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ON A PROPERTY OF METRIC PROJECTIONS ONTO CLOSED SUBSETS OF HILBERT SPACES

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Dedicated to P. L. Butzer on the occasion of his sixtieth birthday

ABSTRACT. Applying the theory of monotone operators to the metric projection P_K of a Hilbert space H onto a nonempty closed subset K of H we prove a kind of connectedness property of the set $\{x \in H; P_K(x)$ is not a singleton or P_K is not upper semi-continuous at $x\}$ which is a typical set for investigations in best approximation. A result of Balaganskii is extended.

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Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and let K be a nonempty closed subset of H. The metric projection $P_K: H \to 2^K$ is defined by

$$P_K(x) := \{k \in K; \|k - x\| = d_K(x)\},$$

where $d_K: H \to \mathbf{R}$ is the distance function of K. If $P_K(x)$ is a singleton for each $x \in H$, then K is called a Chebyshev set.

To motivate the purpose of this note let us first restrict ourselves to Chebyshev sets though our general result is concerned with arbitrary closed subsets of H.

The problem of convexity of a Chebyshev set was studied by many authors under various conditions on the continuity of the metric projection, among others by Klee [11, 12], Vlasov [17] and Asplund [1]. In these investigations, continuity was required at all points of H. In a paper of 1982, Balaganskii [2] considered the aspect of cardinality of the set of points of discontinuity of the metric projection. Typical for his results is the following theorem: If K is a Chebyshev set and if the set of points of discontinuity of P_K is countable, then K is convex. The method of proof in [2] is an inversion method developed by Ficken and later used by Klee [12] and Asplund [1].

In the present note we shall prove a more general result concerning a "connectedness" property of the set of discontinuities of the metric projection from

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which Balaganskii's result involving the cardinality of this set is obtained as a corollary.

Our method of proof uses the theory of monotone operators in Hilbert spaces which turns out to be a very elegant tool. A standard reference for monotone operators in Hilbert space is the monograph of Brézis [7].

The aspect of monotony in best approximation in Hilbert spaces was considered by several authors; see e.g. [10, 6, 4, 5, 9]. The fact that the metric projection P_K is a monotone operator for an arbitrary set K follows immediately from Asplund [1] though it is not definitely stated there. Kenderov [10] explicitly formulated this property and gave an elementary proof. For convex sets this property was well-known much earlier.

To describe the purpose of this paper in more detail let us introduce some notations and preliminaries.

If $x \in H$ and $r > 0$, then the open ball, closed ball, and sphere with center x and radius r will be denoted by $B(x;r)$, $\overline{B}(x;r)$ and $S(x;r)$, respectively. The convex hull and closed convex hull of a set $A \subset H$, will be denoted by $\operatorname{co} A$ and $\overline{\operatorname{co}} A$, respectively; $\delta(A)$ is the diameter of A.

Throughout this note, K will be a nonempty closed subset of H, unless otherwise stated. We already introduced the metric projection of H onto K. Evidently,

$$P_K(x) = \overline{B}(x; d_K(x)) \cap K \ \forall x \in H.$$

As usual, P_K is identified with its graph in $H \times H$. Thus, $(x, k) \in P_K$ if and only if $k \in P_K(x)$. K is said to be proximinal if $P_K(x) \neq \varnothing$ for each $x \in H$.

As a generalization of the metric projection we consider the mapping $\Phi_K \colon H \to 2^H$ defined by

$$\Phi_K(x) := \bigcap_{\varepsilon > 0} \overline{\operatorname{co}}(\overline{B}(x; d_K(x) + \varepsilon) \cap K).$$

This mapping was introduced independently by Berens [4, 5] and Franchetti and Papini [8] for studying approximative properties of sets in normed linear spaces. It was Berens who pointed out and investigated the monotony of Φ_K in Hilbert spaces. We summarize some properties of this mapping; for proofs we mainly refer to [5].

For each $x \in H$, $\Phi_K(x)$ is a nonempty closed convex subset of H and

$$\overline{\operatorname{co}} P_K(x) \subset \Phi_K(x) \subset \overline{B}(x; d_K(x)).$$

Φ_K is a maximally monotone operator, that means first, Φ_K is monotone:

$$\langle y - y', x - x' \rangle \geq 0 \quad \forall (x, y), (x', y') \in \Phi_K,$$

and second, there is no proper monotone extension of Φ_K in $H \times H$. Denoting the inverse of $I + \Phi_K$ by $J := (I + \Phi_K)^{-1}$, we have: J is a contraction and its domain is the whole space H. This follows by Minty's characterization of maximally monotone operators in Hilbert spaces.

Φ_K is even cyclically monotone which means:

$$\sum_{i=1}^{n} \langle x_i - x_{i-1}, y_i \rangle \geq 0$$

for any set of pairs $(x_0, y_0), \ldots, (x_n, y_n) \in \Phi_k$ satisfying $(x_n, y_n) = (x_0, y_0)$, $n \in \mathbb{N}$ arbitrary.

From the inclusion $P_K \subset \Phi_K$, it is also clear that P_K is a monotone, even a cyclically monotone operator.

It follows from a result of Rockafellar on cyclically monotone operators (cf. [7]) that Φ_K is the subdifferential of a continuous convex real-valued function defined on all of H. Indeed, we have

$$\Phi_K = \partial \varphi_K,$$

where $\varphi_K \colon H \to \mathbb{R}$ is the continuous convex function defined by

$$\begin{aligned} \varphi_K(x) &:= \sup\{\langle x, k \rangle - \tfrac{1}{2}\|k\|^2 ; k \in K\} \\ &= \tfrac{1}{2}\|x\|^2 - \tfrac{1}{2} d_K^2(x) \end{aligned}$$

which was introduced by Asplund [1] for studying convexity of Chebyshev sets.

As a consequence of these properties Φ_K is norm to weak upper semicontinuous (usc).

Note that if H is a finite-dimensional space, then $\Phi_K(x) = \operatorname{co} P_K(x)$ for each $x \in H$, while in each infinite dimensional Hilbert space there is a proximinal set K such that $\overline{\operatorname{co}} P_K(x) \neq \Phi_K(x)$ for some $x \in H$, as was proved by Godini [9]; see also Klee [13] for a nonseparable Hilbert space.

The following result relates the aspect of convexity of a Chebyshev set to the maximal monotony of its metric projection.

$(*)$ For a nonempty subset K of H the following statements are equivalent:

(i) P_K is maximally monotone, i.e., $P_K = \Phi_K$.

(ii) K is closed and convex.

(iii) K is a Chebyshev set, and P_K is continuous.

For convenience, continuity is understood here with respect to the norm topology of H. However, there are known various weaker types of continuity of the metric projection under which the equivalence of (ii) and (iii) was established. The implications (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii) were given by Berens and Westphal [6] in 1977. See also [4] for a discussion of this subject.

Stimulated by the "global" result stated in $(*)$ we now consider the set

$$C_K := \{x \in H ; P_K(x) = \Phi_K(x)\}$$

for an arbitrary closed subset K of H. It is equal to the set

$$T_K' := \{x \in H ; \lim_{\varepsilon \to 0+} \delta(\overline{B}(x ; d_K(x) + \varepsilon) \cap K) = 0\}$$

known, in particular, from the work of Soviet mathematicians on approximative properties of sets. Stečkin [16] proved that T_K' is a dense G_δ-set; cf. also

[18; p. 9]. Balaganskii [2], too, worked with the set T'_K. Moreover, one can prove the following alternative representations of the set C_K:

$$\begin{aligned} C_K &= \{x \in H\,; P_K(x) \text{ is a singleton and } P_K \text{ is usc at } x\} \\ &= \{x \in H\,; \varphi_K \text{ is Fréchet differentiable at } x\}. \end{aligned}$$

The latter is, of course, related to Asplund's work [1] on Chebyshev sets.

It is the purpose of this note to prove a type of connectedness property of the complement of C_K. Indeed, we characterize the isolated points of the complement of C_K and show that if a point of $H\backslash C_K$ is not isolated, then it lies on some nonconstant Lipschitz curve which is completely contained in $H\backslash C_K$. Thus, if a path component of $H\backslash C_K$ consists of a single point, this must be isolated.

In the following sections we prove this result and give some applications.

We are most grateful to H. Berens for stimulating discussions.

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Our main result is the following:

Theorem. *Each point $x \in H\backslash C_K$ lies on some nonconstant Lipschitz curve completely contained in $H\backslash C_K$ unless x is an isolated point of $H\backslash C_K$.*

The point x is an isolated point of $H\backslash C_k$ if and only if $P_K(x) = S(x\,; d_K(x))$.

Proof. If $x \in H\backslash C_K$ such that $P_K(x) = S(x\,; d_K(x))$, then for each $z \in \overline{B}(x\,; d_K(x))$, $z \neq x$, we have

$$P_K(z) = \Phi_K(z) = \left\{x + d_K(x)\frac{z-x}{\|z-x\|}\right\},$$

i.e., $z \in C_K$. Thus, x is an isolated point of $H\backslash C_K$.

If $x \in H\backslash C_K$ and $P_K(x) \neq S(x\,; d_K(x))$, there exists a boundary point y of $\Phi_K(x)$ and a unit vector u such that $y \notin P_K(x)$ and $y + \lambda u \notin \Phi_K(x)$ for each $\lambda > 0$. Put $y_\lambda := y + \lambda u$ $(\lambda \geq 0)$. We shall show that $J(x + y_\lambda) \in H\backslash C_K$ for λ sufficiently small. Indeed, fix λ_0 such that $0 < \lambda_0 < \frac{1}{2}d_K(y)$. Then, for $0 \leq \lambda \leq \lambda_0$,

$$|d_K(y_\lambda) - d_K(y)| \leq \|y_\lambda - y\| = \lambda \leq \lambda_0 < \tfrac{1}{2}d_K(y)$$

implying

$$\|y_\lambda - y\| < d_K(y_\lambda)$$

and, as J is a contraction,

$$\begin{aligned} \|(x + y_\lambda - J(x + y_\lambda)) - y_\lambda\| &= \|J(x + y) - J(x + y_\lambda)\| \\ &\leq \|y - y_\lambda\| < d_K(y_\lambda). \end{aligned}$$

Hence, for $0 \leq \lambda \leq \lambda_0$, the element $x + y_\lambda - J(x + y_\lambda)$, by definition an element of $\Phi_K(J(x + y_\lambda))$, does not belong to $P_K(J(x + y_\lambda))$. Noting that $J(x + y_\lambda) = x$ for $\lambda = 0$ and $\neq x$ for $\lambda > 0$ we thus proved that

$$[0\,, \lambda_0] \ni \lambda \to J(x + y_\lambda)$$

is a nonconstant Lipschitz curve in $H\backslash C_K$ with initial point x.

Note that in case of the Euclidean plane the above characterization of isolated points of $H\backslash C_K$ was already observed by Motzkin [14] in 1935.

From the Theorem the central result in Balaganskii's paper is deduced as a corollary.

Corollary 1 (= Theorem 2 of [2]). *If K is a nonconvex subset of H such that $H\backslash K$ is connected, then $H\backslash C_K$ is uncountable unless K is the complement of an open ball.*

Proof. As K is nonconvex, $H\backslash C_K \neq \varnothing$. If the set $H\backslash C_K$ is countable, then, by the theorem, it consists of isolated points, and for any $x \in H\backslash C_K$ we have $P_K(x) = S(x; d_K(x))$. Thus, $S(x; d_K(x)) \subset K$ and $B(x; d_K(x)) \subset H\backslash K$. As $H\backslash K$ is connected, it follows that $H\backslash C_K$ consists of a single point x and $K = H\backslash B(x; d_K(x))$.

More general than Corollary 1 is

Corollary 2 (= Theorem 3 of [2]). *If K is a nonconvex subset of H, then $H\backslash C_K$ is uncountable unless K has the form*

$$K = A\backslash \bigcup_\gamma B_\gamma ,$$

where A is a closed convex body and (B_γ) is a countable family of nonoverlapping open balls contained in A.

Proof. Assume the set $H\backslash C_K$ is countable, and let its points be arranged in an indexed family (x_γ). Then the range of Φ_K is just the set

$$A := K \cup \bigcup_\gamma B(x_\gamma; d_K(x_\gamma)) ,$$

which is closed and thus convex as the closure of the range of a maximally monotone operator is necessarily a convex set. Hence, A satisfies the conditions stated in the corollary for the exceptional case.

If K is a Chebyshev set, then C_K is precisely the set of continuity points of P_K and its complement contains no isolated points. In this case our Theorem reads

Corollary 3. *Let K be a Chebyshev set and x a discontinuity of P_K. Then x lies on some nonconstant Lipschitz curve each point of which is a discontinuity of P_K.*

Finally, Corollary 3 implies Balaganskii's result cited at the beginning.

Corollary 4 (= Theorem 4 of [2]). *Let K be a Chebyshev set. If the set of discontinuities of P_K is countable, then K is convex.*

Let us take a brief glance at the situation in finite-dimensional spaces, in particular in the plane.

If H is the n-dimensional Euclidean space $\mathbf{R}^n$, then

$$C_K = \{x \in \mathbf{R}^n ; P_K(x) \text{ is a singleton}\}$$

is just the uniqueness set with respect to the best approximation from K. It is known that its complement which is denoted by N_K, has Lebesgue measure 0 in $\mathbf{R}^n$. Moreover, N_K is the union of a countable family of compact subsets of finite, $(n-1)$-dimensional Hausdorff measure in $\mathbf{R}^n$. For references see the discussion of these results in [3]. Starting from work of Pauc, Bartke and Berens [3] gave a comprehensive description of the set N_K in the Euclidean plane. Their investigations are based on the initial value problem

$$v'(t) \in v(t) - \Phi_K(v(t)), \quad v(0) = x$$

which has a unique solution $v\colon [0,\infty) \to H$, by the theory of monotone operators [7] and its extension to operators of type $\mathfrak{M}(\omega)$ in the sense of Pazy [15].

Among other things, Bartke and Berens showed that in $\mathbf{R}^2$, N_K is a countable union of rectifiable curves, at least if K is compact.

Of course, we can also obtain and slightly generalize their result by parametrizing the curves in question by means of the operator J. Indeed, if $x \in N_K$ is not an isolated point, then the vectors y and u in the proof of our Theorem can be taken such that u or $-u$ is a canonical unit vector, and such that $w := x + y + \lambda_1 u$ has rational coordinates with respect to the canonical basis of $\mathbf{R}^2$ for some λ_1 satisfying $0 < \lambda_1 < \frac{1}{2} d_K(y)$. Hence,

$$[0, \lambda_1] \ni \lambda \to J(w + \lambda(-u))$$

is a nonconstant Lipschitz curve in N_K with final point x. As the set of such pairs (w, u) is countable, we obtain

Proposition. *If K is a nonempty closed subset of $\mathbf{R}^2$, then N_K is a countable union of isolated points and nonconstant Lipschitz curves.*

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To illustrate the theorem we consider the following simple

Example. Let H be the Hilbert space l^2 of real sequences $x = (x_i)_{i \in \mathbf{N}}$. Let

$$K := \{e^j ; j \in \mathbf{N}\}$$

be the set of canonical unit vectors $e^j = (e_i^j)_{i \in \mathbf{N}}$ for which $e_i^j = 0$ if $i \neq j$ and $e_j^j = 1$. Then for each $x \in l^2$

$$P_K(x) = \{e^j ; j \in \mathbf{N} \ni x_j = \sup_i x_i\}$$

and, as $\varphi_K(x) = \sup_i x_i - \frac{1}{2}$,

$$\Phi_K(x) = \begin{cases} \operatorname{co} P_K(x), & \text{if } \exists j \in \mathbf{N} \ni x_j > 0 \\ \overline{\operatorname{co}}(P_K(x) \cup \{0\}), & \text{if } \exists j \in \mathbf{N} \ni x_j = \sup_i x_i = 0 \\ \{0\}, & \text{if } \forall j \in \mathbf{N} x_j < 0. \end{cases}$$

Thus,

$$C_K = \{x \in l^2 ; \dot{\exists} j \in \mathbf{N} \ni x_j = \sup_i x_i > 0\}$$

and

$$l^2 \backslash C_K = \{x \in l^2 ; \exists j , k \in \mathbf{N} , j \neq k \ni x_j = x_k = \sup_i x_i > 0\}$$
$$\cup \{x \in l^2 ; \sup_i x_i = 0\}.$$

The metric projection P_K is not upper semi-continuous at an $x = (x_i)$ if and only if $\sup_i x_i = 0$ and $x_i < 0$ for infinitely many values of i. Putting

$$V_K := \{x \in l^2 ; P_K \text{ is not usc at } x\}$$

and

$$N_K := \{x \in l^2 ; P_K(x) \text{ is not a singleton}\} ,$$

let us mention that all three sets

$$N_K \backslash V_K , \quad V_K \backslash N_K , \quad N_K \cap V_K$$

are nonempty, their union being the complement of C_K in H.

Now let us consider the behavior of the curves

$$\lambda \to J(x + y + \lambda u)$$

in case $x = 0$. Then $P_K(0) = K$ and

$$\Phi_K(0) = \left\{ y \in l^2 ; \forall i \in \mathbf{N} \; y_i \geq 0 , \sum_{i=1}^{\infty} y_i \leq 1 \right\}.$$

The origin is a support point of $\Phi_K(0)$ not belonging to $P_K(0)$. If we choose a unit vector $u = (u_i)$ such that $u_i \leq 0$ for each $i \in \mathbf{N}$, then

$$[0 , \infty) \ni \lambda \to J(\lambda u) = \lambda u$$

gives a half-line completely contained in $l^2 \backslash C_K$.

On the other hand, if we take a boundary point y of $\Phi_K(0)$, $y \notin K$, such that a canonical unit vector, say e^j, is a suitable choice of u, then, for λ sufficiently large, the associated curve $\lambda \to J(y + \lambda e^j)$ runs in C_K with direction e^j. Indeed, we have to take $y = (y_i) \in \Phi_K(0)$ such that $\sum_{i=1}^{\infty} y_i = 1$ implying that $y + \lambda e^j \notin \Phi_K(0)$ for each $\lambda > 0$. Then

$$J(y + \lambda e^j) = y + (\lambda - 1) e^j \in C_K ,$$

if $\lambda > 1 - y_j + \sup_{i \neq j} y_i$.

To describe the situation for small values of λ let us assume, for convenience, that the sequence (y_i) is, in addition, monotonically decreasing. Let $u = e^1$. If $\lambda > 1 - y_1 + y_2$, then

$$J(y + \lambda e^1) = y + (\lambda - 1) e^1 ,$$

and if

$$1 - \sum_{i=1}^{k} y_i + k y_{k+1} < \lambda \leq 1 - \sum_{i=1}^{k-1} y_i + (k-1) y_k \quad (k \geq 2) ,$$

then

$$J(y+\lambda e^1) = y + \sum_{j=1}^{k} \left[\frac{1}{k} \left(\lambda - 1 + \sum_{i=1}^{k} y_i \right) - y_j \right] e^j$$

is not in C_K since the sequence of coordinates is monotonically decreasing and the first k of them are equal and positive $(k \geq 2)$. Hence,

$$[0, 1 - y_1 + y_2] \ni \lambda \to J(y + \lambda e^1)$$

is a curve in $l^2 \backslash C_K$ composed of a countable family of line segments which is infinite if and only if $y_i > 0$ for each $i \in \mathbb{N}$.

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