SURJECTIVE ISOMETRIES
OF WEIGHTED BERGMAN SPACES

CLINTON J. KOLASKI

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Abstract. Let $\Omega$ be a bounded, simply connected domain in $C^n = R^{2n}$, let $F \in L^1(m_{\Omega})$ be positive and continuous on $\Omega$, and let $B^p_F(\Omega) = L^p(Fdm) \cap H(\Omega)$ ($0 < p < \infty$) denote the weighted Bergman space over $\Omega$. We characterize those automorphisms $\Phi$ of $\Omega$ such that the map $f \rightarrow g \cdot (f \circ \Phi)$ is a surjective isometry of $B^p_F(\Omega)$, including an explicit description of $|g|$.

1. INTRODUCTION

In [2] it was shown that a surjective (linear) isometry of the Bergman space $B^p(\Omega) = L^p(m) \cap H(\Omega)$ ($0 < p < \infty$, $p \neq 2$) must have the form

$$f \rightarrow g \cdot (f \circ \Phi)$$

where $\Phi$ is an automorphism of the bounded Runge domain $\Omega$, and $g$ is related to $\Phi$ by (1.2).

Conversely, given an automorphism $\Phi$ of $\Omega$, if there is a function $g$ in $B^p(\Omega)$ related to $\Phi$ by

$$\int_{\Omega} |(h \circ \Phi)g|^p dm = \int_{\Omega} h dm$$

for every function $h$ continuous on $\overline{\Omega} = \Omega \cup \partial \Omega$, then (1.1) defines a surjective isometry of $B^p(\Omega)$. In (1.2) $dm$ denotes Lebesgue measure.

If the domain $\Omega$ is sufficiently nice, e.g., a ball or polydisc, then for every automorphism $\Phi$ there is a corresponding function $g = g_{\Phi}$ in $B^p(\Omega)$ which satisfies (1.2). It was conjectured in [2] that this would always be the case if the group of automorphisms acted transitively on $\Omega$.

In this paper we shall characterize those automorphism $\Phi$ of a bounded, simply connected domain $\Omega$ which generate surjective isometries of the weighted Bergman space $B^p_F(\Omega) = L^p(Fdm) \cap H(\Omega)$, where the weight function $F \in L^1(m_{\Omega})$ is assumed only to be positive and continuous on $\Omega$. Moreover, when

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the corresponding function \( g_\Phi \) exists, an explicit description of \(|g_\Phi|\) is given. One of the consequences is to answer the above conjecture affirmatively (cf. Remark (ii) and Theorem (2)).

2. Our setting

Let \( n \) be a fixed positive integer, let \( \Omega \) be a bounded domain in \( \mathbb{C}^n = \mathbb{R}^{2n} \), Let \( H(\Omega) \) denote the class of holomorphic functions on \( \Omega \), let \( \text{Aut}(\Omega) \) denote the group of automorphisms (i.e., biholomorphic maps) of \( \Omega \), and let \( m_\Omega \) denote Lebesgue measure on \( \Omega \) normalized so that \( m_\Omega(\Omega) = 1 \). The closure of \( \Omega \) in \( \mathbb{C}^n \) is denoted by \( \overline{\Omega} \).

When referring to the space

\[
\mathbb{B}_F^p(\Omega) = L^p(F dm_\Omega) \cap H(\Omega) \quad (0 < p < \infty)
\]

we shall always assume that the weight function \( F \) satisfies

\[
F \in L^1(m_\Omega) \text{ is POSITIVE and CONTINUOUS on } \Omega,
\]

and that this space is given the usual metric of \( L^p(F dm_\Omega) \).

The following short argument shows \( \mathbb{B}_F^p(\Omega) \) is a closed subspace of \( L^p(F dm_\Omega) \) \( (0 < p < \infty) \).

Let \( a \in \Omega \), choose \( r > 0 \) so that the closed ball \( a + 2r \overline{B} \) lies in \( \Omega \), and let \( \delta > 0 \) denote the minimum of \( F \) on this closed ball.

For \( f \in \mathbb{B}_F^p(\Omega) \), and for \( z \) in the closed ball \( a + r \overline{B} \), the subharmonicity of \(|f|\) gives

\[
|f(z)|^p \leq c \int_{z+r \overline{B}} |f|^p dm \leq \frac{c}{\delta} \int_{z+r \overline{B}} |f|^p F dm \leq \frac{c}{\delta} \int_{\Omega} |f|^p F dm
\]

where \( c \) is a constant independent of \( f \).

If \( \{f_n\} \) is a Cauchy sequence in \( \mathbb{B}_F^p(\Omega) \), then (2.3) implies that \( \{f_n\} \) converges uniformly on compact subsets of \( \Omega \). Hence, \( \mathbb{B}_F^p(\Omega) \) is closed in \( L^p(F dm_\Omega) \) \( (0 < p < \infty) \).

3. Surjective isometries of \( \mathbb{B}_F^p(\Omega) \)

We now turn our attention to determining when an automorphism generates a surjective isometry.

It is easily shown that, given \( \Phi \in \text{Aut}(\Omega) \), if there is a function \( g \) in \( \mathbb{B}_F^p(\Omega) \) \( (0 < p < \infty) \) related to \( \Phi \) by

\[
\int_{\Omega} (h \circ \Phi)|g|^p F dm = \int_{\Omega} h F dm
\]

for every function \( h \) continuous on \( \overline{\Omega} \), then (1.1) defines a surjective isometry of \( \mathbb{B}_F^p(\Omega) \). In the cases considered previously [2, 3] there was always such a function \( g \), but in the present setting this need not be true. Before proving this we shall require a lemma which gives a converse to (3.1).
Lemma 1. Let $p > 0$, and let $\Phi \in \text{Aut}(\Omega)$. If there is a function $g$ in $B_p^p(\Omega)$ such that the map $f \mapsto g \cdot (f \circ \Phi)$ is a surjective isometry of $B_p^p(\Omega)$, then (3.1) holds.

Proof. Let $X$ be the set of all functions continuous on $\overline{\Omega}$ which satisfy (3.1). By our hypothesis $X$ contains all functions for the form $|f|^p$, where $f$ is any analytic polynomial. It is clear from (3.1) that $X$ is a linear space which is closed under complex conjugation. We shall show $X$ is an algebra.

If $p = 2$, then $(f + h)(\bar{f} + \bar{h}) = |f + h|^2 \in X$, for every pair of analytic polynomials $f$ and $h$. It now follows easily that $f\bar{h} \in X$. Since $X$ is clearly closed under the supremum norm, an application of the Stone-Weierstrass theorem completes the proof of the case $p = 2$.

In a similar manner the case $p \neq 2$ follows from [5, Theorem 7.5.3] with $M = \text{algebra of analytic polynomials}, A_f = f \circ \Phi, d\mu_1 = Fdm_{\Omega}$ and $d\mu_2 = |g|^p Fdm_{\Omega}$.

Before we characterize the automorphisms which generate isometries, let us first observe that this class forms a subgroup of $\text{Aut}(\Omega)$.

Proposition 1. The automorphisms of $\Omega$ which generate isometries of $B_p^p(\Omega)$ $(0 < p < \infty)$ form a subgroup of $\text{Aut}(\Omega)$. Moreover, if $\Phi, \Psi \in \text{Aut}(\Omega)$ have (respectively) corresponding functions $g, h \in B_p^p(\Omega)$, then

(a) $h \cdot (g \circ \Psi)$ corresponds to $\Phi \circ \Psi$ and

(b) $1/(g \circ \Phi^{-1})$ corresponds to $\Phi^{-1}$.

Proof. (a) is obvious. To see (b) define $U$ and $T$ on $B_p^p(\Omega)$ by

$$Tf = g \cdot (f \circ \Phi) \text{ and } Uf = [1/(g \circ \Phi^{-1})](f \circ \Phi^{-1}).$$

By our hypothesis, $T$ is a surjective isometry of $B_p^p(\Omega)$, so the range of $T$ is equal to $B_p^p(\Omega)$. It is easy to show that $UT(f) = f$, for all $f \in B_p^p(\Omega)$, and it follows that $U = T^{-1}$; i.e., $U$ is a surjective isometry of $B_p^p(\Omega)$. This proves (b), and the proposition follows.

We are now able to characterize those automorphisms which generate isometries.

Theorem 1. Let $\Omega$ be a bounded, simply connected domain in $\mathbb{C}^n$, and let $\Phi \in \text{Aut}(\Omega)$. Let $p > 0$. Then there is a function $g$ in $B_p^p(\Omega)$ such that the map $f \mapsto g \cdot (f \circ \Phi)$ is a surjective isometry of $B_p^p(\Omega)$ if and only if

$$(3.2) \quad \log(F \circ \Phi/F)$$

is pluriharmonic in $\Omega$. Moreover, if (3.2) holds, then $g$ is unique (modulo a unimodular constant), is never zero in $\Omega$, and $g$ is related to $\Phi$ by

$$(3.3) \quad |g|^p = (F \circ \Phi/F) \cdot |J\Phi|^2$$

where $J\Phi$ denotes the complex Jacobian of $\Phi$.

Proof. Let $p > 0$, and let $\Phi \in \text{Aut}(\Omega)$.
Let us first suppose there is a function $g$ in $B_F^p$ such that the map $f \rightarrow g \cdot (f \circ \Phi)$ defines a surjective isometry of $B_F^p(\Omega)$. Then, by Lemma 1, $\Phi$ and $g$ are related by (3.1). Thus, letting

$$G = |g|^p F/(J_R \Phi)$$

where $J_R \Phi$ denotes the (real) Jacobian of $\Phi$, we find for every continuous function $h$ on $\overline{\Omega}$,

$$\int_{\Omega} h F dm = \int_{\Omega} (h \circ \Phi)|g|^p F dm$$

$$= \int_{\Omega} [h \cdot (G \circ \Phi^{-1})] \circ \Phi \cdot (J_R \Phi) dm$$

$$= \int_{\Omega} h \cdot (G \circ \Phi^{-1}) dm.$$

Consequently, $F = G \circ \Phi^{-1}$; which with (3.4) shows

$$|F \circ \Phi|/F = |g|^p/(J_R \Phi).$$

Because our isometry is surjective, there is a function $f \in B_F^p(\Omega)$ such that $1 = g \cdot (f \circ \Phi)$. It follows that $g$ is never zero in $\Omega$; thus $|g|^p = |g^p|$, where we use the principal branch of the logarithm in defining roots. We recall [5]; there is a holomorphic function $J \Phi$, the complex Jacobian of $\Phi$, which is related to $J_R \Phi$ by $J_R \Phi = |J\Phi|^2$.

We may now write (3.6) in the form

$$|F \circ \Phi|^2/F = |g|^p/(J\Phi)^2,$$

which shows the function $(F \circ \Phi/F)$ is the modulus of a holomorphic function which is never zero in $\Omega$. Thus (3.2) holds.

Conversely, if we assume (3.2) holds, then [5] our assumption that $\Omega$ be simply connected implies there is a function $f$ in $H(\Omega)$ such that $\log(F \circ \Phi/F) = \text{Re}(f)$; thus

$$F \circ \Phi/F = \exp[\text{Re}(f)].$$

If we define

$$g = (J\Phi)^{2/p} \exp(f/p)$$

then $g$ is a holomorphic function on $\Omega$ which (by (3.8)) satisfies (3.3). Moreover,

$$\int_{\Omega} |g|^p F dm = \int_{\Omega} (F \circ \Phi/F) |J\Phi|^2 F dm = \int_{\Omega} F dm < \infty.$$

Hence, $g \in B_F^p(\Omega)$.

Finally an argument similar to (3.10) shows $g$ is related to $\Phi$ by (3.1); therefore, the map $f \rightarrow g \cdot (f \circ \Phi)$ is a surjective isometry of $B_F^p(\Omega)$. As noted above $g$ is never zero in $\Omega$, and (3.7) must hold. Clearly (3.7) gives (3.3), which then shows $g$ is unique, modulo a unimodular constant.
Remarks. (i) The "only if" part is true for any bounded domain.

(ii) If $F \equiv 1$, then (3.2) is satisfied with $f \equiv 0$. Thus every $\Phi \in \text{Aut}(\Omega)$ generates a surjective isometry of $B_F^p(\Omega)$ ($0 < p < \infty$). Moreover, (3.9) shows the corresponding function $g$ is given by $g = (J\Phi)^{2/p}$. This result is generalized in Theorem (2).

(iii) Note that (3.2) is independent of $p$. Thus, if an automorphism $\Phi$ generates an isometry for one $p > 0$, then it generates an isometry for every $p > 0$.

(iv) It is clear from (3.2) that it is the algebraic structure of the weight function $F$, not its behavior near the boundary of $\Omega$, that determines which automorphisms generate isometries. As a simple example let $\Omega$ be the unit disc in $\mathbb{C}$. If $F(z) = (1 - |z|^4)^{-x}$ ($0 < x < 1$), or if $F(z) = \exp[-1/(1 - |z|^2)]$, then direct computation shows that (3.2) holds if and only if the automorphism is a rotation.

(v) Again let $\Omega$ be the unit disc in $\mathbb{C}$. If $F(z) = \exp(z^2 \bar{z} + \bar{z}^2)$, then (3.2) is satisfied only by the identity automorphism.

(vi) If the analytic polynomials are dense in $B_F^p(\Omega)$, then the proof of Theorem (2.1) in [2] can be modified to show that every surjective isometry of $B_F^p(\Omega)$ ($p \neq 2$) is given by (1.1). Question: Is this true when the analytic polynomials are not dense in $B_F^p(\Omega)$?

4. Surjective Isometries of $A^{p,r}(\Omega)$

The "classical" weighted Bergman spaces have the weight function $F$ expressed as a power of the Bergman kernel. To be more precise, let

$$F_r(z) = [B_\Omega(z, z)]^{-r} \quad (z \in \Omega)$$

where $B_\Omega$ denotes the Bergman kernel for the bounded domain $\Omega$, and $r$ is greater than some negative constant $K(\Omega)$. The space $B_F^p(\Omega)$ is usually denoted as $A^{p,r}(\Omega)$ [1, 3, 4].

If the group $\text{Aut}(\Omega)$ acts transitively on $\Omega$ (which is true, for example, when $\Omega$ is a bounded symmetric domain), then [6],

$$B_\Omega(\Phi(z), \Phi(w)) \cdot (J\Phi)(z) \cdot (J\Phi)(w) = B_\Omega(z, w)$$

for all $(z, w) \in \Omega \times \Omega$, and for every $\Phi \in \text{Aut}(\Omega)$.

It now follows easily from (4.1) and (4.2) that

$$F_r \circ \Phi = [J\Phi]^{2r}$$

for every $\Phi \in \text{Aut}(\Omega)$ which, together with Theorem (1), gives

Theorem 2. If the group $\text{Aut}(\Omega)$ acts transitively on $\Omega$, then every $\Phi \in \text{Aut}(\Omega)$ generates a surjective isometry of $A^{p,r}(\Omega)$ for all $p > 0$ and all $r > K(\Omega)$. Moreover, the corresponding function $g$ is given by

$$g = (J\Phi)^{(2r+2)/p}$$
Remark. (vii) If $\Omega$ is taken to be the unit ball in $\mathbb{C}^n$, and if $\Phi \in \text{Aut}(\Omega)$, then [5],

$$(J\Phi)(z) = \frac{(1 - |a|^2)/(1 - \langle z, a \rangle)^2}{(n+1)/2}$$

where $\Phi(a) = 0$. Thus, Theorem 2 contains Theorem 4 of [3] as a special case (recall there $x = (n + 1)r$).

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MINNESOTA-DULUTH, DULUTH, MINNESOTA 55812