HYPO-ANALYTIC PSEUDODIFFERENTIAL OPERATORS

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Abstract. Let $\Omega$ be a hypo-analytic manifold of dimension $m$ equipped with a hypo-analytic structure whose structure bundle $T'$ has dimension $m$. This paper introduces hypo-analytic pseudodifferential operators and it is shown that such operators preserve the hypo-analyticity of a distribution.

1. Introduction

The main concepts relating to hypo-analyticity were introduced by Baouendi, Chang, and Treves in [1], Chapter 1. We shall summarize some of these in this section.

Suppose $\Omega$ is a $C^\infty$ manifold of dimension $m+n$. A hypo-analytic structure on $\Omega$ is the data of an open covering $(U_\alpha)$ of $\Omega$ and for index $\alpha$, of $m$ $C^\infty$ functions $Z^1_\alpha, \ldots, Z^m_\alpha$ satisfying the following two conditions:

(i) $dZ^1_\alpha, \ldots, dZ^m_\alpha$ are linearly independent at each point of $U_\alpha$;
(ii) if $U_\alpha \cap U_\beta \neq \emptyset$, there are open neighborhoods $\mathcal{O}_\alpha$ of $Z_\alpha(U_\alpha \cup U_\beta)$ and $\mathcal{O}_\beta$ of $Z_\beta(U_\alpha \cap U_\beta)$ and a holomorphic map $F^\alpha_\beta$ of $\mathcal{O}_\alpha$ onto $\mathcal{O}_\beta$, such that

$$Z_\beta = F^\alpha_\beta \circ Z_\alpha \quad \text{on} \quad U_\alpha \cap U_\beta.$$

When the $Z^I$ are real-valued and $n = 0$, such a structure specializes to a real analytic structure. A distribution $h$ defined in an open neighborhood of a point $p_0$ of $\Omega$ is hypo-analytic at $p_0$ if there is a hypo-analytic local chart $(U_\alpha, Z_\alpha)$ whose domain contains $p_0$ and a holomorphic function $\tilde{h}_\alpha$ defined on an open neighborhood of $Z_\alpha(p_0)$ in $C^m$ such that $h = \tilde{h}_\alpha \circ Z$ in a neighborhood of $p_0$.

By a hypo-analytic local chart we mean an $(m+1)$-tuple $(U, Z^1, \ldots, Z^m)$ [abbreviated $(U, Z)$] consisting of an open subset $U$ of $\Omega$ and of $m$ hypo-analytic functions $Z^1, \ldots, Z^m$ whose differentials are linearly independent at every point of $U$.

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In general, the mapping \( Z = (Z^1, \ldots, Z^m): U \to \mathbb{C}^m \) is not a local embedding. However, when \( \dim \Omega = m \), this mapping is a local diffeomorphism. Throughout this paper we will assume that the dimension of \( \Omega \) is \( m \).

2. Preliminaries

We will reason in a hypo-analytic local chart \((U, Z)\) in \( \Omega \). We shall assume that the open set \( U \) has been contracted sufficiently so that the mapping \( Z = (Z^1, \ldots, Z^m): U \to \mathbb{C}^m \) is a diffeomorphism of \( U \) onto \( Z(U) \) and that \( U \) is the domain of local coordinates \( x_j \) \((1 \leq j \leq m)\) all vanishing at a "central point" which will be denoted by \( 0 \). We will suppose \( Z(0) = 0 \) and denote by \( Z_x \) the Jacobian matrix of the \( Z^j \) with respect to the \( x^k \). Substitution of \( Z_x(0)^{-1}Z(x) \) for \( Z(x) \) will allow us to assume that \( Z_x(0) = \) the identity matrix. This will permit us to take the real part of the \( Z^j \) \((j = 1, \ldots, m)\) as coordinates and write in these new coordinates

\[
Z^j = x^j + \sqrt{1} \Phi^j(x), \quad j = 1, \ldots, m,
\]

where \( \Phi = (\Phi^1, \ldots, \Phi^m) \) is real-valued whose differential at the origin is 0. Moreover, the functions \( Z^j \) are selected so that all the derivatives of order two of the \( \Phi^j \) vanish at the origin. Indeed if this is not already so it suffices to replace each \( Z^j \) by

\[
Z^j - \sqrt{-1}/2 \sum \sum \frac{\partial^2 \Phi^j}{\partial x^k \partial x^l}(0)Z^kZ^l.
\]

We will use \( Z^*_x \) to denote the transpose of the inverse of the matrix \( Z_x \).

Since the first and second derivatives of all the \( \Phi^j \) are zero at the origin, after contracting \( U \) if necessary, we can find a number \( c, 0 < c < 1 \) such that for all \( x, y \in U \) and for all \( \xi \in \mathbb{R}^m \),

\[
|\text{Im} Z^*_x(x)\xi| \leq c|\text{Re} Z^*_x(x)\xi|
\]

and

\[
(2.1) \quad \text{Re} \left\{ \sqrt{-1}Z^*_x(x)\xi \cdot (Z(x) - Z(y)) - (Z^*_x(x)\xi) \cdot (Z(x) - Z(y)) \right\} \leq -c|\xi| |Z(x) - Z(y)|^2,
\]

where \( \langle \zeta \rangle = (\zeta_1^2 + \cdots + \zeta_m^2)^{1/2} \).

3. HYPO-ANALYTIC PSEUDODIFFERENTIAL OPERATORS

Our aim is to introduce pseudodifferential operators that are naturally associated with hypo-analytic structures. This definition generalizes analytic pseudodifferential operators (for a treatment of the analytic theory see [6]).

Definition 3.1. Let \( d \) be a real number. We denote by \( \mathcal{S}^d(U, U) \) the space of holomorphic functions \( \tilde{a}(z, w, \theta) \) in a product set \( \mathcal{E} \times \mathcal{E} \times \mathcal{E} \) with \( \mathcal{E} \) an open neighborhood of \( Z(U) \) and \( \mathcal{E} \) an open cone in \( C_m \setminus \{0\} \) containing \( R_m \setminus \{0\} \), which have the following property:
Given any compact subset $K$ of $\mathcal{C}$ and any closed cone $\mathcal{C}' \subset \mathcal{C}$ whose interior contains $R_m \setminus \{0\}$, there is a constant $r > 0$ such that for all $z, w$ in $K$ and all $\theta$ in $\mathcal{C}'$, we have:

$$|\tilde{a}(z, w, \theta)| \leq r(1 + |\theta|)^d.$$  

**Definition 3.2.** We say that a $C^\infty$ function $a(x, y, \theta)$ in $U \times U \times R_m$ is a hypo-analytic amplitude of degree $d$ and we write $a \in S^d(U, U)$ if there is $\tilde{a} \in S^d(U, U)$ such that

$$a(x, y, \theta) = \tilde{a}(Z(x), Z(y), \theta), \quad \text{for all } x \in U, \ y \in U, \ 0 \not= \theta \in R_m.$$  

Let $a(x, y, \theta) = \tilde{a}(Z(x), Z(y), \theta)$ be a hypo-analytic amplitude of degree $d \in \mathbb{R}$ in $U \times U$. For any $\varepsilon > 0$ and $u \in C_0^0(U)$ we define the linear operator

$$A^\varepsilon u(x) = \left( \frac{1}{4\pi^3} \right)^{m/2} \int_U \int_{R_m} \exp(\sqrt{-1} \xi \cdot (Z(x) - Z(y)) - \varepsilon |\xi|^2) a(x, y, \xi) u(y) dZ(y) d\xi.$$  

We contract $U$ sufficiently so that for every $x, y \in U$ and $\xi \in R_m$ the point $Z^*_x(x)\xi + \sqrt{-1}(Z^*_x(x)\xi)(Z(x) - Z(y))$ will remain in the cone in which $a(x, y, \cdot)$ is defined. We observe that each $A^\varepsilon u$ is a hypo-analytic function.

**Theorem 3.1.** When $\varepsilon \to 0$, $A^\varepsilon$ converges to a continuous linear operator $A: E'(U) \to D'(U)$ which maps $C_c^\infty(U)$ into $C^\infty(U)$ continuously.

**Proof.** We deform the path of $\xi$-integration from $R_m$ to the image of $R_m$ under the map

$$\xi \rightarrow \zeta(\xi) = Z^*_x(x)\xi + \sqrt{-1}(Z^*_x(x)\xi)(Z(x) - Z(y)).$$

Thus

$$A^\varepsilon u(x) = \left( \frac{1}{4\pi^3} \right)^{m/2} \int_U \int_{R_m} \exp(\sqrt{-1} Z^*_x(x)\xi \cdot (Z(x) - Z(y)))$$

$$- (Z^*_x(x)\xi)(Z(x) - Z(y))^2 - \varepsilon(\zeta(\xi))^2)$$

$$\times a(x, y, \zeta(\xi)) u(y) \det \left( \frac{\partial \xi}{\partial \zeta} \right) dZ(y) d\xi.$$  

If the amplitude $a$ has degree $d < -m - 1$ and $u \in C_c(U)$, condition (2.1) will imply that $A^\varepsilon u$ converges uniformly on compact subsets of $U$ to a continuous function $Au$. Moreover, in this case, $A: C_c^0(U) \to C^0(U)$ will be a continuous operator.

In general, if the degree of $a = d$, we consider the holomorphic functions

$$A^\varepsilon u(z) = \left( \frac{1}{4\pi^3} \right)^{m/2} \int_U \int_{R_m} \exp(\sqrt{-1} \xi \cdot (z - Z(y)) - \varepsilon |\xi|^2) \tilde{a}(z, Z(y), \xi) u(y) dZ(y) d\xi.$$
We denote the Laplacian \( \sum_{j=1}^{m} D_{z_j}^2 \) by \( \Delta_z \) and write

\[
A^S u(z) = \left( \frac{1}{4\pi^3} \right)^{m/2} \int_U \int_{R_m} (1 - \Delta_z)^k \left\{ e^{\sqrt{-1} \xi(\bar{z} - Z(y)) - \bar{\varepsilon}|\xi|^2} \right\} \frac{\bar{\partial}}{(1 + |\xi|^2)^k} u(y) \, dZ(y) \, d\xi.
\]

In the latter, we apply the transposed Leibniz formula to get

\[
\left\{ (1 - \Delta_z)^k e^{\sqrt{-1} \xi(\bar{z} - \omega)} \right\} \bar{a}(z, \omega, \xi) = \sum_{|\alpha + \beta| \leq 2k} c_{\alpha, \beta} \left( \frac{\partial}{\partial z} \right)^\alpha \left( \frac{\partial}{\partial \xi} \right)^\beta \bar{a}(z, \omega, \xi),
\]

where the \( c_{\alpha, \beta} \) are integers.

We can thus write \( A^S u(x) = \sum_{|\alpha| \leq 2k} M^\alpha(A^S u(x)) \), where the \( A^\alpha \) are defined like \( A^S \) except that their amplitudes have degree \( \leq d - 2k \). For \( k \) sufficiently large we have shown that \( A^S u \) converges to a continuous function. Therefore, \( A^S u \) converges to \( Au \) in the space \( \mathcal{D}'(U) \). Suppose now \( u \in \mathcal{D}'(U) \). We choose continuous functions \( u_\alpha \in C^\alpha_c(U) \) that satisfy \( u = \sum_{|\alpha| \leq k} M^\alpha(u_\alpha) \). We may integrate by parts to get, for each \( \alpha \), \( A^\alpha(M^\alpha u_\alpha) = A^\alpha(u_\alpha) \), where the degree of the amplitude of \( A^\alpha \) is \( \leq |\alpha| + d \). Thus \( A^S u = \sum_{|\alpha| \leq k} M^\alpha(u_\alpha) \) which brings us to a situation already considered. We conclude that \( A^S u \rightarrow Au \) in \( \mathcal{D}'(U) \).

Suppose now \( u \in C^\infty_c(U) \). We denote \( \sum_{j=1}^{m} M_j^2 \) by \( M_\delta \), where the \( M_j \) are vector fields satisfying \( M_j Z^k = \delta_j \). Integration by parts gives

\[
(3.2) \quad A^S u(x) = \left( \frac{1}{4\pi^3} \right)^{m/2} \int_U \int_{R_m} \exp(\sqrt{-1} \xi \cdot (Z(x) - Z(y)) - \varepsilon|\xi|^2) \times \frac{(1 - \Delta_M)^k \{a(x, y, \xi)u(y)\}}{(1 + |\xi|^2)^k} \, dZ(y) \, d\xi.
\]

After deforming contour as in (3.1), we see that \( A^S u \) converges in \( C^\alpha_c(U) \) to the continuous function \( Au \). Moreover, the same convergence also occurs for \( M_\alpha(A^S u) \) for all \( \alpha \). It follows that \( Au \in C^\infty(U) \). Finally, we will show that the operator \( A: C^\infty_c(U) \rightarrow C^\infty(U) \) is continuous. Let \( u \in C^\infty_c(U) \). We can write (3.2) as

\[
(3.3) \quad Au(x) = \left( \frac{1}{4\pi^3} \right)^{m/2} \int_U \int_{R_m} \exp(\sqrt{-1} Z(x) \xi \cdot (Z(x) - Z(y)) - \langle Z^\ast(x) \xi \rangle (Z(x) - Z(y))^2) \times \frac{(1 - \Delta_M)^k \{a(x, y, \xi)u(y)\}}{(1 + |\xi|^2)^k} \, \det \frac{\partial^\xi}{\partial \xi} \, dZ(y) \, d\xi.
\]

We note that both the exponential term and \( \det \frac{\partial^\xi}{\partial \xi} \) are bounded. Suppose now the sequence \( u_n \rightarrow u \) in \( C^\infty_c(U) \). Then (3.3) shows that \( Au_n \rightarrow Au \) in
To conclude the proof, it suffices to show that for every multi-index $\alpha$, the sequence $M^{\alpha}(Au_n)$ is uniformly convergent on compact subsets of $U$.

For each $\alpha$, there is an amplitude $b^{\alpha}$ of degree $\leq d + |\alpha|$ such that

\[
M^{\alpha}(A^\varepsilon u)(x) = \left(\frac{1}{4\pi^3}\right)^{m/2} \int_{R_m} \int_U \exp\left(\sqrt{-1}\xi \cdot (Z(x) - Z(y)) - \varepsilon|\xi|^2\right)b^{\alpha}(x, y, \xi)u(y)\,dZ(y)\,d\xi.
\]

By what we have already seen, the right-hand side converges in the space $C^0_c(U)$.

**Definition 3.3.** The operator $A: \mathcal{E}'(U) \to \mathcal{E}'(U)$ of Theorem 3.1 will be called a hypo-analytic pseudodifferential operator.

**Example.** A hypo-analytic differential operator on $\Omega$ may be defined as a linear differential operator $P$ on $\Omega$ satisfying the following property:

For every open subset $\Omega'$ of $\Omega$ and every hypo-analytic function $f$ on $\Omega'$, $Pf$ is hypo-analytic on $\Omega'$. In the hypo-analytic chart $(U, Z)$, let $M_{j}$ $(1 \leq j \leq m)$ be the vector fields satisfying

\[
M_{j}Z^k = \delta^k_j.
\]

Then a hypo-analytic differential operator $P$ takes the form

\[
P = \sum_{|\alpha| \leq k} a_{\alpha}M^{\alpha},
\]

where each $a_{\alpha}$ is a hypo-analytic function on $U$. We will show that such an operator is an example of a hypo-analytic pseudodifferential operator. For $u \in E'(U)$ and $\varepsilon > 0$ let

\[
u^\varepsilon(x) = \left(\frac{1}{4\pi^3}\right)^{m/2} \int_{R_m} \int_U \exp\left(\sqrt{-1}\xi \cdot (Z(x) - Z(y)) - \varepsilon|\xi|^2\right)u(y)\,dZ(y)\,d\xi.
\]

In [1], the authors observed that $u^\varepsilon \to u$ in the space $\mathcal{D}'(U)$. Write $P$ as $\sum_{|\alpha| \leq k} b_{\alpha}N^{\alpha}$ where each $b_{\alpha}$ is hypo-analytic and $N_{j} = -\sqrt{-1}M_{j}$ for each $j$. We then have

\[
N_{j}u^\varepsilon(x) = \left(\frac{1}{4\pi^3}\right)^{m/2} \int_{R_m} \int_U \exp\left(\sqrt{-1}\xi \cdot (Z(x) - Z(y)) - \varepsilon|\xi|^2\right)\xi_{j}u(y)\,dZ\,d\xi
\]

for each $j$ and therefore

\[
Pu^\varepsilon(x) = \left(\frac{1}{4\pi^3}\right)^{m/2} \int_{R_m} \int_U \exp\left(\sqrt{-1}\xi \cdot (Z(x) - Z(y)) - \varepsilon|\xi|^2\right)\left(\sum_{|\alpha| \leq k} b_{\alpha}(x)\xi^{\alpha}\right)u(y)\,dZ\,d\xi.
\]
When \( \varepsilon \to 0 \), we get \( Pu = Au \), where \( A \) is the hypo-analytic pseudodifferential operator whose amplitude is

\[
\sum_{|\alpha| \leq k} b_{\alpha}(x) \xi^{\alpha}.
\]

4. Pseudolocal Property

The aim of this section is to show that hypo-analytic pseudodifferential operators map hypo-analytic functions to hypo-analytic functions.

Since the first and second derivatives of \( \Phi \) vanish at the origin, after shrinking \( U \) if necessary, we may assume that for all \( x, y \) in \( U \),

\[
|\Phi(x) - \Phi(y)| \leq |x - y|/2
\]

and

\[
(4.1) \quad |\Phi(y)| \leq 1/2|y|^2.
\]

We shall need the following lemma.

**Lemma 4.1.** Let \( A \) be a hypo-analytic pseudodifferential operator with amplitude \( a(x, y, \xi) = \tilde{a}(Z(x), Z(y), \xi) \) and let \( u \) be in \( E' \). If \( u \) vanishes in some neighborhood of 0, \( Au \) is hypo-analytic at 0.

**Proof.** For each \( \varepsilon > 0 \) we consider the holomorphic function

\[
\tilde{A}^\varepsilon u(z) = \left( \frac{1}{4\pi^3} \right)^{m/2} \int_U \int_{R_m} \exp(-i\xi \cdot (z - Z(y))) - \varepsilon |\xi|^2 \tilde{a}(z, Z(y), \xi) u(y) dZ(y) d\xi.
\]

We deform the path of \( \xi \)-integration from \( R_m \) to the image of \( R_m \) under the map \( \zeta(\xi) = \xi + \sqrt{-1}|\xi|((z - Z(y)) \) and write

\[
\tilde{A}^\varepsilon u(z) = \left( \frac{1}{4\pi^3} \right)^{m/2} \int_U \int_{R_m} \exp(-i\xi \cdot (z - Z(y))) - |\xi|(z - Z(y))^2 - \varepsilon \zeta(\xi))
\]

\[
\times \tilde{a}(z, Z(y), \zeta(\xi)) u(y) \frac{\partial \zeta}{\partial \xi} dZ(y) d\xi.
\]

Let \( Q(z) = \text{Re} \left\{ \sqrt{-1} \xi \cdot (z - Z(y)) - |\xi|((z - Z(y))^2) \right\} \). Using (4.1) we have

\[
Q(0) = \xi \cdot \phi(y) - |\xi|(|y|^2 - |\phi(y)|^2) \leq -\frac{1}{4}|y|^2|\xi|.
\]

Let \( d \) be a positive number such that \( y \in \text{supp} \ u \Rightarrow |y| > d \). Then \( Q(0) \leq -\frac{1}{4}d^2|\xi| \) which by continuity implies that \( Q(z) \leq -\frac{1}{4}d^2|\xi| \) for \( z \) in a sufficiently small neighborhood of 0. Therefore, as \( \varepsilon \to 0 \), \( \tilde{A}^\varepsilon u(z) \) converges uniformly on some neighborhood of 0. It follows that \( Au(x) = \tilde{A}u(Z(x)) \) is hypo-analytic at 0.
Theorem 4.2. Suppose $A$ is a hypo-analytic pseudodifferential operator and $u \in \mathcal{E}'(U)$. If $u$ is hypo-analytic at $0$ then $Au$ is hypo-analytic at $0$.

Proof. Let $\tilde{u}$ be a holomorphic function such that $u(y) = \tilde{u}(Z(y))$ for $y$ near $0$. In the integral for $\tilde{A}^* u(z)$ we deform the "y-contour" from $U$ to the image of $U$ under the map $Z(y) \rightarrow \tilde{Z}(y) = Z(y) - \sqrt{-1} \chi(y) d\xi/|\xi|$, where $d$ is a sufficiently small positive number, $\chi \in C^\infty_c(U)$, $0 \leq \chi \leq 1$, $\chi \equiv 1$ near $0$ and $\text{supp} \chi$ sufficiently small. We may thus write

$$\tilde{A}^* u(z) = \left( \frac{1}{4\pi^2} \right)^{m/2} \int_{R_m^*} \int_U \exp(\sqrt{-1}\xi(z - Z(y))) - d\chi(y)|\xi| - \epsilon|\xi|^2\right)$$

$$\times a(z, \tilde{Z}(y), \xi) \tilde{u}(\tilde{Z}(y)) d\tilde{Z}(y) d\xi.$$

We next deform the $\xi$-integration to the image of $R_m$ under the map $\xi \rightarrow \xi(z) = \tilde{\xi} + \sqrt{-1}|\xi|(z - Z(y))$. We will show that $\tilde{A}^* u(z)$ converges uniformly near $z = 0$. To prove this, we will estimate

$$Q(z) = \text{Re}\{\sqrt{-1}\xi(z - Z(y)) - |\xi|(z - Z(y))^2 - d\chi(y)(\xi(\xi))\}.$$

Lemma 4.1 allows us to shrink the support of $u$ so that when $y \in \text{supp} u$ and $z$ is small enough, $|\xi|/2 \leq \text{Re}(\xi(\xi))$. Moreover, for such $z$ and $y$ we have: $Re\{\sqrt{-1}\xi(z - Z(y)) - |\xi|(z - Z(y))^2\} \leq (|y|^2/8 + 3|z|)|\xi|$. Therefore for $z$ near $0$, $Q(z) \leq -(|y|^2/8 + d\chi(y)/2 - 3|z|)|\xi|$. This estimate together with the fact that $\chi(y) \equiv 1$ near $y = 0$ yield the required result.

In [1] the authors microlocalized hypo-analyticity by first adapting Sato's definition and then showing its equivalence with the one derived from the Fourier–Bros–Iagolnitzer transform [4]. The operators defined in this paper also preserve microlocal hypo-analyticity [3].

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References


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