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# SURJECTIVE ISOMETRIES OF WEIGHTED BERGMAN SPACES

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ABSTRACT. Let $\Omega$ be a bounded, simply connected domain in $\mathbf{C}^n = R^{2n}$, let $F \in L^1(m_\Omega)$ be positive and continuous on $\Omega$, and let $B_F^p(\Omega) = L^p(Fdm) \cap H(\Omega)$ $(0 < p < \infty)$ denote the weighted Bergman space over $\Omega$. We characterize those automorphisms $\Phi$ of $\Omega$ such that the map $f \to g \cdot (f \circ \Phi)$ is a surjective isometry of $B_F^p(\Omega)$, including an explicit description of $|g|$.

## 1. INTRODUCTION

In [2] it was shown that a surjective (linear) isometry of the Bergman space $B^p(\Omega) = L^p(m) \cap H(\Omega)$ $(0 < p < \infty,\ p \neq 2)$ must have the form

$$f \to g \cdot (f \circ \Phi) \tag{1.1}$$

where $\Phi$ is an automorphism of the bounded Runge domain $\Omega$, and $g$ is related to $\Phi$ by (1.2).

Conversely, given an automorphism $\Phi$ of $\Omega$, if there is a function $g$ in $B^p(\Omega)$ related to $\Phi$ by

$$\int_\Omega (h \circ \Phi)|g|^p dm = \int_\Omega h dm \tag{1.2}$$

for every function $h$ continuous on $\overline{\Omega} = \Omega \cup \partial\Omega$, then (1.1) defines a surjective isometry of $B^p(\Omega)$. In (1.2) $dm$ denotes Lebesque measure.

If the domain $\Omega$ is sufficiently nice, e.g., a ball or polydisc, then for every automorphism $\Phi$ there is a corresponding function $g = g_\Phi$ in $B^p(\Omega)$ which satisfies (1.2). It was conjectured in [2] that this would always be the case if the group of automorphisms acted transitively on $\Omega$.

In this paper we shall characterize those automorphism $\Phi$ of a bounded, simply connected domain $\Omega$ which generate surjective isometries of the weighted Bergman space $B_F^p(\Omega) \equiv L^p(Fdm) \cap H(\Omega)$, where the weight function $F \in L^1(m_\Omega)$ is assumed only to be positive and continuous on $\Omega$. Moreover, when

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the corresponding function $g_\Phi$ exists, an explicit description of $|g_\Phi|$ is given. One of the consequences is to answer the above conjecture affirmatively (cf. Remark (ii) and Theorem (2)).

## 2. Our setting

Let $n$ be a fixed positive integer, let $\Omega$ be a bounded domain in $\mathbf{C}^n = R^{2n}$, Let $H(\Omega)$ denote the class of holomorphic functions on $\Omega$, let $\mathrm{Aut}(\Omega)$ denote the group of automorphisms (i.e., biholomorphic maps) of $\Omega$, and let $m_\Omega$ denote Lebesgue measure on $\Omega$ normalized so that $m_\Omega(\Omega) = 1$. The closure of $\Omega$ in $\mathbf{C}^n$ is denoted by $\overline{\Omega}$.

When referring to the space

$$B_F^p(\Omega) \equiv L^p(Fdm_\Omega) \cap H(\Omega) \ (0 < p < \infty) \tag{2.1}$$

we shall always assume that the weight function $F$ satisfies

$$F \in L^1(m_\Omega) \text{ is POSITIVE and CONTINUOUS on } \Omega, \tag{2.2}$$

and that this space is given the usual metric of $L^p(Fdm_\Omega)$.

The following short argument shows $B_F^p(\Omega)$ is a closed subspace of $L^p(Fdm_\Omega)$ $(0 < p < \infty)$.

Let $a \in \Omega$, choose $r > 0$ so that the closed ball $a + 2r\overline{B}$ lies in $\Omega$, and let $\delta > 0$ denote the minimum of $F$ on this closed ball.

For $f \in B_F^p(\Omega)$, and for $z$ in the closed ball $a + r\overline{B}$, the subharmonicity of $|f|^p$ gives

$$|f(z)|^p \le c \int_{z+r\overline{B}} |f|^p dm \le \frac{c}{\delta} \int_{z+r\overline{B}} |f|^p F dm \le \frac{c}{\delta} \int_\Omega |f|^p F dm \tag{2.3}$$

where $c$ is a constant independent of $f$.

If $\{f_j\}$ is a Cauchy sequence in $B_F^p(\Omega)$, then (2.3) implies that $\{f_j\}$ converges uniformly on compact subsets of $\Omega$. Hence, $B_F^p(\Omega)$ is closed in $L^p(Fdm_\Omega)$ $(0 < p < \infty)$.

## 3. Surjective isometries of $B_F^p(\Omega)$

We now turn our attention to determining when an automorphism generates a surjective isometry.

It is easily shown that, given $\Phi \in \mathrm{Aut}(\Omega)$, if there is a function $g$ in $B_F^p(\Omega)$ $(0 < p < \infty)$ related to $\Phi$ by

$$\int_\Omega (h \circ \Phi)|g|^p F dm = \int_\Omega hF dm \tag{3.1}$$

for every function $h$ continuous on $\overline{\Omega}$, then (1.1) defines a surjective isometry of $B_F^p(\Omega)$. In the cases considered previously [2, 3] there was always such a function $g$, but in the present setting this need not be true. Before proving this we shall require a lemma which gives a converse to (3.1).

**Lemma 1.** *Let* $p > 0$, *and let* $\Phi \in \mathrm{Aut}(\Omega)$. *If there is a function* $g$ *in* $B_F^p(\Omega)$ *such that the map* $f \to g \cdot (f \circ \Phi)$ *is a surjective isometry of* $B_F^p(\Omega)$, *then* (3.1) *holds.*

*Proof.* Let $X$ be the set of all functions continuous on $\overline{\Omega}$ which satisfy (3.1). By our hypothesis $X$ contains all functions for the form $|f|^p$, where $f$ is any analytic polynomial. It is clear from (3.1) that $X$ is a linear space which is closed under complex conjugation. We shall show $X$ is an algebra.

If $p = 2$, then $(f + h)(\bar{f} + \bar{h}) = |f + h|^2 \in X$, for every pair of analytic polynomials $f$ and $h$. It now follows easily that $f\bar{h} \in X$. Since $X$ is clearly closed under the supremum norm, an application of the Stone-Weierstrass theorem completes the proof of the case $p = 2$.

In a similar manner the case $p \neq 2$ follows from [5, Theorem 7.5.3] with $M =$ algebra of analytic polynomials, $Af = f \circ \Phi$, $d\mu_1 = F dm_\Omega$ and $d\mu_2 = |g|^p F dm_\Omega$.

Before we characterize the automorphisms which generate isometries, let us first observe that this class forms a subgroup of $\mathrm{Aut}(\Omega)$.

**Proposition 1.** *The automorphisms of* $\Omega$ *which generate isometries of* $B_F^p(\Omega)$ $(0 < p < \infty)$ *form a subgroup of* $\mathrm{Aut}(\Omega)$. *Moreover, if* $\Phi, \Psi \in \mathrm{Aut}(\Omega)$ *have* (*respectively*) *corresponding functions* $g, h \in B_F^p(\Omega)$, *then*

(a) $h \cdot (g \circ \Psi)$ *corresponds to* $\Phi \circ \Psi$ *and*

(b) $1/(g \circ \Phi^{-1})$ *corresponds to* $\Phi^{-1}$.

*Proof.* (a) is obvious. To see (b) define $U$ and $T$ on $B_F^p(\Omega)$ by

$$Tf = g \cdot (f \circ \Phi) \text{ and } Uf = [1/(g \circ \Phi^{-1})](f \circ \Phi^{-1}).$$

By our hypothesis, $T$ is a surjective isometry of $B_F^p(\Omega)$, so the range of $T$ is equal to $B_F^p(\Omega)$. It is easy to show that $UT(f) = f$, for all $f \in B_F^p(\Omega)$, and it follows that $U = T^{-1}$; i.e., $U$ is a surjective isometry of $B_F^p(\Omega)$. This proves (b), and the proposition follows.

We are now able to characterize those automorphisms which generate isometries.

**Theorem 1.** *Let* $\Omega$ *be a bounded, simply connected domain in* $\mathbf{C}^n$, *and let* $\Phi \in \mathrm{Aut}(\Omega)$. *Let* $p > 0$. *Then there is a function* $g$ *in* $B_F^p(\Omega)$ *such that the map* $f \to g \cdot (f \circ \Phi)$ *is a surjective isometry of* $B_F^p(\Omega)$ *if and only if*

$$\log(F \circ \Phi / F) \tag{3.2}$$

*is pluriharmonic in* $\Omega$. *Moreover, if* (3.2) *holds, then* $g$ *is unique* (*modulo a unimodular constant*), *is never zero in* $\Omega$, *and* $g$ *is related to* $\Phi$ *by*

$$|g|^p = (F \circ \Phi / F) \cdot |J\Phi|^2 \tag{3.3}$$

*where* $J\Phi$ *denotes the complex Jacobian of* $\Phi$.

*Proof.* Let $p > 0$, and let $\Phi \in \mathrm{Aut}(\Omega)$.

Let us first suppose there is a function $g$ in $B_F^p$ such that the map $f \to g \cdot (f \circ \Phi)$ defines a surjective isometry of $B_F^p(\Omega)$. Then, by Lemma 1, $\Phi$ and $g$ are related by (3.1). Thus, letting

$$G = |g|^p F/(J_R \Phi) \tag{3.4}$$

where $J_R\Phi$ denotes the (real) Jacobian of $\Phi$, we find for every continuous function $h$ on $\overline{\Omega}$,

$$\begin{aligned} \int_\Omega hFdm &= \int_\Omega (h \circ \Phi)|g|^p Fdm \\ &= \int_\Omega \{[h \cdot (G \circ \Phi^{-1})] \circ \Phi\}(J_R\Phi)dm \\ &= \int_\Omega h \cdot (G \circ \Phi^{-1})dm. \end{aligned} \tag{3.5}$$

Consequently, $F = G \circ \Phi^{-1}$; which with (3.4) shows

$$(F \circ \Phi)/F = |g|^p/(J_R\Phi). \tag{3.6}$$

Because our isometry is surjective, there is a function $f \in B_F^p(\Omega)$ such that $1 = g \cdot (f \circ \Phi)$. It follows that $g$ is never zero in $\Omega$; thus $|g|^p = |g^p|$, where we use the principal branch of the logarithm in defining roots. We recall [5]; there is a holomorphic function $J\Phi$, the complex Jacobian of $\Phi$, which is related to $J_R\Phi$ by $J_R\Phi = |J\Phi|^2$.

We may now write (3.6) in the form

$$(F \circ \Phi)/F = |g^p/(J\Phi)^2|, \tag{3.7}$$

which shows the function $(F \circ \Phi/F)$ is the modulus of a holomorphic function which is never zero in $\Omega$. Thus (3.2) holds.

Conversely, if we assume (3.2) holds, then [5] our assumption that $\Omega$ be simply connected implies there is a function $f$ in $H(\Omega)$ such that $\log(F \circ \Phi/F) = \mathrm{Re}(f)$; thus

$$F \circ \Phi/F = \exp[\mathrm{Re}(f)]. \tag{3.8}$$

If we define

$$g = (J\Phi)^{2/p} \exp(f/p) \tag{3.9}$$

then $g$ is a holomorphic function on $\Omega$ which (by (3.8)) satisfies (3.3). Moreover,

$$\int_\Omega |g|^p Fdm = \int_\Omega (F \circ \Phi/F)|J\Phi|^2 Fdm = \int_\Omega Fdm < \infty. \tag{3.10}$$

Hence, $g \in B_F^p(\Omega)$.

Finally an argument similar to (3.10) shows $g$ is related to $\Phi$ by (3.1); therefore, the map $f \to g \cdot (f \circ \Phi)$ is a surjective isometry of $B_F^p(\Omega)$. As noted above $g$ is never zero in $\Omega$, and (3.7) must hold. Clearly (3.7) gives (3.3), which then shows $g$ is unique, modulo a unimodular constant.

*Remarks.* (i) The "only if" part is true for any bounded domain.

(ii) If $F \equiv 1$, then (3.2) is satisfied with $f \equiv 0$. Thus every $\Phi \in \mathrm{Aut}(\Omega)$ generates a surjective isometry of $B_F^p(\Omega)$ $(0 < p < \infty)$. Moreover, (3.9) shows the corresponding function $g$ is given by $g = (J\Phi)^{2/p}$. This result is generalized in Theorem (2).

(iii) Note that (3.2) is independent of $p$. Thus, if an automorphism $\Phi$ generates an isometry for one $p > 0$, then it generates an isometry for every $p > 0$.

(iv) It is clear from (3.2) that it is the algebraic structure of the weight function $F$, not its behavior near the boundary of $\Omega$, that determines which automorphisms generate isometries. As a simple example let $\Omega$ be the unit disc in $\mathbf{C}$. If $F(z) = (1 - |z|^4)^{-x}$ $(0 < x < 1)$, or if $F(z) = \exp[-1/(1 - |z|^2)]$, then direct computation shows that (3.2) holds if and only if the automorphism is a rotation.

(v) Again let $\Omega$ be the unit disc in $\mathbf{C}$. If $F(z) = \exp(z^2\bar{z} + z\bar{z}^2)$, then (3.2) is satisfied only by the identity automorphism.

(vi) If the analytic polynomials are dense in $B_F^p(\Omega)$, then the proof of Theorem (2.1) in [2] can be modified to show that every surjective isometry of $B_F^p(\Omega)$ $(p \neq 2)$ is given by (1.1). Question: Is this true when the analytic polynomials are not dense in $B_F^p(\Omega)$?

## 4. Surjective isometries of $A^{p,r}(\Omega)$

The "classical" weighted Bergman spaces have the weight function $F$ expressed as a power of the Bergman kernel. To be more precise, let

$$F_r(z) = [B_\Omega(z, z)]^{-r} \qquad (z \in \Omega) \tag{4.1}$$

where $B_\Omega$ denotes the Bergman kernel for the bounded domain $\Omega$, and $r$ is greater than some negative constant $K(\Omega)$. The space $B_{F_r}^p(\Omega)$ is usually denoted as $A^{p,r}(\Omega)$ [1, 3, 4].

If the group $\mathrm{Aut}(\Omega)$ acts transitively on $\Omega$ (which is true, for example, when $\Omega$ is a bounded symmetric domain), then [6],

$$B_\Omega(\Phi(z), \Phi(w)) \cdot (J\Phi)(z) \cdot (\overline{J\Phi})(w) = B_\Omega(z, w) \tag{4.2}$$

for all $(z, w) \in \Omega \times \Omega$, and for every $\Phi \in \mathrm{Aut}(\Omega)$.

It now follows easily from (4.1) and (4.2) that

$$(F_r \circ \Phi)/F_r = |J\Phi|^{2r} \tag{4.3}$$

for every $\Phi \in \mathrm{Aut}(\Omega)$ which, together with Theorem (1), gives

**Theorem 2.** *If the group* $\mathrm{Aut}(\Omega)$ *acts transitively on* $\Omega$, *then every* $\Phi \in \mathrm{Aut}(\Omega)$ *generates a surjective isometry of* $A^{p,r}(\Omega)$ *for all* $p > 0$ *and all* $r > K(\Omega)$. *Moreover, the corresponding function* $g$ *is given by*

$$g = (J\Phi)^{(2r+2)/p} \tag{4.4}$$

*Remark.* (vii) If $\Omega$ is taken to be the unit ball in $\mathbf{C}^n$, and if $\Phi \in \mathrm{Aut}(\Omega)$, then [5],

$$(J\Phi)(z) = [(1 - |a|^2)/(1 - \langle z, a\rangle)^2]^{(n+1)/2}$$

where $\Phi(a) = 0$. Thus, Theorem 2 contains Theorem 4 of [3] as a special case (recall there $x = (n+1)r$).

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