THE RANK IN HOMOGENEOUS SPACES OF NONPOSITIVE CURVATURE

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Abstract. Given a solvable and simply connected Lie group $G$ with Lie algebra $\mathfrak{g}$ and a left invariant metric of nonpositive curvature without flat factor, we prove that $\text{rank}(G) \leq \dim a$, where $a$ is the orthogonal complement of $[\mathfrak{g}, \mathfrak{g}]$ in $\mathfrak{g}$. In particular, if $H$ is a simply connected homogeneous space of nonpositive curvature satisfying the visibility axiom then $H$ has rank one.

INTRODUCTION

Let $G$ be a solvable and simply connected Lie group with a left invariant metric of nonpositive curvature. If $\mathfrak{g}$ is the Lie algebra of $G$, then $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus a$ where $a$, the orthogonal complement of $[\mathfrak{g}, \mathfrak{g}]$ in $\mathfrak{g}$ with respect to the metric, is an abelian subalgebra of $\mathfrak{g}$.

The rank of $G$ ($\text{rank}(G)$) is defined as the minimum of the dimensions of the spaces of parallel Jacobi vector fields along the geodesics through the identity of $G$. This definition coincides with the usual one in the symmetric case.

In this paper we show that if $G$ does not have de Rham flat factor, then $\text{rank}(G)$ is at most $\dim a$ (Theorem 1.3). This bound for $\text{rank}(G)$ is the best possible since for symmetric $G$, it coincides with $\dim a$ (Remark 1.4).

As a consequence, we obtain that if $H$ is a simply connected homogeneous space of nonpositive curvature satisfying the visibility axiom then $H$ has rank one (Corollary 1.6). This fact was proved in [5, Theorem 2.6] for $\dim H \leq 4$.

Finally, we show that the strict inequality in Theorem 1.3 may occur. In particular, we obtain examples of rank 1-homogeneous spaces of nonpositive curvature having planes of zero curvature (hence, they are not symmetric). Moreover, they do not satisfy the visibility axiom.

PRELIMINARIES

Let $H$ be a simply connected homogeneous Riemannian manifold of nonpositive curvature ($K \leq 0$). If $\gamma$ is a unit geodesic in $H$, $\text{rank}(\gamma)$ is defined...
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to be the dimension of the vector space of parallel Jacobi vector fields on \( \gamma \). The *rank of \( H \) (denoted by \( \text{rank}(H) \)) is the minimum of \( \text{rank}(\gamma) \) over all unit geodesic \( \gamma \) of \( H \) such that \( \gamma(0) = p \) for some \( p \) in \( H \). This definition was introduced in [3] (\( H \) not necessarily a homogeneous space) and it coincides with the usual one if \( H \) is a symmetric space (see [5, Preliminaries]). It is clear that \( \text{rank}(H) \geq 1 \) and \( \text{rank}(H) = \dim H \) if and only if \( H \) is flat.

Being \( H \) homogeneous it admits a simply transitive solvable Lie group of isometries (see [1, Proposition 2.5]) and hence \( H \) is isometric to a solvable Lie group \( G \) with a left invariant metric of nonpositive curvature.

For \( G \) as above, \( G \) satisfies the *visibility axiom* if and only if \( G \) admits a left invariant metric of negative curvature (see [4, Corollary 2.3]). Hence, if \( G \) has sectional curvature \( K \leq 0 \), \( G \) satisfies the visibility axiom and it also follows, by the definition of rank, that \( G \) has rank one.

We recall that if \( X, Y, Z \in \mathfrak{g} \), the Lie algebra of \( G \), the Riemannian connection \( \nabla \) is given by

\[
2\langle \nabla_X Y, Z \rangle = \langle [X, Y], Z \rangle - \langle [Y, Z], X \rangle + \langle [Z, X], Y \rangle
\]

and the sectional curvature \( K \) at \( e \), the identity of \( G \), is defined by

\[
|X \wedge Y|^2 K(X, Y) = \langle R(X, Y)Y, X \rangle
\]

where \( R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]} \).

In general, in a Lie group \( G \), if for any \( g \in G \), \( L_g \) and \( R_g \) denote the left and right translations respectively and \( I_g = L_g \circ R_{g^{-1}} \), then the adjoint representation of \( G \) defined by \( \text{Ad}(g) = (dI_g)_{e} \) satisfies \( I_g(\exp X) = \exp(\text{Ad}(g)X) \) and \( \text{Ad}(\exp X) = \text{Exp}(\text{ad}_X) \) for every \( X \) in \( \mathfrak{g} \), where \( \exp: \mathfrak{g} \to G \) is the exponential map of \( G \) and \( \text{Exp} \) denote the exponential map in \( \mathfrak{g} \).

1. THE RANK OF \( G \)

Let \( G \) be a solvable and simply connected Lie group \( G \) with a left invariant metric of nonpositive curvature. If \( \mathfrak{g} \) is the Lie algebra of \( G \), then \( \mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus a \) where \( a \), the orthogonal complement of \( [\mathfrak{g}, \mathfrak{g}] \) with respect to the metric, is an abelian subalgebra of \( \mathfrak{g} \). (see [1, Theorem 5.2].)

If \( \mathfrak{g}^\kappa \) is the complexification of \( \mathfrak{g} \) then we have a decomposition in direct sum, \( \mathfrak{g}^\kappa = \bigoplus \mathfrak{g}^\kappa_\lambda \), where \( \mathfrak{g}^\kappa_\lambda = \{ U \in \mathfrak{g}^\kappa : (\text{ad}_H - \lambda(H))I_k : U = 0 \) for some \( k \geq 1 \) and for all \( H \in a \} \) is the associated root space for the root \( \lambda \in (a^*)^\kappa \) under the abelian action of \( a \) on \( \mathfrak{g} \). If \( \lambda = \alpha \pm i \beta \) is a root of \( a \) in \( \mathfrak{g} \) (that is \( \mathfrak{g}^\kappa_\lambda \neq 0 \)), the generalized root space is defined by \( \mathfrak{g}_{\alpha, \lambda} = \mathfrak{g}_{\alpha, -\lambda} = \mathfrak{g} \cap (\mathfrak{g}^\kappa_\lambda \oplus \mathfrak{g}^\kappa_{-\lambda}) \) and \( \mathfrak{g} \) is the direct sum of the generalized root spaces \( \mathfrak{g}_{\alpha, \lambda} \).

We assume that \( G \) has no de Rham flat factor; it follows from [2, Theorem 4.6] that the factors \( \mathfrak{g}_0 = \sum \mathfrak{g}_{0, \alpha} \) and \( a_0 = \{ H \in a : \alpha(H) = 0 \) for all roots \( \alpha + i \beta \} \) are zero. Then \( \mathfrak{g} = \sum_{\alpha \neq 0} \mathfrak{g}_{0, \alpha} \).

The following lemma is the key for the proof of Theorem 1.3.
Lemma 1.1. If $H \in \mathfrak{a}$ satisfies $\alpha(H) > 0$ for $\alpha + i\beta$ a root then

$$\lim_{t \to +\infty} \text{Exp}(-t \text{ad}_H)X = 0 \text{ for all } X \in \mathfrak{g}_{\alpha,\beta}'. $$

Proof. Let $\lambda = \alpha + i\beta$ be a root of $\alpha$ in $\mathfrak{g}'. $ By definition of $\mathfrak{g}'^c$, $N = (\text{ad}_H - \lambda(H)I)|_{\mathfrak{g}'^c}$ is a nilpotent operator on $\mathfrak{g}'^c$. Then $\text{ad}_H |_{\mathfrak{g}'^c} = \lambda(H)I + N$ and $\text{Exp}(-t \text{ad}_H)|_{\mathfrak{g}'^c} = e^{-t\lambda(H)} \text{Exp}(-tN);$ since $|e^{-it\beta(H)}| = 1$ it follows that

$$\lim_{t \to +\infty} \text{Exp}(-t \text{ad}_H)X = 0 \text{ if and only if } \lim_{t \to +\infty} e^{-t\alpha(H)} \text{Exp}(-tN) = 0 \text{ in } GL(\mathfrak{g}'^c).$$

We compute this limit in each matricial coordinate $(ij).$ Since $N$ is nilpotent,

$$\text{Exp}(-tN) = \sum_{k=0}^{s} \frac{(-1)^k t^k}{k!} N^k (N^{s+1} = 0)$$

and

$$\text{Exp}(-tN)_{ij} = \sum_{k=0}^{s} \frac{(-1)^k t^k}{k!} (N^k)_{ij} = P^s_{ij}(t)$$

is a polynomial in $t$ of degree $s \geq 0$. Then, $\lim_{t \to +\infty} e^{-t\alpha(H)} (\text{Exp}(-tN))_{ij} = \lim_{t \to +\infty} e^{-t\alpha(H)} P^s_{ij}(t) = 0$ since $\alpha(H) > 0.$

Hence, $\lim_{t \to +\infty} \text{Exp}(-t \text{ad}_H)U = 0$ for all $U \in \mathfrak{g}'^c$ such that $\lambda = \alpha + i\beta$ is a root and consequently, $\lim_{t \to +\infty} \text{Exp}(-t \text{ad}_H)X = 0$ for all $X \in \mathfrak{g}_{\alpha,\beta}'$. 

A Jacobi vector field $J$ on a geodesic $\gamma$ is said to be stable if there exists a constant $c > 0$ such that $|J(t)| \leq c$ for all $t \geq 0$. We recall that if $\gamma$ is a geodesic of $G$, for every tangent vector $v$ at $\gamma(0)$ there exists a unique stable Jacobi vector field $J$ on $\gamma$ such that $J(0) = v$ (see [6, Lemma 2.2]). It is obvious that every parallel vector field on a geodesic $\gamma$ is stable.

Lemma 1.2. Let $H \in \mathfrak{a}$ be such that $\alpha(H) > 0$ for all roots $\alpha + i\beta$ (such an $H$ exists by [1, Proposition 5.6] since $G$ has no flat factor). Then the stable Jacobi vector field $J$ on the geodesic $\gamma_H(t) = \exp tH$ with $J(0) = X \in \mathfrak{g}'$ is given by $J(t) = \tilde{X}_{\exp tH}$, where $\tilde{X}$ is the right invariant vector field on $G$ such that $\tilde{X}_e = X.$

Proof. It follows from Lemma 1.1 since $\tilde{X}$ is a Jacobi field on $\gamma_H$ and $|\tilde{X}_{\exp tH}| = |\text{Ad}(\exp -tH)X| = |\text{Exp}(-t \text{ad}_H)X|.$

Theorem 1.3. Let $G$ be a solvable and simply connected Lie group with a left invariant metric of nonpositive curvature. If $G$ has no de Rham flat factor then $1 \leq \text{rank}(G) \leq \text{dim } \mathfrak{a}.$

Proof. Let $\mathfrak{a}' = \{H \in \mathfrak{a}: \alpha(H) > 0 \text{ for all roots } \alpha + i\beta\}. $ We will show that if $H \in \mathfrak{a}'$ then there is no parallel Jacobi vector field $J$ on $\gamma_H$ with $J(0) \neq 0$ in $\mathfrak{g}'. $ In fact, if such a $J$ exists, $J$ is a stable vector field on $\gamma_H$ with $J(0) = X \in \mathfrak{g}'$ and from Lemma 1.2, $J(t) = \tilde{X}_{\exp tH}$ for all $t \in \mathbb{R}$. This is a contradiction since $\lim_{t \to +\infty} |\tilde{X}_{\exp tH}| = 0$ and $|J(t)| = |X|$ for all $t \in \mathbb{R}.$
Therefore, \( J \) is a parallel vector field on \( \gamma_H \) if and only if \( J(t) = (dL_{\exp H})_e Z \) with \( Z \in a \) \( (\nabla_H Z = 0) \); hence the dimension of the space of parallel Jacobi vector fields on \( \gamma_H \) equals \( \dim a \) and consequently \( \operatorname{rank}(\gamma_H) = \dim a \) for all \( H \in a' \). Hence, \( \operatorname{rank}(G) \leq \dim a \).

We note that if \( G \) admits de Rham flat factor then \( \operatorname{rank}(G) \leq \dim a + \dim \mathcal{G}_0 \) (see [2, Theorem 4.6]).

**Remark 1.4.** This bound for \( \operatorname{rank}(G) \) is the best possible since in the symmetric case \( (\nabla R = 0) \), \( \operatorname{rank}(G) \) coincides with \( \dim a \). In fact, \( G \) being a symmetric space of noncompact type \( (G \) has no flat factor) it follows from [7, §6, Chapter V] that \( \operatorname{rank}(G) \) is the maximal dimension of a flat Euclidean isometrically imbedded in \( G \) as a complete totally geodesic submanifold. Consequently \( \operatorname{rank}(G) \geq \dim a \), since \( \exp(a) \) satisfies the above conditions (see [4, §2]). Hence, Theorem 1.3 implies that \( \operatorname{rank}(G) = \dim a \).

**Theorem 1.5.** Let \( G \) be a solvable and simply connected Lie group with a left invariant metric of nonpositive curvature. If \( G \) satisfies the visibility axiom then \( G \) has rank one.

**Proof.** It is a direct corollary of Theorem 1.3, since if \( G \) satisfies the visibility axiom then \( G \) has no flat factor and \( \dim a = 1 \) (see [4, Theorem 2.3]).

**Corollary 1.6.** If \( H \) is a Riemannian simply connected homogeneous space of nonpositive curvature satisfying the visibility axiom then \( H \) has rank one. In particular, if \( H = G/T \) admits a \( G \)-invariant metric of negative curvature, \( H \) has rank one.

**Proof.** The first assertion is immediate and the last one follows from [4, Theorem 2.1].

## 2. Example

In this section, we exhibit a Lie group \( G \) with a left invariant metric of nonpositive curvature such that \( \operatorname{rank}(G) = 1 \) and \( \dim a = 2 \), thus showing that strict inequality in Theorem 1.3 occurs. In this example the commutator subalgebra \( \mathcal{G}' \) is not abelian, in contrast to the case presented in [5, Example 3.3].

First we give a formula for the sectional curvature for a special case.

**Lemma 2.1.** Let \( \mathcal{G} \) be a solvable Lie algebra with an inner product \( \langle \, , \rangle \) such that \( a \), the orthogonal complement of \( \mathcal{G}' \) is abelian. If \( \operatorname{ad}_{\mathcal{H} |_{\mathcal{G}'}} \) is symmetric with respect to \( \langle \, , \rangle \) for all \( H \in a \), then

\[
\langle R(X + H , Y + T)(Y + T) , X + H \rangle \\
= \langle R(X,Y)Y , X \rangle - [[H , Y] - [T , X]]^2 - \langle [H , Y] - [T , X] , Y \rangle \\
\]

for all \( X , Y \in \mathcal{G}' , H , T \in a \).
Proof. Let $X, Y \in \mathfrak{g}$, $H, T \in a$. Since $R(T, H) = 0$ and $\nabla_H = 0$ for all $H, T \in a$, by the definition of $R$ we have,

$$
\langle R(X + H, Y + T)(Y + T), X + H \rangle = \langle R(X, Y), X \rangle + \langle R(X, T), X \rangle + \langle R(T, Y), T \rangle + \langle R(H, Y), Y \rangle + 2\langle R(H, Y), T \rangle + 2\langle R(T, Y), X \rangle + 2\langle R(H, Y), X \rangle + 2\langle R(T, Y), X \rangle.
$$

By using the connection formula,

$$
2\langle \nabla_{[H, Y]} Y, X \rangle = \langle [[T, X], X], Y \rangle - \langle [X, [T, X]], Y \rangle + \langle [Y, [T, X]], X \rangle
$$

we get,

$$
\langle R(X + H, Y + T)(Y + T), X + H \rangle = \langle R(X, Y), X \rangle + \langle R(X, T), X \rangle + \langle R(H, Y), Y \rangle + 2\langle R(H, Y), T \rangle + 2\langle R(T, Y), X \rangle
$$

and the formula follows.

Example 2.2. Let $\mathfrak{g}$ be the Lie algebra of dimension five that is generated by $\{e_1, e_2, \ldots, e_5\}$ and Lie bracket

$$
[e_1, e_2] = e e_3, \quad \gamma > 0
$$

$$
[e_1, e_3] = [e_2, e_3] = [e_4, e_5] = 0
$$

$$
[e_4, e_1] = \gamma e_1, \quad [e_4, e_2] = -\gamma e_2, \quad [e_4, e_3] = 0
$$

$$
[e_5, e_1] = \alpha e_1, \quad [e_5, e_2] = \beta e_2, \quad [e_5, e_3] = (\alpha + \beta)e_3
$$

where $\gamma \neq 0$, $\alpha > 0$, $\beta > 0$ are chosen such that $\gamma^2 - \alpha \beta < 0$.

Note that $\mathfrak{g}'$ is nonabelian, it is spanned by $\{e_1, e_2, e_3\}$ and its center $z$ is $\text{Re}_3$.

Let $\langle \cdot, \cdot \rangle$ be the inner product in $\mathfrak{g}$ such that $\langle e_i, e_j \rangle = \delta_{ij}$ $i, j = 1 \ldots 5$ and observe that $\text{ad}_{e_1} \mathfrak{g}'$, $\text{ad}_{e_2} \mathfrak{g}'$ are symmetric with respect to $\langle \cdot, \cdot \rangle$.

For each $\varepsilon \geq 0$ let $(\mathfrak{g}_\varepsilon, \langle \cdot, \cdot \rangle)$ denote the Lie algebra with the same inner product as the one in $\mathfrak{g}$ and the same Lie bracket except for $[e_1, e_2]_{\varepsilon} = \varepsilon e_3$. We note that $\mathfrak{g}' = \mathfrak{g}_0', \mathfrak{g}_\varepsilon = \mathfrak{g}_0$ and $\alpha = a_\varepsilon$ as vector spaces.
By a straightforward computation, using the connection and the curvature formulas we get

\[\begin{align*}
\nabla_{e_i} e_1 &= \gamma e_4 + \alpha e_5, &\nabla_{e_i} e_2 &= \frac{1}{2}\varepsilon e_3, &\nabla_{e_i} e_3 &= -\frac{1}{2}\varepsilon e_2 \\
\nabla_{e_i} e_4 &= -\gamma e_1, &\nabla_{e_i} e_5 &= -\alpha e_1, &\nabla_{e_i} e_4 &= \gamma e_2 \\
\nabla_{e_i} e_5 &= -\beta e_2, &\nabla_{e_i} e_4 &= 0, &\nabla_{e_i} e_5 &= -(\alpha + \beta)e_3
\end{align*}\]

(1)

\[\begin{align*}
K_\varepsilon(e_1, e_2) &= -\frac{3}{4}\varepsilon^2 + \gamma^2 - \alpha\beta \\
K_\varepsilon(e_2, e_3) &= \frac{1}{4}\varepsilon^2 - \beta(\alpha + \beta) \\
K_\varepsilon(e_1, e_3) &= \frac{1}{4}\varepsilon^2 - \alpha(\alpha + \beta)
\end{align*}\]

(2)

(3) If \(X = ae_1 + be_2\) and \(Y\) in \(z^\perp\) (the orthogonal complement of \(z\) in \(\mathfrak{g}'\)) is independent with \(X\), up to a positive constant,

\[K_\varepsilon(X, Y + ce_3) = |X \wedge Y|^2 K_\varepsilon(e_1, e_2) + c^2[a^2 K_\varepsilon(e_1, e_3) + b^2 K_\varepsilon(e_2, e_3)].\]

Now, if \(X, Y \in \mathfrak{g}', H, T \in \sigma\) and \(\{X + H, Y + T\}\) are orthonormal vectors in \(\mathfrak{g}\), it follows from Lemma 2.1 that

(4) \(K_\varepsilon(X + H, Y + T) = \langle R_\varepsilon(X, Y)Y, X \rangle - |[H, Y] - [T, X]|^2 + \varepsilon f(X, Y, H, T)\)

where \(R_\varepsilon\) is the curvature tensor for \((\mathfrak{g}_\varepsilon, \langle, \rangle)\) and \(f\) is a continuous function of \(X, Y, H, T\). Moreover, if \(G_{5,2}(\mathfrak{g})\) is the Grassmann manifold of 2-planes of \(\mathfrak{g}\), the curvature function \((\varepsilon, \pi) \mapsto K_\varepsilon(\pi)\) is uniformly continuous for \(0 \leq \varepsilon \leq 1\), \(\pi \in G_{5,2}(\mathfrak{g})\). In fact, since any 2-plane \(\pi \subset \mathfrak{g}\) is spanned by orthonormal vectors \(\{X + H, Y + T\}\) with \(X \in z^\perp\), by using (3) and (4) it is an easy computation to show that \(K_\varepsilon(\pi)\) is continuous; \(G_{5,2}(\mathfrak{g})\) being compact, the assertion follows.

As \(\varepsilon \to 0\), \((\mathfrak{g}_\varepsilon, \langle, \rangle)\) tends to a well defined limit algebra \((\mathfrak{g}_0, \langle, \rangle)\) such that \(\mathfrak{g}_0\) is abelian. Then,

(i) If \(X, Y \in \mathfrak{g}_0', H, T \in \sigma\) and \(\{X + H, Y + T\}\) are orthonormal vectors,

\[K_0(X + H, Y + T) = \langle R_0(X, Y)Y, X \rangle - |[H, Y] - [T, X]|^2.\]

(ii) If \(X = ae_1 + be_2\) and \(Y \in z^\perp\) is independent with \(X\), by the formula given in (3), up to a positive constant,

\[K_0(X, Y + ce_3) = |X \wedge Y|^2(y^2 - \alpha\beta) - (\alpha + \beta)c^2(a^2\alpha + b^2\beta) < 0.\]

In particular, \((\mathfrak{g}_0, \langle, \rangle)\) has sectional curvature \(K_0 \leq 0\).

(iii) Since any 2-plane \(\pi \subset \mathfrak{g}_0'\) contains a vector in \(z^\perp\), we have that there exists a number \(r > 0\) such that \(K_0(\pi) < -r\) for all 2-plane \(\pi \subset \mathfrak{g}_0'.\) (The curvature function \(K_0\) is continuous and negative on the Grassmann manifold of 2-planes in \(\mathfrak{g}_0'\) which is compact.)
From the uniform continuity stated above, there exist \( \varepsilon_0 > 0 \) such that, if \( \varepsilon \leq \varepsilon_0 \) then

\[
K_\varepsilon(\pi) < K_0(\pi) + r \quad \text{for all } 2\text{-plane } \pi \subset \mathcal{V}.
\]

Now, let \( \pi \) be a 2-plane in \( \mathcal{V} \) spanned by orthonormal vectors \( \{X + H, Y + T\} \) with \( X, Y \in \mathcal{V}' \), \( H, T \in \mathcal{V} \). First we assume that \( X, Y \) are independent. Hence, by using (i) and (iii),

\[
K_0(\pi) = |X \wedge Y|^2 K_0(X, Y) - \langle [H, Y] - [T, X] \rangle^2 \leq K_0(X, Y) < -r.
\]

Consequently, (iv) implies that,

\[
K_\varepsilon(\pi) < K_0(\pi) + r < K_0(X, Y) + r < 0 \quad \text{if } \varepsilon < \varepsilon_0.
\]

If \( Y \) is a multiple of \( X \), \( K_\varepsilon(\pi) = -\langle [H, Y] - [T, X] \rangle^2 \leq 0 \) (the other terms in (4) are zero).

Hence, if \( \varepsilon \leq \varepsilon_0 \) \((\mathcal{V}, \langle , \rangle)\) has sectional curvature \( K_\varepsilon \leq 0 \). Moreover, if \( X = a e_1 + b e_2 \) with \( a \neq 0 \) \( b \neq 0 \) is a unit vector, then for \( Y \in \mathcal{V}' \), \( T \in \mathcal{V} \) such that \( \{X, Y + T\} \) are orthonormal we have, if \( Y \neq 0 \), \( K_\varepsilon(X, Y + T) < 0 \) ((v)) if \( T \neq 0 \), \( K_\varepsilon(X, T) = -\langle [X, T] \rangle^2 < 0 \). In fact, \( [X, T] = a(T, \gamma e_4 + \alpha e_5)e_1 + b(T, -\gamma e_4 + \beta e_5)e_2 \) is nonzero, since \( a \neq 0 \), \( b \neq 0 \) and \( \gamma e_4 + \alpha e_5, -\gamma e_4 + \beta e_5 \) are linearly independent in \( \mathcal{V} \).

Hence, if \( \varepsilon \leq \varepsilon_0 \) and \( G_\varepsilon \) is the simply connected Lie group associated to \((\mathcal{V}, \langle , \rangle)\), the geodesic \( \gamma \) in \( G_\varepsilon \) with \( \gamma'(0) = X \) has rank one; therefore, \( \text{rank}(G_\varepsilon) = 1 \).

**References**


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