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A NOTE ON THE DIFFERENTIAL EQUATIONS OF GLEICK-LORENZ

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ABSTRACT. It is shown that for the Gleick-Lorenz equations, every solution in the positive octant blows up.

We consider the nonlinear system of differential equations

$$\begin{aligned} \dot{x}_1 &= 10(x_2 - x_1) &&= F_1(x) \\ \dot{x}_2 &= x_1 x_3 + 28x_1 - x_2 &&= F_2(x) \\ \dot{x}_3 &= x_1 x_2 - (8/3)x_3 &&= F_3(x) \end{aligned}$$

for $x = (x_1, x_2, x_3)$ in the positive orthant $\mathbf{R}^3_+$, attributed to E. N. Lorenz by J. Gleick [Gleick 1987, p. 323]. Although Gleick describes their dynamics as chaotic [Gleick 1987, p. 30], in a simulation by C. Deno [Deno 1988] the forward orbit of any point other than the origin blows up. We rigorously verify this dynamic behavior.

For vectors u, f we write $u > v$ in case $u_i > v_i$ for all i.

The system is cooperative in $\mathbf{R}^3_+$, i.e. $\partial F_i/\partial x_j \geq 0$ for $i \neq j$. Therefore the Müller-Kamke theorem on differential inequalities implies that if $x(t)$ and $y(t)$ are solutions with $x(0) > y(0) \geq 0$ then $x(t) > y(t)$ for all $t \geq 0$ at which both solutions are defined [Müller 1926, Kamke 1932; or see Coppel 1965].

It is easily verified that for any solution $x(t)$ with $x(0) > 0$, there is a solution $y(t)$ such that $x(0) > y(0) > 0$ and $F(y(0)) > 0$. It follows from the theory of cooperative systems that each $y_i(t)$ is strictly increasing for $t \geq 0$ [Selgrade 1980]. Since there are no equilibria except the origin, $y(t)$ cannot converge; therefore some $y_i(t) \to \infty$ and $\|y(t)\| \to \infty$. The Müller-Kamke theorem now implies $\|x(t)\| \to \infty$.

For any solution $z(t)$ with $z(0) \geq 0$, $z \neq 0$ it is easily seen that $z(t) > 0$ for all $t > 0$, and the preceding argument shows that $\|z(t)\| \to \infty$.

It should be noted that these equations differ from those of [Lorenz 1963] in the sign of the term $x_1 x_3$.

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