FIXED POINTS OF AUTOMORPHISMS OF COMPACT Riemann Surfaces and Higher-Order Weierstrass Points

RYUTARO HORIUCHI AND TOMIHIKO TANIMOTO

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Dedicated to Professor Kôtarô Oikawa for his 60th birthday

Abstract. A sufficient condition for fixed points of an automorphism of prime order on a compact Riemann surface to be higher-order Weierstrass points is given. This leads us to a complete study of the cases where the prime orders are small.

1. Let $M$ be a compact Riemann surface of genus $g > 2$. We denote $\text{Aut} M$ the group of conformal automorphisms of $M$, $\nu(T)$ the number of fixed points of an automorphism $T \in \text{Aut} M$ and $H^q(M)$ the space of holomorphic $q$-differentials on $M$.

Lewittes proved that if $\nu(T) > 5$, then every fixed point is a 1-Weierstrass point [5], and in this relation, some cases have been studied by Accola [1], Duma [2], Farkas and Kra [3] for higher-order Weierstrass points (see Corollaries 1, 2, 3, 4 below). Guerrero [4] proved that if $\nu(T) = 1$ and the fixed point is not a 1-Weierstrass point, then $T$ has order 6, $g \equiv 1 \mod 6$ and the fixed point is a $q$-Weierstrass point for all $q \geq 2$. It is known that if the order of $T$ is prime, then $\nu(T) \geq 2$ [3]. Guerrero also gave examples of Riemann surfaces with automorphisms of prime order whose two fixed points are not $q$-Weierstrass points for $q \geq 2$.

The purpose of this paper is to give a sufficient condition for fixed points to be $q$-Weierstrass points ($q \geq 2$) and to supplement the results mentioned above. We will show that if $\nu(T)(2s + 1 - n) \neq 2(n\delta - r)$, then the fixed points of $T$ are $q$-Weierstrass points, and study the case where $\nu(T) \geq 3$ and the order of $T$ is 5.

2. For $T \in \text{Aut} M$, let $e$ be the rotation constant of $T$ at a fixed point of $T$, i.e. locally $T^{-1}: z \to ez$. There is a basis for the space of holomorphic $q$-differentials such that the linear map induced by $T$ on this space is given by the matrix $\text{diag}(e^{\gamma_1-1+q}, e^{\gamma_2-2+q}, \ldots, e^{\gamma_d-1+q})$ for each $q \geq 2$, where $d = (2q - 1)(g - 1)$, and $1 = \gamma_1 < \gamma_2 < \cdots < \gamma_d < 2q(g - 1) + 2$ is the $q$-gap.
sequence at the fixed point. If there exists such a $\gamma_j$ with $\gamma_j > j$ for at least one $j$ in the $q$-gap sequence at a point, then the point is called a $q$-Weierstrass point. So, for a fixed point which is not a $q$-Weierstrass point, we have the matrix

$$\text{diag}(e^\theta, e^{\theta+1}, \ldots, e^{d-1+q}).$$

The multiplicity of the eigenvalue 1 is equal to $\dim H_q^q(T)(M)$ (for more details, see Farkas and Kra [3]).

3. Now we give a sufficient condition for fixed points to be $q$-Weierstrass point ($q \geq 2$).

**Theorem 1.** Assume that for $T \in \text{Aut } M$ of prime order $n$, there is a fixed point of $T$ which is not a $q$-Weierstrass point for some $q \geq 2$. Let $q - 1 = kn + s$ ($0 \leq s \leq n - 1$), $g - 1 = mn + t$ ($0 \leq t \leq n - 1$), $(2q - 1)(g - 1) = [(2q - 1)(g - 1)/n]n + r$ ($0 \leq r \leq n - 1$) and $\delta = [(r + s)/n]$. Then we have

$$\nu(T)(2s - (n - 1)) = 2(n\delta - r).$$

**Proof.** The representation of $T$ on $H^q_q(T)(M)$ is

$$\text{diag}(e^\theta, \ldots, e^{(2q-1)(g-1)+q-1}),$$

where $\theta = e^{2\pi i/n}$, and the multiplicity of the eigenvalue 1 is

$$\left\lfloor \frac{(2q-1)(g-1) + q - 1}{n} \right\rfloor - \left\lfloor \frac{q - 1}{n} \right\rfloor,$$

and is also equal to $\dim H_q^q(T)(M)$.

We set

$$\delta' = \left(\left\lfloor \frac{(2q-1)(g-1) + q - 1}{n} \right\rfloor - \left\lfloor \frac{q - 1}{n} \right\rfloor\right) - \left\lfloor \frac{(2q-1)(g-1)}{n} \right\rfloor$$

$$= \left\lfloor \frac{(2s + 1)t + s}{n} \right\rfloor - \left\lfloor \frac{(2s + 1)t}{n} \right\rfloor.$$

Now we have

$$\delta' = \left\lfloor \frac{pn + r + s}{n} \right\rfloor - \left\lfloor \frac{pn + r}{n} \right\rfloor = \left\lfloor \frac{r + s}{n} \right\rfloor,$$

and thus we have $\delta = \delta' = 0$ or 1.

Substituting the relation

$$\left\lfloor \frac{(2q-1)(g-1)}{n} \right\rfloor = \dim H_q^q(T)(M) - \delta$$

$$= (2q - 1)(g - 1) + \nu(T)[q(1 - 1/n)] - \delta,$$
where $g$ is the genus of the Riemann surface $M/(T)$ and the Riemann-Hurwitz formula

$$ g - 1 = n(g - 1) + \frac{1}{2} \nu(T)(n - 1) $$

into the relation (2), we get

$$ \nu(T)(2s - (n - 1)) = 2(n\delta - r). $$

4. In the case $n = 2$, under the same hypothesis as in the above theorem, we have $\nu = 2$, which means that the genus $g \equiv 0 \mod n$ as is seen from the Riemann-Hurwitz relation. In the case $\nu = 2$, it was shown by Guerrero [4] that there exists an automorphism of prime order $n$ on a Riemann surface of genus $n$ whose two fixed points are not $q$-Weierstrass points ($q \geq 2$).

**Corollary 1 (Duma [2]).** Let $T \in \text{Aut} M$ be of order 2. If $\nu(T) \geq 3$, then every fixed point is a $q$-Weierstrass point ($q \geq 2$).

**Corollary 2 (Farkas and Kra [3]).** Let $T \in \text{Aut} M$ be of prime order $n$. If $\nu(T) \geq 3$, then every fixed point of $T$ is a $q$-Weierstrass point for $q \geq 2$, $q \equiv 1 \mod n$.

**Proof.** If we set $s = 0$ in Theorem 1, then $\delta = [(r + s)/n] = 0$ so that $\nu(T) = 2r/(n-1) \leq 2$. This contradiction proves that the fixed points are $q$-Weierstrass points with $q \equiv 1 \mod n$.

**Corollary 3 (Accola [1]).** Let $T \in \text{Aut} M$ be of prime order $n$. If $\nu(T) \geq 3$, then every fixed point of $T$ is an $n$-Weierstrass point.

**Proof.** If we set $s = n - 1$ in the above theorem, then

$$ \nu(T) = 2(n\delta - r)/(n - 1) \leq 2. $$

This contradiction shows that fixed points are $q$-Weierstrass points with $q \equiv 0 \mod n$.

5. Now we can improve Corollary 2 and Corollary 3 to some extent:

**Theorem 2.** Let $T \in \text{Aut} M$ be of prime order $n$. If $\nu(T) \geq 3$, then every fixed point of $T$ is a $q$-Weierstrass point for $q \geq 2$, $q - 1 \equiv s \mod n$, where $s$ satisfies the inequalities

$$ \frac{\nu}{2(\nu - 1)}(n - 1) < s \quad \text{or} \quad s < \frac{\nu - 2}{2(\nu - 1)}(n - 1). $$

**Proof.** If $\delta = 0$ in (1), then we have $\nu(T)(n - 1 - 2s) = 2r \geq 0$ and $\nu(T)(n - 1) - 2(\nu(T) - 1)s = 2(r + s) \leq 2(n - 1)$ so that

$$ \frac{\nu(T) - 2}{2(\nu(T) - 1)}(n - 1) \leq s \leq \frac{n - 1}{2}. $$

If $\delta = 1$ in (1), then we have

$$ \nu(T)(2s - (n - 1)) = 2(n - r) > 0 $$
and
\[ 2(\nu(T) - 1)s - \nu(T)(n - 1) = 2(n - (r + s)) \leq 0 \]
so that
\[ \frac{n}{2} \leq s \leq \frac{\nu(T)}{2(\nu(T) - 1)}(n - 1). \]
Since \( (\nu(T) - 2)/2(\nu(T) - 1) \leq 1/2 \) and \( \frac{1}{2} \leq \nu(T)/2(\nu(T) - 1) \), the theorem is now proven.

From this theorem, we can obtain the following:

**Corollary 4 (Duma [2]).** Let \( T \in \text{Aut} M \) be of order 3. If \( \nu(T) \geq 3 \), then every fixed point is a \( q \)-Weierstrass point \( (q \geq 2) \) except for \( q \equiv 2 \pmod{3} \).

The remaining case \( q \equiv 2 \pmod{3} \) will be settled, following Guerrero’s example [4].

The hyperelliptic Riemann surface defined by the equation
\[ w^2 = (1 + z^3 + z^6 + z^9), \]
has genus \( g = 4 \), and has an automorphism with three fixed points, two of which over \( z = 0 \) can be shown to be non-5-Weierstrass points.

6. As for the case \( n = 5 \), the cases \( s = 0 \) and \( s = 4 \) are settled by Theorem 2.

In the case \( s = 2 \), there exists a hyperelliptic Riemann surface of genus 5 with an automorphism of order 5 whose two fixed points are not \( q \)-Weierstrass points \( (q \equiv 3 \pmod{5}) \) (Guerrero [4]).

In the case \( s = 1 \), assume that a fixed point of \( T \) is not a \( q \)-Weierstrass point, then we have \( \nu(T) = r = 3 \), provided that \( \nu(T) \geq 3 \).

In the case \( s = 3 \), we have \( \nu(T) = 3, r = 2 \) under the same assumptions as in the case \( s = 1 \).

We can show that the hyperelliptic Riemann surface defined by the equation
\[ w^2 = 1 + z^5 \]
is of genus two and has three fixed points, two of which over \( z = 0 \) are not \( q \)-Weierstrass points for \( q = 4, 7 \).

Thus we have the next corollary.

**Corollary 5.** Let \( T \in \text{Aut} M \) be of order 5. If \( \nu(T) \geq 3 \), then every fixed point is a \( q \)-Weierstrass point \( (q \geq 2) \), except for the following cases:

1. \( q \equiv 2 \pmod{5} \) and \( \nu(T) = r = 3 \),
2. \( q \equiv 3 \pmod{5} \),
3. \( q \equiv 4 \pmod{5} \) and \( \nu(T) = 3, r = 2 \).

**References**


**Department of Mathematics, Kyoto Sangyo University, Kamigamo Kitaku Kyoto 603, Japan**