ALGEBRAS OF OPERATORS ISOMORPHIC TO THE CIRCULANT ALGEBRA

ALAN C. WILDE

(Communicated by Louis J. Ratliff, Jr.)

Abstract. The algebra of $n \times n$ circulant matrices has a specific structure. This paper displays different operators on linear vector spaces that have the same structure, i.e. are isomorphic.

1. Introduction

Complex $n \times n$ circulant matrices are a matrix representation of the group ring (over $\mathbb{C}$) of the cyclic group. P. J. Davis [1] also proves that the set of circulants with complex entries have an idempotent basis. This paper displays algebras of operators which are isomorphic to the algebra of $n \times n$ complex circulant matrices.

§2 reviews properties of circulants and introduces a cyclic group of automorphisms on the circulant algebra generalizing conjugation. The group ring over $\mathbb{C}$ of this group is isomorphic to that of circulants themselves.

In §3, functional equations, whose solutions are functions $\mathbb{C}^n \to \mathbb{C}$, are solved using cyclic and idempotent linear operators on the space (labeled $U$) of functions $\mathbb{C}^n \to \mathbb{C}$. Again, this algebra of linear operators is isomorphic to $n \times n$ circulants.

§4 displays cyclic and idempotent linear operators on the space $V$ of functions on $n \times n$ complex circulants. Furthermore, §4 shows a relationship between the operators on $V$ and those on $U$.

Finally, §5 shows a linear involution on $V$ whose group ring is isomorphic to $2 \times 2$ complex circulant matrices.

2×2 circulant matrices and 2-dimensional complex analysis to \( n \times n \) circulant matrices. More work continuing the present paper is forthcoming.

2. Properties of circulants

An \( n \times n \) circulant matrix is a square matrix like the following:

\[
X = \begin{bmatrix}
x_0 & x_1 & x_2 & \cdots & x_{n-1} \\
x_{n-1} & x_0 & x_1 & \cdots & x_{n-2} \\
x_{n-2} & x_{n-1} & x_0 & \cdots & x_{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
x_1 & x_2 & x_3 & \cdots & x_0
\end{bmatrix}
\]

Let \( A_n \) denote the set of circulant matrices with complex entries, or \( F = \mathbb{C} \). Let \( K \) denote the circulant matrix with \( x_1 = 1 \) and \( x_j = 0 \) for \( j \neq 1 \). Then \( K^h \) (the \( h \)th power of \( K \), \( 1 \leq h \leq n \)) is the circulant matrix with \( x_h = 1 \) and \( x_j = 0 \) for all \( j \neq h \), \( K^0 = I \) (the identity matrix), \( K^1 = K \), and \( K^n = I \). So \( X \) can be written as

\[
X = \sum_{h=0}^{n-1} x_h K^h
\]

for \( x_0, x_1, \ldots, x_{n-1} \in \mathbb{C} \). In other words, \( K^0 = I, K, \ldots, K^{n-1} \) forms a basis for the circulants \( A_n \). Let \( a \) denote any of the \( n \)th roots of one, or \( a = e^{2\pi i/n} \). Let

\[
y_h = \sum_{j=0}^{n-1} a^{hj} x_j \quad \text{for} \quad h = 0, 1, \ldots, n-1.
\]

Then, as well known (see [1]), the numbers \( y_0, y_1, \ldots, y_{n-1} \) are the eigenvalues of the circulant matrix \( X \), each \( y_h \) having the corresponding eigenvector \( \text{Col}(1, a^h, a^{2h}, \ldots, a^{(n-1)h}) \).

Circulant matrices have also another basis \( E_0, E_1, \ldots, E_{n-1} \) defined by

\[
E_h = \frac{1}{n} \sum_{j=0}^{n-1} a^{-hj} K^j \quad \text{for} \quad h = 0, 1, \ldots, n-1.
\]

As shown in [1], these matrices have the following properties:

\[
E_h^2 = E_h \quad \text{for} \quad h = 0, 1, \ldots, n-1;
\]

\[
E_h E_i = 0 \quad \text{for} \quad h \neq i \quad \text{and}
\]

\[
E_0 + E_1 + \cdots + E_{n-1} = I.
\]

Also,

\[
K^h = \sum_{j=0}^{n-1} a^{hj} E_j \quad \text{for} \quad h = 1, 2, \ldots, n-1.
\]
Thus, the idempotents $E_0, E_1, \ldots, E_{n-1}$ form a basis for $A_n$, and it is shown in [1] that for $X$ in equation (2) we also have

$$X = \sum_{h=0}^{n-1} y_h E_h = y_0 E_0 + y_1 E_1 + \cdots + y_{n-1} E_{n-1}. \quad (6)$$

We have seen that every circulant, or $X \in A_n$, can be written in one and only one way in the form (2), or

$$X = x_0 I + x_1 K + x_2 K^2 + \cdots + x_{n-1} K^{n-1}. \quad (7)$$

Let us now define the function $\theta : A_n \to A_n$ by taking (see Wilde [5])

$$\theta(X) = x_0 I + x_1 (aK) + x_2 (aK)^2 + \cdots + x_{n-1} (aK)^{n-1} \quad (8)$$

i.e., $\theta$ replaces $K$ by $aK$ in (2), and by composition

$$\theta^k(X) = x_0 I + x_1 aK + x_2 (aK)^2 + \cdots + x_{n-1} (aK)^{n-1}, \quad (9)$$

i.e., $\theta^k$ replaces $x_h$ by $a^k x_h$ for $h = 0, 1, \ldots, n-1$. Also, $\theta^n(X) = X$. Then $\theta$ is an automorphism in $A_n$ that preserves $C$, $C$ being embedded in $A_n$ by the correspondence $z \to zI$ for $z \in C$. We have seen that, for any circulant $X \in A_n$, or $X = x_0 I + x_1 K + \cdots + x_{n-1} K^{n-1}$, $x_0, x_1, \ldots, x_{n-1} \in C$, if we take the numbers $y_h = \sum_{j=0}^{n-1} a^j x_j$, $h = 0, 1, \ldots, n-1$, then relation (6) holds, or $X = y_0 E_0 + y_1 E_1 + \cdots + y_{n-1} E_{n-1}$, and then

$$\theta(X) = y_1 E_0 + y_2 E_1 + \cdots + y_{n-1} E_{n-2} + y_0 E_{n-1}, \quad (10)$$

i.e., $\theta$ shifts the eigenvalues over one space.

To generalize $\text{Re} \, z$ and $i \, \text{Im} \, z$, let $q_0, q_1, \ldots, q_{n-1}$ be the functions $A_n \to A_n$ defined by

$$q_h = \frac{1}{n} \sum_{j=0}^{n-1} a^{-hj} \theta^j \quad \text{for} \ h = 0, 1, \ldots, n-1. \quad (10)$$

Then

$$q_h^2 = q_h \quad \text{for} \ h = 0, 1, \ldots, n-1; \quad (10.1)$$

$$q_h q_j = 0 \quad \text{for} \ h \neq j; \quad (10.2)$$

$$q_0 + q_1 + \cdots + q_{n-1} = \theta^0; \quad \text{and} \quad (10.3)$$

$$q_0 + a^h q_1 + a^{2h} q_2 + \cdots + a^{(n-1)h} q_{n-1} = \theta^h. \quad (10.4)$$

Also,

$$q_h(x_0 I + x_1 K + \cdots + x_{n-1} K^{n-1}) = x_h K^h \quad \text{for} \ h = 0, 1, \ldots, n-1. \quad (11)$$

Equations (10), (10.1)–(10.3), and (11) are proved in Wilde [5]. The algebra generated by $I, K, K^2, \ldots, K^{n-1}$ and $\theta^0, \theta^1, \ldots, \theta^{n-1}$ over $C$ are isomorphic and can be called circulant algebras.
If \( f \) is an entire function \( C \rightarrow C \), then
\[
f(z_0 I + z_1 K + \cdots + z_{n-1} K^{n-1}) = \sum_{h=0}^{n-1} \left[ \frac{1}{n} \sum_{k=0}^{n-1} a^{-hk} f \left( \sum_{j=0}^{n-1} a^j z_j \right) \right] K^h
\]
for all \( z_0, z_1, \ldots, z_{n-1} \in C \) as proved by Wilde [6].

3. Functional equations

For any entire function \( f: C \rightarrow C \), equation (12) can be written as follows:

\[
f(z_0 I + z_1 K + \cdots + z_{n-1} K^{n-1}) = \sum_{h=0}^{n-1} F_h(z_0, z_1, \ldots, z_{n-1}) K^h
\]
where
\[
F_h(z_0, z_1, \ldots, z_{n-1}) = \frac{1}{n} \sum_{k=0}^{n-1} a^{-hk} f \left( \sum_{j=0}^{n-1} a^j z_j \right),
\]
\( h = 0, 1, \ldots, n-1 \). The reader may also verify that for each \( h \), \( F_0, F_1, \ldots, F_{n-1} \) satisfy the functional equation

\[
F(z_0, az_1, a^2 z_2, \ldots, a^{n-1} z_{n-1}) = a^h F(z_0, z_1, \ldots, z_{n-1})
\]
for \( a = e^{2\pi i/n} \), all \( h = 0, 1, \ldots, n-1 \), and all \( z_0, z_1, \ldots, z_{n-1} \in C \).

Equation (15) is related to the circulant algebra also in another way. Let \( U = \{F|F: C^n \rightarrow C\} \) and let \( C \) be the operator \( C: U \rightarrow U \), linear in \( F \), defined by

\[
C(F)(z_0, z_1, \ldots, z_{n-1}) = F(z_0, az_1, a^2 z_2, \ldots, a^{n-1} z_{n-1}),
\]
i.e. \( C \) assigns to each function \( F(z_0, \ldots, z_{n-1}) \) in \( U \) the function \( F(z_0, az_1, \ldots, a^{n-1} z_{n-1}) \) obtained by substituting \( a^j z_j \) for \( z_j \), \( j = 0, 1, \ldots, n-1 \). If we denote \( C^k \) the operation \( C \) composed with itself \( k \) times, then

\[
C^k(F)(z_0, z_1, \ldots, z_{n-1}) = F(z_0, a^k z_1, a^{2k} z_2, \ldots, a^{(n-1)k} z_{n-1}).
\]
By Wilde [2], \( C^j = C^n \) if and only if \( n \) divides \( j \); and linear combinations of \( C^0, C^1, C^2, \ldots, C^{n-1} \) over \( C \) form a circulant algebra. Equation (15) can now be written in the form

\[
C(F) = a^h F.
\]
Also, we may define the operators \( M_0, M_1, \ldots, M_{n-1}: U \rightarrow U \) by taking

\[
M_h = \frac{1}{n}(C^0 + a^{-h} C^1 + a^{-2h} C^2 + \cdots + a^{-(n-1)h} C^{n-1})
\]
for \( h = 0, 1, \ldots, n - 1 \). These operators have the following properties:

\begin{align}
(20.1) & \quad M_h^2 = M_h \quad \text{for } h = 0, 1, \ldots, n - 1; \\
(20.2) & \quad M_h M_j = 0 \quad \text{if } h \neq j; \\
(20.3) & \quad M_0 + M_1 + \cdots + M_{n-1} = C^0; \quad \text{and} \\
(20.4) & \quad M_0 + a^h M_1 + a^{2h} M_2 + \cdots + a^{(n-1)h} M_{n-1} = C^h 
\end{align}

for \( h = 0, 1, \ldots, n - 1 \). (Properties (20.1)-(20.3) are proved in Wilde [7]. Properties (20.1)-(20.4) are similar to those of the functions \( E_0, E_1, \ldots, E_{n-1} \) and operators \( q_0, q_1, \ldots, q_{n-1} \) in \( \S 2 \).

By Wilde [7], a function \( F \) in \( U \) satisfies equation (15) (or (18)) if and only if \( F \in \text{Ran} M_h \). Moreover, properties (20.1)-(20.3) above imply that \( U = \text{Ran} M_0 \oplus \text{Ran} M_1 \oplus \cdots \oplus \text{Ran} M_{n-1} \) (a direct sum), as proved by Wilde in [7]. Each function \( F_h, h = 0, 1, \ldots, n - 1 \), defined by equation (14) is in \( \text{Ran} M_h \) and in addition

\begin{equation}
F_0 + F_1 + \cdots + F_{n-1} = f(z_0 + z_1 + \cdots + z_{n-1}).
\end{equation}

Thus

\begin{equation}
F_h = M_h(f(z_0 + z_1 + \cdots + z_{n-1})).
\end{equation}

### 4. Other circulant algebras

By equations (19) and (17), if a function \( g \) maps \( \mathbb{C}^n \) into \( \mathbb{C} \), then

\begin{equation}
M_h(g)(z_0, z_1, \ldots, z_{n-1}) = \frac{1}{n} \sum_{k=0}^{n-1} a^{-hk} g(z_0, a^k z_1, a^{2k} z_2, \ldots, a^{(n-1)k} z_{n-1})
\end{equation}

for \( h = 0, 1, \ldots, n - 1 \).

For \( f: \mathbb{A}_n \to \mathbb{A}_n \), there exist functions \( f_0, f_1, \ldots, f_{n-1}: \mathbb{C}^n \to \mathbb{C} \) such that

\begin{equation}
f \left( \sum_{h=0}^{n-1} z_h K^h \right) = \sum_{h=0}^{n-1} f_h(z_0, z_1, \ldots, z_{n-1}) K^h.
\end{equation}

Hence, from equation (20.3),

\begin{equation}
f \left( \sum_{h=0}^{n-1} z_h K^h \right) = \sum_{h=0}^{n-1} \sum_{i=0}^{n-1} M_{h+i}(f_h) K^h
\end{equation}

\begin{equation}
= \sum_{i=0}^{n-1} \sum_{h=0}^{n-1} M_{h+i}(f_h) K^h.
\end{equation}

Let \( V = \{f|f: \mathbb{A}_n \to \mathbb{A}_n \} \). For \( f \in V \), let us define \( p_i(f) \) and \( g_i \) such that

\begin{equation}
p_i(f) \left( \sum_{h=0}^{n-1} z_h K^h \right) = \sum_{h=0}^{n-1} M_{h+i}(f_h) K^h
\end{equation}
and

\[ g_i = \sum_{h=0}^{n-1} M_{h+i}(f_h), \]

for \( i = 0, 1, \ldots, n - 1 \) and \( h + i \) taken modulo \( n \). By equations (20.1) and (20.2),

\[ M_{h+i}(g_i) = M_{h+i}(f_h) \]

for \( h = 0, 1, \ldots, n - 1; \ i = 0, 1, \ldots, n - 1; \) and \( h + i \) taken modulo \( n \). Substituting (28) into (26) yields

\[ p_i(f) \left( \sum_{h=0}^{n-1} z_h K^h \right) = \sum_{h=0}^{n-1} M_{h+i}(g_i) K^h, \]

for \( i = 0, 1, \ldots, n - 1 \) and \( h + i \) taken modulo \( n \).

Using equation (26), we can prove

\[ p_i^2 = p_i \quad \text{for } i = 0, 1, \ldots, n - 1; \]

\[ p_i p_j = 0 \quad \text{if } i \neq j; \]

\[ (p_0 + p_1 + \cdots + p_{n-1})(f) = f, \quad f \in V, \]

i.e., \( p_0, p_1, \ldots, p_{n-1} \) are orthogonal projections on \( V \), adding to the identity function on \( V \), and so generating over \( \mathbb{C} \) a circulant algebra.

Also, we derive another formula for the projections \( p_i(f) \). By equations (26), (23), (8), and (24),

\[ p_i(f) \left( \sum_{h=0}^{n-1} z_h K^h \right) = \sum_{h=0}^{n-1} M_{h+i}(f_h) K^h \]

\[ = \sum_{h=0}^{n-1} \left[ \frac{1}{n} \sum_{k=0}^{n-1} a^{-(i+h)k} f_h(z_0, a^k z_1, \ldots, a^{(n-1)k} z_{n-1}) \right] K^h \]

\[ = \frac{1}{n} \sum_{k=0}^{n-1} a^{-ik} \theta^{-k} f \left( \sum_{h=0}^{n-1} a^{hk} z_h K^h \right) \]

\[ = \frac{1}{n} \sum_{k=0}^{n-1} a^{-ik} \theta^{-k} f \theta^k \left( \sum_{h=0}^{n-1} z_h K^h \right), \]

since \( \theta \) is a function \( A_n \to A_n \), namely one-to-one and onto. This result can be rewritten in the form

\[ p_i(f) = \frac{1}{n} \sum_{k=0}^{n-1} a^{-ik} \theta^{-k} f \theta^k, \]

for all functions \( f \in V \). Finally, it can be shown that \( f \theta = \theta^i f \) if and only if \( f \in \text{Ran} p_i \).
Now we want to show that for each \( f \in V \) and every \( i = 0,1,\ldots,n-1 \), there exists only one function \( g_i \) such that equation (29) holds. Indeed, suppose there exists another function \( g^*_i : C^n \to C \) such that \( p_i(f) = \sum_{h=0}^{n-1} M_{h+i}(g_i)K^h = \sum_{h=0}^{n-1} M_{h+i}(g^*_i)K^h \). Since \( I, K, K^2, \ldots, K^{n-1} \) is a basis of \( A_n, M_{h+i}(g_i) = M_{h+i}(g^*_i) \) for \( h = 0, 1, \ldots, n-1 \). By equation (20.3),

\[
g_i = \sum_{h=0}^{n-1} M_{h+i}(g_i) = \sum_{h=0}^{n-1} M_{h+i}(g^*_i) = g^*_i.
\]

So \( g_i \) is unique. Thus, there is an isomorphism between functions \( g_i : C^n \to C \) and functions \( \sum_{h=0}^{n-1} M_{h+i}(g_i)K^h \) in \( \text{Ran} p_i \).

A result of all this is as follows: let \( W = \{ f : C \to C \text{ with } f \text{ an entire function} \} \) and \( U = \{ f : C^n \to C \} \). Let \( I \) be a monomorphism \( W \to U^n \) defined by \( I(f) = (f(z_0 + \cdots + z_{n-1}), 0, \ldots, 0) \). Let \( \overline{\psi} \) be a monomorphism \( W \to V \) defined by

\[
\overline{\psi}(f) \left( \sum_{h=0}^{n-1} z_hK^h \right) = \sum_{h=0}^{n-1} M_h(f(z_0 + \cdots + z_{n-1}))K^h = f \left( \sum_{h=0}^{n-1} z_hK^h \right),
\]

which follows from equations (13), (14), and (22). Then there exists an isomorphism \( \psi : U^n \to V \) defined by

\[
\psi(g_0, \ldots, g_{n-1}) \left( \sum_{h=0}^{n-1} z_hK^h \right) = \sum_{i=0}^{n-1} \sum_{h=0}^{n-1} M_{h+i}(g_i)K^h
\]

such that the following diagram commutes:

```
  W
   ↓ I
   ↓ \overline{\psi}
U^n   \downarrow \psi
     \quad V
```

5. A LINEAR INVOLUTION

Suppose \( g \) is a function \( A_n \to A_n \) given by

\[
g = \sum_{i=0}^{n-1} g_iK^i
\]

where \( g_i \) is given by equation (27). Written out, we have

\[
g = \sum_{i=0}^{n-1} \left[ \sum_{h=0}^{n-1} M_{h+i}(f_h) \right] K^i.
\]

Let \( V \) denote the space of functions \( A_n \to A_n \). If \( f \) is an element of \( V \), then there exist a set of \( n \) functions \( f_0, f_1, \ldots, f_{n-1} \) mapping \( C^n \) into \( C \) such
that \( f = \sum_{h=0}^{n-1} f_h K^h \) (like equation (24)). Let us switch the \( h \) and the \( i \) in the right-hand side of equation (33). Then let \( \varphi \) be the function \( V \to V \) defined by

\[
\varphi \left( \sum_{h=0}^{n-1} f_h K^h \right) = \sum_{h=0}^{n-1} \left( \sum_{i=0}^{n-1} M_{h+i}(f_i) \right) K^h. 
\]

We use equations (20.1)-(20.3) to show that

\[
\varphi \left( \varphi \left( \sum_{h=0}^{n-1} f_h K^h \right) \right) = \sum_{h=0}^{n-1} \left( \sum_{i=0}^{n-1} \sum_{k=0}^{n-1} M_{h+i+k}(f_k) \right) K^h 
\]

\[
= \sum_{h=0}^{n-1} \sum_{i=0}^{n-1} M_{h+i}(f_i) K^h 
\]

\[
= \sum_{h=0}^{n-1} f_h K^h,
\]

i.e., \( \varphi^2 = \varphi^0 \) (the identity function on \( V \)). Thus \( \varphi \) is a linear involution on \( V \). Consider the set \( B = \{a_0 \varphi^0 + a_1 \varphi \mid a_0, a_1 \in \mathbb{C} \} \), i.e., linear combinations over \( \mathbb{C} \) of \( \varphi^0 \) and \( \varphi \) (since \( \varphi^2 = \varphi^0 \)). Then \( B \) is a \( 2 \times 2 \) complex circulant algebra; \((\varphi^0 + \varphi)/2 \) and \((\varphi^0 - \varphi)/2 \) are idempotent elements of \( B \), i.e., they are projections on \( V \). If \( f \) is in \( V \), then \( \varphi(f) = f \) if and only if \( f \in \text{Ran}(\varphi^0 + \varphi)/2 \); and \( \varphi(f) = -f \) if and only if \( f \in \text{Ran}(\varphi^0 - \varphi)/2 \). Also, \( V = \text{Ran}(\varphi^0 + \varphi)/2 \oplus \text{Ran}(\varphi^0 - \varphi)/2 \) (a direct sum).

If \( n = 2 \), then \( K^2 = I \). Let \( f \) and \( g \) be two functions \( \mathbb{C}^2 \to \mathbb{C} \). Then

\[
\varphi[f(z_0, z_1) + K g(z_0, z_1)] = I[f(z_0, z_1) + f(z_0, -z_1) + g(z_0, z_1) - g(z_0, -z_1)]/2 
\]

\[
+ K[f(z_0, z_1) - f(z_0, -z_1) + g(z_0, z_1) + g(z_0, -z_1)]/2.
\]

Note that, if \( f_i \) is a function \( \mathbb{C}^n \to \mathbb{C} \) for each \( i \), we have by equation (34) that

\[
\varphi(f_i K^i) = \sum_{h=0}^{n-1} M_{h+i}(f_i) K^h.
\]

Indeed, since (by equation (11))

\[
q_i \left( \sum_{h=0}^{n-1} f_h K^h \right) = f_i K^i, \quad \varphi \text{ is an isomorphism}
\]

\[
q_i(V) \to p_i(V) \quad \text{for each } i.
\]

Equation (34) implies that

\[
\varphi[M_{h+i}(f_h) K^h] = M_{h+i}(f_h) K^i
\]
where $h$ and $i$ vary from 0 to $n - 1$ and $h + i$ is taken modulo $n$. Since

$$\varphi(f) = f \quad \text{if and only if} \quad f \in \text{Ran}(\varphi^0 + \varphi)/2,$$

the function

$$(1/2)(\varphi^0 + \varphi)[M_{h+i}(f_h)K^h] = (1/2)M_{h+i}(f_h)(K^h + K^i)$$

is a fixed point of $\varphi$.

**Bibliography**

5. Alan C. Wilde, *Cauchy-Riemann conditions for algebras isomorphic to the circulant algebra*, J. Univ. of Kuwait (Science), 14 (1987), 189–204.

**Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109**