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ON THE ALMOST EVERYWHERE EXISTENCE OF THE ERGODIC HILBERT TRANSFORM

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ABSTRACT. Let $(X, \mathfrak{M}, \mu)$ be a finite measure space, T an invertible measure-preserving transformation and v a positive measurable function. For $p = 1$, we prove that the ergodic Hilbert transform $Hf(x) = \lim_{n\to\infty} \sum_{i=-n}^{n}{}' f(T^i x)/i$ exists a.e. for every f in $L^1(v d\mu)$ if and only if $\inf_{i\geq 0} v(T^i x) > 0$ a.e. We also solve the problem for $1 < p \leq 2$. In this case the condition is $\sup_{k\geq 1} k^{-1} \sum_{i=0}^{k-1} v^{-1/(p-1)}(T^i x) < \infty$ a.e. If the transformation T is ergodic, the characterizing conditions become that $1/v \in L^\infty$ and $v^{-1/(p-1)} \in L^1(\mu)$, respectively. These characterizations, together with some recent results, give, for $1 \leq p \leq 2$, that $Hf(x)$ exists a.e. for every f in $L^p(v d\mu)$ if and only if the sequence of the Césàro-averages $k^{-1}(f(x) + f(Tx) + \dots f(T^{k-1}x))$ converge a.e. for every f in $L^p(v d\mu)$. This equivalence has recently been obtained by Jajte for a unitary operator, not necessarily positive, acting on L^2.

1. INTRODUCTION

Let $(X, \mathfrak{M}, \mu)$ be a σ-finite measure space and let T be an invertible measure preserving transformation from X into itself. We consider *the ergodic Hilbert transform*, associated to T, defined by

$$Hf(x) = \lim_{n\to\infty} \sum_{i=-n}^{n}{}' \frac{f(T^i x)}{i}, \tag{1.1}$$

where f denotes a measurable function and the prime denotes omission of the 0th term.

In [1], Cotlar shows that if $\mu(X) = 1$, then, the ergodic Hilbert transform of each function $f \in L^1(\mu)$ exists almost everywhere. The above also remains true when $\mu(X) = \infty$. (For a simple proof see [8]).

Our aim in this paper is to characterize those *positive measurable functions* v for which there exists Hf, in the pointwise sense, for every $f \in L^p(v d\mu)$, $1 \leq p < \infty$.

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In the case $p = 1$ and $\mu(X) < \infty$, we prove that the characterizing condition is given by $\inf_{i\geq 0} v(T^i x) > 0$ a.e., which is equivalent, in this case, to $\inf_{-\infty<i<\infty} v(T^i x) > 0$ a.e.. However, the last condition is only sufficient but not necessary for the existence of the ergodic Hilbert transform when $\mu(X) = \infty$.

In the case $p > 1$ and $\mu(X) < \infty$, we consider *the ergodic maximal operator*

$$Mf(x) = \sup_{n\geq 1} |A_n f|, \tag{1.2}$$

associated to the Césàro-averages

$$A_n f(x) = \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x) \tag{1.3}$$

and we show that condition $Mv^{-1/(p-1)}(x) < \infty$ a.e. implies the existence of $Hf(x)$ a.e. for each $f \in L^p(vd\mu)$ and moreover such a condition is also necessary when $1 < p \leq 2$.

In what follows, we will alway assume that $T: X \to X$ is an invertible measure preserving transformation, v is a positive measurable function and H, M, A_n are the operators defined by (1.1)–(1.3), respectively. On the other hand, throughout this paper, we denote by H^* *the ergodic maximal Hilbert transform*

$$H^* f(x) = \sup_{n\geq 1} \left| \sum_{i=-n}^{n}{}' \frac{f(T^i x)}{i} \right|$$

and by f^* *the ergodic maximal function*

$$f^* = \sup_{n,m\geq 0} \left| \frac{1}{n+m+1} \sum_{i=-n}^{m} f(T^i x) \right|.$$

Finally, we will consider two sets as equals if they agree up to a set of measure zero.

2. The main result

Theorem 2.1. *Let $(X, \mathfrak{M}, \mu)$ be a finite measure space and $1 \leq p \leq 2$. The following statements are equivalent:*

(a) *There exists $Hf(x)$ a.e. for every $f \in L^p(vd\mu)$.*

(b) *$H^* f(x) < \infty$ a.e. for every $f \in L^p(vd\mu)$.*

(c) *There exists a positive measurable function u such that for every f and $\lambda > 0$*

$$\int_{\{x\in X:\ H^* f(x)>\lambda\}} u\, d\mu \leq \lambda^{-p} \int_X |f|^p v\, d\mu.$$

(d) *$\inf_{-\infty<i<\infty} v(T^i x) > 0$ a.e. if $p = 1$; $(v^{-1/(p-1)})^*(x) < \infty$ a.e. if $p > 1$.*

(e) $\inf_{i\geq 0} v(T^i x) > 0$ *a.e. if* $p = 1$; $Mv^{-1/(p-1)}(x) < \infty$. *a.e. if* $p > 1$.

Proof. It is clear that (a) implies (b) and (d) implies (e). On the other hand, (b) implies (c) by *Nikishin's theorem* (see [2]). Then, it remains to prove (e) $\Rightarrow$ (a) and (c) $\Rightarrow$ (d).

Proof of (e) $\Rightarrow$ (a). If $p = 1$, it follows from (e) that $X = \bigcup_{s\geq 1} X_s$, where

$$X_s = \{x \in X: \sup_{i\geq 0} v^{-1}(T^i x) < s\}.$$

The X_s are absorbing sets, that is $X_s \subset T^{-1}(X_s)$, and, since $\mu(X) < \infty$, we have that the X_s are invariant sets, i.e., $T(X_s) = X_s$. On the other hand, it is easy to see that $L^1(X_s, v d\mu) \subset L^1(X_s, \mu)$ and thus (a) follows from this and Cotlar's result.

Now, let $p > 1$ and $\sigma = v^{-1/(p-1)}$. Since $\limsup_{n\to\infty} A_n\sigma(x) < \infty$ a.e., we decompose X into the invariant sets $X_s = \{x \in X: \limsup_{n\to\infty} A_n\sigma(x) < s\}$.

For each positive integer N let $\sigma_N = \min(\sigma, N)$. Since $\sigma_N \in L^1(X_s, \mu)$ and $\mu(X_s)$ is finite, it follows from *Birkhoff's ergodic theorem* that there exists an invariant function σ_N^0 in $L^1(X_s, \mu)$ such that $\lim_{k\to\infty} A_k\sigma_N(x) = \sigma_N^0(x)$ for almost every x in X_s. Moreover

$$\int_{X_s} \sigma_N \, d\mu = \int_{X_s} \sigma_N^0 \, d\mu,$$

since X_s is invariant. Therefore,

$$\int_{X_s} \sigma_N \, d\mu \leq \int_{X_s} \limsup_{k\to\infty} A_k\sigma \, d\mu < s\mu(X_s).$$

Then, it follows from the Lebesgue's monotone convergence theorem that $\sigma \in L^1(X_s, \mu)$.

On the other hand, Hölder's inequality shows

$$\int_{X_s} |f| \, d\mu \leq \left(\int_{X_s} |f|^p v \, d\mu\right)^{1/p} \left(\int_{X_s} \sigma \, d\mu\right)^{1/p'},$$

where p' is the conjugate exponent of p. Therefore, $L^p(X_s, v d\mu) \subset L^1(X_s, \mu)$ and, thus, we obtain (a).

Proof of (c) $\Rightarrow$ (d). In order to prove this implication we will need the following definition and the following Lemma proved in [4].

Definition 2.2. Let s be a positive integer. The measurable set $B \subset X$ is the base (with respect to T) of an ergodic rectangle of length s if $T^i B \cap T^j B = \emptyset$, $i \neq j$, $0 \leq i, j \leq s-1$. In such case, the set $\mathfrak{R} = \bigcup_{0\leq i\leq s-1} T^i B$ will be called ergodic rectangle with base B and length s.

Lemma 2.3. *Let Y be a measurable subset of X and let k be a positive integer. Then, there exists a countable family $\{B_i, i \in \mathbb{Z}^+\}$ of sets of finite measure such that:*

(i) $Y = \bigcup_{i=0}^{\infty} B_i$.

(ii) $B_i \cap B_j = \varnothing$ *if* $i \neq j$.

(iii) *For every* i, B_i *is the base of an ergodic rectangle of length* $s(i) \leq k$ *and such that if* $s(i) < k$, *then* $T^{s(i)}A = A$ *for every measurable set* $A \subset B_i$.

For every positive integer k let $\{B_{i,k}, i \in \mathbb{Z}\}$ be the sequence given by Lemma 2.3 for an integer sufficiently large, for example $4k$, and the set X.

We consider $X = Y \cup Z$, where

$$Y = \bigcap_{k=1}^{\infty} \left(\bigcup_{\{i:\, s(i,k)=4k\}} B_{i,k} \right) \tag{2.4}$$

and

$$Z = \bigcup_{k=1}^{\infty} \left(\bigcup_{\{i:\, s(i,k)<4k\}} B_{i,k} \right). \tag{2.5}$$

First, we notice that $\inf_{-\infty<i<\infty} v(T^i x) > 0$ for almost every x in Z and, therefore, $(v^{-1/(p-1)})^*(x) < \infty$ a.e. x in Z if $p > 1$, even if (c) does not hold.

Let $B_{i,k}$ be a base of a rectangle with length $s(i,k) < 4k$ for some integer k. We have

$$B_{i,k} = \bigcup_{(n_1, n_2, \dots, n_{s(i,k)}) \in \mathbb{Z}^{s(i,k)}} D_{n_1, n_2, \dots, n_{s(i,k)}}, \tag{2.6}$$

where

$$D_{n_1, n_2, \dots, n_{s(i,k)}} = \{x \in B_{i,k} : 2^{n_1} \leq v(x) < 2^{n_1+1}, 2^{n_2} \leq v(Tx) < 2^{n_2+1}, \dots,$$
$$2^{n_{s(i,k)}} \leq v(T^{s(i,k)-1}x) < 2^{n_{s(i,k)}+1}\}$$

By Lemma (2.3) we have that $T^{s(i,k)}A = A$ where $A = D_{n_1, n_2, \dots, n_{s(i,k)}}$ and, therefore for almost every x in A

$$\inf_{-\infty<i<\infty} v(T^i x) \geq \min\{2^{n_j}, 1 \leq j \leq n_{s(i,k)}\} > 0.$$

Thus, by (2.5) and (2.6) we get $\inf_{-\infty<i<\infty} v(T^i x) > 0$ a.e. in Z.

Now, we will prove that condition (d) holds for almost every x in Y. We may assume without loss of generality that $u \in L^1(\mu)$ and $u \leq v$.

Let us fix a positive integer k and let $B_{i,k}$ be a base of a rectangle with length $s(i,k) = 4k$. If $p = 1$ we consider any subset E of $B_{i,k}$ with positive measure and any integer h. It is easy to see that if $x \in E$ and j is an integer such that $h+1 \leq j \leq h+k$, then $H^* \chi_{T^h E}(T^j x) \geq 1/k$, where $\chi_{T^h E}$ denotes the characteristic function of $T^h E$. Therefore, it follows from (c) that

$$\int_{\bigcup_{j=h+1}^{h+k} T^j E} u \, d\mu \leq k \int_{T^h E} v \, d\mu \tag{2.7}$$

for every integer h. Since T preserves the measure, it follows from (2.7) that

$$\int_E A_{k+1}u(T^h x)\,d\mu(x) \leq \int_E v(T^h x)\,d\mu(x)$$

and, therefore, for a.e. $x \in B_{i,k}$ we have

$$A_{k+1}u(T^h x) \leq v(T^h x), \text{ for every integer } h. \tag{2.8}$$

Now, it is easy to see that (2.8) holds for almost every x in Y and any k.

On the other hand, *Birkhoff's ergodic theorem* asserts that the averages $A_{k+1}u$ converge a.e. to a positive invariant function u_0. Thus, if we let k tend to infinity in (2.8) we get $0 < u_0(x) \leq v(T^h x)$, for a.e. x in Y and any h, which proves (d), in the case $p = 1$.

Now, we consider $p > 1$ and $\sigma = v^{-1/(p-1)}$. Then, for each integer r we define

$$E_{i,k,r} = \{x \in B_{i,k} \colon 2^{r+1} < A_k\sigma(x) \leq 2^{r+2}\}$$

and for a measurable set $E \subset E_{i,k,r}$, with $\mu(E) > 0$, let $\mathfrak{R}$ be the rectangle with base E and length k, i.e., $\mathfrak{R} = \bigcup_{0\leq i\leq k-1} T^i E$.

If f is a non-negative function with support in $\mathfrak{R}$ we have $H^* f(T^j x) \geq 2^{-1}A_k f(x)$ for $x \in E$ and $k \leq j \leq 2k-1$. Then, for $f = \sigma\chi_{\mathfrak{R}}$ we obtain

$$Q = \bigcup_{j=k}^{2k-1} T^j E \subset \{x \in X \colon H^*(\sigma\chi_{\mathfrak{R}})(x) > 2^r\}$$

and, consequently, it follows from (c) that $2^{rp}\int_Q u\,d\mu \leq \int_{\mathfrak{R}} \sigma\,d\mu$, that is,

$$2^{rp}\int_E \sum_{i=k}^{2k-1} u(T^i x)\,d\mu(x) \leq \int_E \sum_{i=0}^{k-1} \sigma(T^i x)\,d\mu(x).$$

Therefore, by the definition of $E_{i,k,r}$ we have

$$\int_E (A_k\sigma(x))^p A_k u(T^k x)\,d\mu(x) \leq 4^p \int_E A_k\sigma(x)\,d\mu(x),$$

and, thus, we obtain that there exists a constant C such that for almost every x in $B_{i,k}$

$$(A_k\sigma(x))^{p-1} A_k u(T^k x) \leq C,$$

and, therefore, for a.e. x in Y and every positive integer k.

On the other hand, since $u \in L^1(\mu)$ and $\mu(X) < \infty$, we get

$$\lim_{k\to\infty} A_k u(T^k x) > 0$$

a.e. and, therefore, for almost every in Y there exists a real $\alpha(x) > 0$ such that for every positive integer k

$$(A_k\sigma(x))^{p-1} \leq C/\alpha(x). \tag{2.9}$$

It is obvious that we can obtain an inequality as (2.9) with T^{-1} instead of T. Therefore, we have $\sigma^*(x) < \infty$ a.e. in Y and, thus, the proof of Theorem 2.1 is complete.

Remarks. 1. In the case $\mu(X) = \infty$, condition (d), for $p = 1$, of Theorem 2.1 implies the existence of the ergodic Hilbert transform, since the sets $\{x \in X\colon \sup_{-\infty \le i \le \infty} v^{-1}(T^i x) < s\}$ are invariant and, then, it is enough to take these sets instead of X_s in the proof of (e) $\Rightarrow$ (a) in Theorem 2.1. However, (a) does not imply (d), as the following example shows.

(2.10) **Example.** Let X be the set of the integers with the counting measure μ, T the shift transformation $Tk = k + 1$ and v the function defined by $v(2^n) = v(-2^n) = 1/n$ for every positive integer n and $v(x) = 1$ otherwise.

Let f be a nonnegative function in $L^1(v\,d\mu)$. We define the function g given by $g(x) = f(x)$ if $x = \pm 2^n$ for some n and $g(x) = 0$ otherwise. Since $f - g$ is in $L^1(\mu)$ there exists $H(f-g)(x)$ for every x. We will now prove that $Hg(x)$ also exists for every x.

Let $x \ge 2$ and s an integer such that $2^{s-1} \le x < 2^s$. For each positive integer k let

$$a_k(x) = \sum_{i=1}^{k} \frac{g(x+i)}{i} \quad \text{and} \quad b_k(x) = \sum_{i=1}^{k} \frac{g(x-i)}{i}.$$

We have that $a_k(x) = \sum_{j=s}^{n} g(2^j)/(2^j - x)$, for some integer n, and therefore for every k

$$a_k(x) = \sum_{j=2}^{n} g(2^j) v(2^j) \frac{j}{2^j - x} \le \|f\|_{1,v\,d\mu} \sum_{j=s}^{\infty} \frac{j}{2^j - x} < \infty,$$

which proves that the sequence $\{a_k(x)\}$ converges as $k \to \infty$.

On the other hand, for k sufficiently large, $b_k(x)$ is dominated by a sum of the type

$$\sum_{j=1}^{s-1} g(2^j)/(x - 2^j) + \sum_{j=1}^{n} g(-2^j)/(x + 2^j),$$

for some positive integer n, and since

$$\sum_{j=1}^{n} g(-2^j)/(x + 2^j) \le \|f\|_{1,v\,d\mu} \sum_{j=1}^{\infty} j/(x + 2^j) < \infty,$$

we obtain that $\{b_k(x)\}$ also converges as $k \to \infty$.

Therefore, we get that there exists $Hg(x)$ for $x \ge 2$. The existence for $x < 2$ can be proved in a similar way. Thus, we conclude that $Hf(x)$ exists for every x and any $f \in L^1(v\,d\mu)$ but $\inf_{-\infty < i < \infty} v(T^i x) = 0$. Furthermore, in this example, we have that $\inf_{i \le 0} v(T^i x) = 0$ and $\inf_{i \ge 0} v(T^i x) = 0$, which shows that condition "$\inf_{i \le 0} v(T^i x) > 0$ or $\inf_{i \ge 0} v(T^i x) > 0$" is not necessary for the existence of the ergodic Hilbert transform in the case $\mu(X) = \infty$.

2. If the transformation T is *ergodic*, condition (d) and (e) of Theorem 2.1, in the case $p = 1$, become that $1/v \in L^\infty$, since the function $u(x) = \inf_{-\infty<i<\infty} v(T^i x)$ is invariant and, therefore, essentially constant. In this case, the proof of (e) $\Rightarrow$ (a) is trivial since the inclusion $L^p(v\,d\mu) \subset L^1(\mu)$ is obvious. If T is ergodic and $p > 1$, conditions (d) and (e) reduce to saying that $\sigma = v^{-1/(p-1)} \in L^1(\mu)$, since if (e) holds, then X agrees with some invariant set $X_s = \{x \in X\colon \limsup_{n\to\infty} A_n\sigma(x) < s\}$ and, on the other hand, $\sigma \in L^1(X_s, \mu)$, as we saw in the proof of (e) $\Rightarrow$ (a). Moreover, if T is ergodic and the measure space is nonatomic, then the set Z defined in (2.5) is empty, since it follows easily from Lemma 2.3 that, for every positive integer k, the set X can be written as a countable union of bases of rectangles of length $4k$ (see Corollary 2.12 in [4]).

3. In the case $p > 1$, the proofs of (e) $\Rightarrow$ (a) and (c) $\Rightarrow$ (d) do not use that $p \le 2$. The condition $p \le 2$ is only used in implication (b) $\Rightarrow$ (c), when we apply Nikishin's theorem. Thus, we leave as an open problem to see whether or not condition (a) implies conditions (d) when $p > 2$. On the other hand, the result (e) $\Rightarrow$ (a) can be extended to the context of *Orlicz spaces.*

We consider an *N-function* ϕ, that is, a function $\phi(s) = \int_0^s \varphi$ where $\varphi\colon [0,\infty) \to \mathbf{R}$ is continuous from the right, nondecreasing such that $\varphi(s) > 0, s > 0, \varphi(0) = 0$ and $\varphi(s) \to \infty$ for $s \to \infty$, and, associated to ϕ, we consider the Orlicz class $L_\phi(\mu) = \{f \in \mathfrak{M}\colon \int_X \phi(|f|)\,d\mu < \infty\}$, where $\mathfrak{M}$ denotes the space of measurable and a.e. finite functions from X to $\mathbf{R}$ (or to $\mathbb{C}$). Thus, if $\phi(s) = s^p, p > 1$, then $L_\phi(\mu) = L^p(\mu)$.

Associated to φ we have $\rho\colon [0,\infty) \to \mathbf{R}$ defined by $\rho(t) = \sup\{s\colon \varphi(s) \le t\}$ (*the generalized inverse function of* φ), which has the same aforementioned properties of φ. Then, the N-function ψ defined by $\psi(t) = \int_0^t \rho$ is called the *complementary* N*-function of* ϕ. Thus, if $\phi(s) = p^{-1}s^p, p > 1$, then $\psi(t) = q^{-1}t^{p'}$ where p' is the conjugate exponent of p.

Young's inequality asserts that $st \le \phi(s) + \psi(t)$ for $s, t \ge 0$, equality holding if $t = \varphi(s)$ or else $s = \rho(t)$. (See II-13 in [7].)

Then, the proof of (e) $\Rightarrow$ (a), in the case $p > 1$, can be easily adapted, taking $\sigma = \rho(1/v)$ instead of $\sigma = v^{-1/(p-1)}$, in the following way:

We consider $X_s = \{x \in X\colon \limsup_{n\to\infty} A_n\sigma(x) < s\}$ and we obtain that $\sigma \in L^1(X_s, \mu)$ using the same arguments used there. On the other hand, taking into account Young's inequality and the equality cases in this inequality, we have

$$\int_{X_s} |f|\,d\mu \le \int_{X_s} \phi(|f|)v\,d\mu + \int_{X_s} \psi(1/v)v\,d\mu \le \int_{X_s} \phi(|f|)v\,d\mu + \int_{X_s} \sigma\,d\mu$$

and therefore we obtain $L_\phi(X_s, v\,d\mu) \subset L^1(X_s, \mu)$. Thus, we conclude that *if* $(X, \mathfrak{M}, \mu)$ *is a finite measure space and* ϕ *an* N*-function, the condition* $M\rho(1/v)(x) < \infty$ *a.e. implies that* $Hf(x)$ *exists a.e. for every* $f \in L_\phi(v\,d\mu)$.

We remark that, if T is *ergodic*, the condition $M\rho(1/v)(x) < \infty$ a.e. becomes that $\rho(1/v) \in L^1(\mu)$.

4. The characterizations of the positive functions v for which the Cesàro-averages $A_n f$ converge a.e. for every f in $L^p(v\,d\mu)$ are studied in [5] for $p = 1$ and in [6] for $1 < p < \infty$, with T not necessarily invertible and assuming $\mu(X) < \infty$. The characterizing conditions are precisely the conditions included in (e) of Theorem 2.1. Therefore, we have the following corollary:

Corollary 2.11. *Let $(X, \mathfrak{M}, \mu)$ be a finite measure space and $1 \leq p \leq 2$. Then, $Hf(x)$ exists a.e. for every $f \in L^p(v\,d\mu)$ if, and only if, the Cesàro-averages $A_n f$ converge a.e. for every $f \in L^p(v\,d\mu)$.*

The above result was proved in [3] for unitary operators (not necessarily positive) acting on L^2. Corollary 2.12 is also closely related to the following result, obtained by Wos in [9].

Theorem. *Let $(X, \mathfrak{M}, \mu)$ be a finite measure space and T an ergodic invertible measure preserving transformation from X into itself. If f is a measurable function such that the averages $A_n f$ converge a.e., then, the ergodic Hilbert transform $Hf(x)$ exists a.e.*

Our result is similar to Wos' but as far as we know the converse of this theorem is an open problem. Of course, there are at least two more differences: the transformation in the paper of Wos is ergodic but he assumes the existence of the limit for a single function and not for all the functions in a certain class $L^p(v\,d\mu)$ as we do.

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