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FIXED POINTS OF AUTOMORPHISMS OF COMPACT RIEMANN SURFACES AND HIGHER-ORDER WEIERSTRASS POINTS

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Dedicated to Professor Kôtaro Oikawa for his 60th birthday

ABSTRACT. A sufficient condition for fixed points of an automorphism of prime order on a compact Riemann surface to be higher-order Weierstrass points is given. This leads us to a complete study of the cases where the prime orders are small.

1. Let M be a compact Riemann surface of genus $g \geq 2$. We denote $\operatorname{Aut} M$ the group of conformal automorphisms of M, $\nu(T)$ the number of fixed points of an automorphism $T \in \operatorname{Aut} M$ and $H^q(M)$ the space of holomorphic q-differentials on M.

Lewittes proved that if $\nu(T) \geq 5$, then every fixed point is a 1-Weierstrass point [5], and in this relation, some cases have been studied by Accola [1], Duma [2], Farkas and Kra [3] for higher-order Weierstrass points (see Corollaries 1, 2, 3, 4 below). Guerrero [4] proved that if $\nu(T) = 1$ and the fixed point is not a 1-Weierstrass point, then T has order 6, $g \equiv 1 \bmod 6$ and the fixed point is a q-Weierstrass point for all $q \geq 2$. It is known that if the order of T is prime, then $\nu(T) \geq 2$ [3]. Guerrero also gave examples of Riemann surfaces with automorphisms of prime order whose two fixed points are not q-Weierstrass points for $q \geq 2$.

The purpose of this paper is to give a sufficient condition for fixed points to be q-Weierstrass points $(q \geq 2)$ and to supplement the results mentioned above. We will show that if $\nu(T)(2s + 1 - n) \neq 2(n\delta - r)$, then the fixed points of T are q-Weierstrass points, and study the case where $\nu(T) \geq 3$ and the order of T is 5.

2. For $T \in \operatorname{Aut} M$, let ε be the rotation constant of T at a fixed point of T, i.e. locally $T^{-1}\colon z \to \varepsilon z$. There is a basis for the space of holomorphic q-differentials such that the linear map induced by T on this space is given by the matrix $\operatorname{diag}(\varepsilon^{\gamma_1 - 1 + q}, \varepsilon^{\gamma_2 - 2 + q}, \dots, \varepsilon^{\gamma_d - 1 + q})$ for each $q \geq 2$, where $d = (2q-1)(g-1)$, and $1 = \gamma_1 < \gamma_2 < \cdots < \gamma_d < 2q(g-1) + 2$ is the q-gap

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sequence at the fixed point. If there exists such a γ_j with $\gamma_j > j$ for at least one j in the q-gap sequence at a point, then the point is called a q-Weierstrass point. So, for a fixed point which is not a q-Weierstrass point, we have the matrix

$$\operatorname{diag}(\varepsilon^q, \varepsilon^{q+1}, \dots, \varepsilon^{d-1+q}).$$

The multiplicity of the eigenvalue 1 is equal to $\dim H^q_{\langle T\rangle}(M)$ (for more details, see Farkas and Kra [3]).

3. Now we give a sufficient condition for fixed points to be q-Weierstrass point $(q \geq 2)$.

Theorem 1. *Assume that for $T \in \operatorname{Aut} M$ of prime order n, there is a fixed point of T which is not a q-Weierstrass point for some $q \geq 2$. Let $q-1 = kn+s$ $(0 \leq s \leq n-1)$, $g-1 = mn+t$ $(0 \leq t \leq n-1)$, $(2q-1)(g-1) = [(2q-1)(g-1)/n]n + r$ $(0 \leq r \leq n-1)$ and $\delta = [(r+s)/n]$. Then we have*

$$\nu(T)(2s-(n-1)) = 2(n\delta - r). \tag{1}$$

Proof. The representation of T on $H^q(M)$ is

$$\operatorname{diag}(\varepsilon^q, \dots, \varepsilon^{(2q-1)(g-1)+q-1}),$$

where $\varepsilon = e^{2\pi i/n}$, and the multiplicity of the eigenvalue 1 is

$$\left[\frac{(2q-1)(g-1)+q-1}{n}\right] - \left[\frac{q-1}{n}\right],$$

and is also equal to $\dim H^q_{\langle T\rangle}(M)$.

We set

$$\begin{aligned}\delta' &= \left(\left[\frac{(2q-1)(g-1)+q-1}{n}\right] - \left[\frac{q-1}{n}\right]\right) - \left[\frac{(2q-1)(g-1)}{n}\right] \\ &= \left[\frac{(2s+1)t+s}{n}\right] - \left[\frac{(2s+1)t}{n}\right].\end{aligned}$$

Now we have

$$(2q-1)(g-1) = \left[\frac{(2q-1)(g-1)}{n}\right]n + r \qquad (0 \leq r \leq n-1), \tag{2}$$

and

$$(2q-1)(g-1) \equiv (2s+1)t \equiv r \pmod{n}.$$

If we write $(2s+1)t = pn + r$, then

$$\delta' = \left[\frac{pn+r+s}{n}\right] - \left[\frac{pn+r}{n}\right] = \left[\frac{r+s}{n}\right],$$

and thus we have $\delta = \delta' = 0$ or 1.

Substituting the relation

$$\begin{aligned}\left[\frac{(2q-1)(g-1)}{n}\right] &= \dim H^q_{\langle T\rangle}(M) - \delta \\ &= (2q-1)(\tilde{g}-1) + \nu(T)[q(1-1/n)] - \delta,\end{aligned}$$

where $\tilde{g}$ is the genus of the Riemann surface $M/\langle T\rangle$ and the Riemann-Hurwitz formula

$$g-1=n(g-1)+\tfrac{1}{2}\nu(T)(n-1)$$

into the relation (2), we get

$$\nu(T)(2s-(n-1))=2(n\delta-r).$$

4. In the case $n=2$, under the same hypothesis as in the above theorem, we have $\nu=2$, which means that the genus $g\equiv 0 \bmod n$ as is seen from the Riemann-Hurwitz relation. In the case $\nu=2$, it was shown by Guerrero [4] that there exists an automorphism of prime order n on a Riemann surface of genus n whose two fixed points are not q-Weierstrass points $(q\geq 2)$.

Corollary 1 (Duma [2]). *Let $T\in \operatorname{Aut} M$ be of order 2. If $\nu(T)\geq 3$, then every fixed point is a q-Weierstrass point $(q\geq 2)$.*

Corollary 2 (Farkas and Kra [3]). *Let $T\in \operatorname{Aut} M$ be of prime order n. If $\nu(T)\geq 3$, then every fixed point of T is a q-Weierstrass point for $q\geq 2$, $q\equiv 1$* (mod n).

Proof. If we set $s=0$ in Theorem 1, then $\delta=[(r+s)/n]=0$ so that $\nu(T)=2r/(n-1)\leq 2$. This contradiction proves that the fixed points are q-Weierstrass points with $q\equiv 1 \bmod n$.

Corollary 3 (Accola [1]). *Let $T\in \operatorname{Aut} M$ be of prime order n. If $\nu(T)\geq 3$, then every fixed point of T is an n-Weierstrass point.*

Proof. If we set $s=n-1$ in the above theorem, then

$$\nu(T)=2(n\delta-r)/(n-1)\leq 2.$$

This cotradiction shows that fixed points are q-Weierstrass points with $q\equiv 0 \bmod n$.

5. Now we can improve Corollary 2 and Corollary 3 to some extent:

Theorem 2. *Let $T\in \operatorname{Aut} M$ be of prime order $n\geq 3$. If $\nu(T)\geq 3$, then every fixed point of T is a q-Weierstrass point for $q\geq 2$, $q-1\equiv s$* (mod n), *where s satisfies the inequalities*

$$\frac{\nu}{2(\nu-1)}(n-1)<s \qquad \text{or} \qquad s<\frac{\nu-2}{2(\nu-1)}(n-1).$$

Proof. If $\delta=0$ in (1), then we have $\nu(T)(n-1-2s)=2r\geq 0$ and $\nu(T)(n-1)-2(\nu(T)-1)s=2(r+s)\leq 2(n-1)$ so that

$$\frac{\nu(T)-2}{2(\nu(T)-1)}(n-1)\leq s\leq \frac{n-1}{2}.$$

If $\delta=1$ in (1), then we have

$$\nu(T)(2s-(n-1))=2(n-r)>0$$

and

$$2(\nu(T)-1)s-\nu(T)(n-1)=2(n-(r+s))\leq 0$$

so that

$$\frac{n}{2}\leq s\leq \frac{\nu(T)}{2(\nu(T)-1)}(n-1).$$

Since $(\nu(T)-2)/2(\nu(T)-1)\leq 1/2$ and $\frac{1}{2}\leq \nu(T)/2(\nu(T)-1)$, the theorem is now proven.

From this theorem, we can obtain the following:

Corollary 4 (Duma [2]). *Let $T\in \operatorname{Aut} M$ be of order 3. If $\nu(T)\geq 3$, then every fixed point is a q-Weierstrass point $(q\geq 2)$ except for $q\equiv 2 \pmod 3$.*

The remaining case $q\equiv 2 \bmod 3$ will be settled, following Guerrero's example [4].

The hyperelliptic Riemann surface defined by the equation

$$w^2=(1+z^3+z^6+z^9),$$

has genus $g=4$, and has an automorphism with three fixed points, two of which over $z=0$ can be shown to be non-5-Weierstrass points.

6. As for the case $n=5$, the cases $s=0$ and $s=4$ are settled by Theorem 2.

In the case $s=2$, there exists a hyperelliptic Riemann surface of genus 5 with an automorphism of order 5 whose two fixed points are not q-Weierstrass points $(q\equiv 3 \bmod 5)$ (Guerrero [4]).

In the case $s=1$, assume that a fixed point of T is not a q-Weierstrass point, then we have $\nu(T)=r=3$, provided that $\nu(T)\geq 3$.

In the case $s=3$, we have $\nu(T)=3$, $r=2$ under the same assumptions as in the case $s=1$.

We can show that the hyperelliptic Riemann surface defined by the equation

$$w^2=1+z^5$$

is of genus two and has three fixed points, two of which over $z=0$ are not q-Weierstrass points for $q=4, 7$.

Thus we have the next corollary.

Corollary 5. *Let $T\in \operatorname{Aut} M$ be of order 5. If $\nu(T)\geq 3$, then every fixed point is a q-Weierstrass point $(q\geq 2)$, except for the following cases:*

(1) $q\equiv 2 \bmod 5$ *and* $\nu(T)=r=3$,

(2) $q\equiv 3 \bmod 5$

(3) $q\equiv 4 \bmod 5$ *and* $\nu(T)=3$, $r=2$.

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