0(l^2)} \|\sigma\|_2. \]

The continuity of the inclusion is obvious. But the space \( l^2(I^2) \) is isometric with \( l_2 \). Thus, the space \( \bigoplus_N l^2 \), as a projective limit of \( l^2 \) is a closed subspace of the topological product \( \prod_I l^2 \) [2, 19.10.3].

**Remark.** Since \( l^p(I^p) \) is isometric with \( l^p \), \( 1 \leq p < +\infty \), the preceding proof serves for the spaces \( l^p \), and with minor modifications for the nonseparable spaces \( l^p(I) \), \( 1 \leq p < +\infty \). Therefore it covers the situation for all Hilbert spaces.

**Proposition 2.** Let \( I \) be uncountable. Then \( \bigoplus_I l^2 \) is not a subspace of any product \( \prod_I l^2 \).

**Proof.** The latter space has separable associated Banach spaces while the former does not. \( \Box \)

**Proposition 3.** Let \( H \) be a Hilbert space, and \( I \) an uncountable set. Then \( \bigoplus_I H \) is not a subspace of any product of copies of \( H \).

**Proof.** We may write \( H = l^2(I) \) with \( I \) uncountable, by the remarks previous to Proposition 1, and Proposition 2. Let \( I \) be uncountable with \( d = \text{card} I \).

**Step 1.** Let \( A \in L(l^2(I), l^1(I)) \) represented by a matrix \( (a_{i,j})_{(i,j) \in I \times I} \) in the form:

\[ A(x_j) = (y_i) \quad \text{with} \quad y_i = \sum_{j \in I} a_{i,j} x_j. \]

Suppose that \( A \) has (a) a row of zeros or (b) a column of zeros. Then \( A \) cannot be part of a factorization of a diagonal operator \( D_\sigma : l_1(I) \to l_1(I), \sigma \in l^{+}\infty(I) \) through \( l^2(I) \). In case (a) since then all the vectors in \( \text{IM} A \) would have some coordinate zero, and \( \text{IM} AB \neq \text{IM} D_\sigma \). In case (b) it is \( A' : l^{+}\infty(I) \to l^2(I) \) which has a row of zeros and cannot be injective; since \( D_{\sigma^{-1}} \) is injective, the factorization \( D_{\sigma^{-1}} = D_{\sigma}' = B'A' \) is impossible, and thus \( D_{\sigma} = AB \) is impossible too.

From all this it follows that a nonzero element must exist in each row and in each column of \( A \). Therefore the set \( \{(i,j) \in I \times I : a_{i,j} \neq 0\} \) is uncountable, and we may assume \( a_{i,j} > 0 \) for uncountable many pairs \( (i,j) \). Thus an \( \varepsilon > 0 \) must exist such that \( a_{i,j} \geq \varepsilon \) for an uncountable set \( Z \subset I \times I \). Moreover these indexes of \( Z \) need to be scattered through infinitely many rows and columns of \( I \times I \); because if we suppose that they are "concentrated" in, let us say, a single column, then those vectors of \( l^2(I) \) with the corresponding index zero have zero as the image by \( A \). Since \( B \) can be considered surjective (see Step 2), \( A \) would not be a part of a factorization of \( D_\sigma \). If they are "concentrated" on a row we obtain the same result by transposition. Therefore we can choose a countable set \( Z_0 = \{(i_n, j_n) \in Z, n \in N\} \) such that \( i_n \neq i_m \) and \( j_n \neq j_m \).
whenever \( n \neq m \). Choose then an element \((z_j) \in l^2(I)\setminus l^1(I)\) with \( z_j \geq 0 \ \forall j \) and \( z_j \neq 0 \) if some couple \((i,j) \in Z_0\). If \( Az = y \), then we find that for each pair \((i,j) \in Z_0\):

\[
y_i = \sum_{k \in I} a_{ik} z_k \geq \varepsilon z_j
\]

whence

\[
\sum_{i \in I} |y_i| \geq \varepsilon \sum_{j \in I} |z_j| = +\infty
\]

and \( A \) cannot be an operator from \( l^2(I) \) into \( l^1(I) \). In this way we have essentially proved that:

**Step 2.** The diagonal operator \( D_\sigma : l^1(I) \to l^1(I) \), \( \sigma \in l^1(I) \), cannot be subfactorized through \( l^2(I) \): the above manipulations settle the case of factorization. For subfactorizations we use orthogonal projection onto \( IM \) to obtain a factorization through a Hilbert space. If this is nonseparable the calculations of step 1 apply. If it is \( l^2 \) then \( D_\sigma = AB \), \( B \in L(l^1(I), l^2) \) and \( A \in L(l^1(l^1), l^2) \) is clearly false since the image of \( D_\sigma \) cannot be contained in any \( l^1(N) = N \) a countable subset of \( I \).

**Step 3.** It is not hard to check that \( \varphi_d \) has a fundamental system of neighborhoods of 0 with associated Banach spaces isometric with \( l^1(I) \). Under this isometry the linking maps are the diagonal operators \( D_\sigma \), \( \sigma \in l^1(I) \).

**Step 4.** Let us assume that \( \varphi_d \) is a subspace of some product \( l^2(I)^J \). There is a fundamental system of neighborhoods of 0 in \( l^2(I)^J \), \( \mathcal{U} \), with associated Banach spaces isometric with \( l^2(I) \). Thus an embedding of \( \varphi_d \) into \( l^2(I)^J \) would imply for \( U \in \mathcal{U} \) the subfactorization

\[
l^1(I) \to (\varphi_d)_{\mathcal{U} \cap \varphi_d} \to l^1(I)
\]

of \( D_\sigma \), which we know is not possible.

**Step 5.** We complete the proof of our Proposition 3. Since the embedding of \( \varphi_d \) into \( H^J \) is not possible when \( d \) is uncountable, the embedding of \( \bigoplus I H \) into \( H^J \) is impossible too. \( \square \)

**Remark.** The result in Step 4 also holds when \( \varphi_d \) and \( l^2(J) \) with different index sets are considered. It is obviously true when \( \text{card} J < d \) and, reasoning as in Step 2, when \( \text{card} J > d \).

**Remark.** For general operators \( T : l^p(I) \to l^q(J) \), \( I, J \) uncountable and \( p > q \) we can obtain (compare with the final part of Step 1):

1. \( \forall j \in I \ \text{card}\{i \in I : a_{ij} \neq 0\} \leq \aleph_0 \)
2. \( \forall i \in I \ \text{card}\{j \in I : a_{ij} \neq 0\} \leq \aleph_0 \) if \( p > 1 \).
From this it follows that if \( \text{card}\{(i,j) \in I \times I : a_{ij} \neq 0\} > \aleph_0 \) then these indexes cannot be "concentrated" in a countable number of rows or columns. We could then proceed as in Step 1 to obtain a contradiction. Therefore:

(\*) \[ \text{card}\{(i,j) \in I \times I : a_{ij} \neq 0\} \leq \aleph_0 \]

and then IM \( T \subset l_q(N) \). We have

**Lemma.** Let \( I, J \) be uncountable sets, \( p > q \leq 1 \) and \( T : l_p(I) \to l_q(J) \) a continuous operator. Then IM \( T \subset l_q(N) \).

In the advance of this paper [4] the above lemma was incorrectly stated due to the omission of the hypothesis \( p > 1 \). The next counterexample shows that in that way it is no longer true: consider a partition \( I = \bigcup_{n=1}^{\infty} I_n \), with \( I_n \) uncountable for all \( n \). Take \( I_0 \) a countable subset of \( I \), and define

\[ a_{ij} = \begin{cases} i^{-4} & \text{when } i \in I_0 \text{ and } j \in I, \\ 0 & \text{otherwise}. \end{cases} \]

This matrix defines an operator from \( l_1(I) \) to \( l_{1/2}(I) \) for which (\*) does not hold.

**References**