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A DUAL TO BAER'S LEMMA

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ABSTRACT. Let A be an abelian group. We investigate the splitting of sequences $(*)$ $0 \to P \to G \to H \to 0$ with P A-projective: Examples show that restrictions on G and H must be imposed to obtain a dual to Baer's Lemma. A characterization of the splitting of sequences like $(*)$ where G is A-reflexive and $R_A(H) = 0$ is given in terms of A and $E(A)$, when A is slender and nonmeasurable. Furthermore, we consider related problems and present applications of our results.

1. INTRODUCTION AND NOTATION

There are only very few results that guarantee the splitting of an exact sequence $0 \to B \to C \to G \to 0$ of torsion-free abelian groups. Perhaps the most frequently used of these is Baer's Lemma [F, Proposition 86.5]. Arnold, Lady, and Albrecht succeeded in [AL, A3, and A4] in extending Baer's Lemma to a situation more general than the one in [F].

We consider an abelian group A. An abelian group G is *A-projective* if it is isomorphic to a direct summand of $\oplus_I A$ for some index-set I. The smallest cardinality possible for I is the *A-rank of* G. The *A-socle of* G, denoted by $S_A(G)$, is the subgroup of G which is generated by $\{\phi(A) | \phi \in \mathrm{Hom}(A, G)\}$. Dually, we define the *A-radical of* G, $R_A(G)$, by $R_A(G) = \cap\{\ker \phi | \phi \in \mathrm{Hom}(G, A)\}$. Finally, A is *self-small* if the functor $\mathrm{Hom}(A, -)$ preserves direct sums of copies of A.

Generalized Baer's Lemma [A4, Corollary 2.2]. *The following conditions are equivalent for a self-small abelian group* A:

a) *Every exact sequence* $P \to F \to 0$ *of A-projective groups such that* F *has finite A-rank splits.*
b) *Every exact sequence* $0 \to B \xrightarrow{\alpha} G \xrightarrow{\beta} P \to 0$, *such that* P *is A-projective of finite A-rank, and* $\alpha(B) + S_A(G) = G$, *splits.*

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c) $M \otimes_{E(A)} A \neq 0$ *for all non-zero, finitely generated right modules over the endmorphism ring,* $E(A)$, *of* A.

The proof of this result used the following pair of functors between the category of abelian groups and the category of right $E(A)$-modules: For an abelian group G, composition of maps induces a right $E(A)$-module structure on $H_A(G) = \mathrm{Hom}(A, G)$. For a right $E(A)$-module M, set $T_A(M) = M \otimes_{E(A)} A$. There are natural maps $\theta_G : T_A H_A(G) \to G$ and $\phi_M : M \to H_A T_A(M)$ where $\theta_G(\alpha \otimes a) = \alpha(a)$ and $[\phi_M(m)](a) = m \otimes a$ for all $\alpha \in H_A(G)$, $m \in M$ and $a \in A$. Arnold and Murley in [AM] showed that if $E(A)$ is discrete in the finite topology, T_A is a category equivalence between the categories of projective right $E(A)$-modules, $\mathfrak{p}_E$, and the A-projective groups, $\mathfrak{p}_A$ whose inverse is H_A. Moreover θ_G is the natural isomorphism for G in $\mathfrak{p}_A$ and ϕ_M is the natural isomorphism for M in $\mathfrak{p}_E$.

In this paper, we give a dualization of the generalized Baer's Lemma. To simplify our notation, we write $G^* = \mathrm{Hom}(G, A)$ and $M^* = \mathrm{Hom}_{E(A)}(M, A)$ for abelian groups G and left $E(A)$-modules M. There are natural maps $\psi_G : G \to G^{**}$ (the group G^* carries a natural left $E(A)$-module structure) and $\psi_M : M \to M^{**}$ defined by $\psi_G(g)(\phi) = \phi(g)$ for all $g \in G$ and $\phi \in G^*$ (with ψ_M defined similarly). The group G is *A-reflexive* if ψ_G is an isomorphism. Huber and Warfield showed in [HW], that every direct summand of A^I is A-reflexive if A is slender and $|A|$ as well as $|I|$ are nonmeasurable. In this case, ψ_P is an isomorphism if P is a projective module of nonmeasurable cardinality.

In §2, we consider the dual of condition a) of the Generalized Baer's Lemma. We say that A has the *(finite) dual Baer splitting property* if every exact sequence $0 \to P \to G \to H \to 0$, such that $P \oplus Q = A^I$, where $|I|$ is nonmeasurable (finite), G is A-reflexive, and $R_A(H) = 0$, splits. The restriction $R_A(H) = 0$ is necessary for our dual in view of Example 3.6 if we do not want to impose any immediate restrictions on A.

We show that a slender abelian group A of nonmeasurable cardinality has the dual Baer-splitting property exactly if a submodule M of a finitely generated projective left $E(A)$-module is projective whenever M^* is A-projective of finite A-rank. We conclude this section with numerous examples.

§3 presents a dual to condition b) of the Generalized Baer Lemma. The most natural dualization would be to investigate the groups A such that every pure exact sequence

$$0 \to P \xrightarrow{\alpha} G \xrightarrow{\beta} B \to 0, \tag{1.1}$$

such that P is A-projective of finite A-rank, and $\alpha(P) \cap R_A(G) = 0$, splits. However, Example 3.4 shows that there exists a nonsplitting sequence like (1.1) for every cotorsion-free group A. Hence, we restrict our discussion to the case that $R_A(G) = 0$.

Even in this case, it becomes necessary to restrict the generality of G as is demonstrated in Proposition 3.5 and Example 3.6. We choose G from the class of all groups admitting an embedding $\alpha : G \to A^I$ such that $\alpha(S_A(G))$ is a pure subgroup of A^I. We say that G has a *strongly pure A-socle* in this case. For reasons of simplicity we restrict ourselves, in this introduction, to mentioning the case that $\mathbf{Q} \otimes_Z E(A)$ is a semi-simple, finite-dimensional $\mathbf{Q}$-vector-space, and A is flat as an $E(A)$-moduel. We show that such a group A has the finite dual Baer splitting property precisely when any A-projective pure subgroup with finite A-rank of a group G with strongly pure A-socle is a direct summand. This happens if and only if a submodule of a finitely generated projective right $E(A)$-module is either projective or has infinite projective dimension.

Although we obtain a dual version of the generalized Baer's Lemma, our examples show that the situation is not nearly as satisfactory in the dual case as it was in [AL and A4]. While [A5, Theorem 2.8] shows that every cotorsion-free ring R can be realized as the endomorphism ring of a cotorsion-free abelian group G, which has the Baer splitting property, the (finite) dual splitting property restricts the generality of $E(G)$.

2. A dual to Baer's lemma

In this section, we consider the duals of conditions a) and b) in Baer's Lemma.

Definition. An abelian group A has the (finite) dual Baer splitting property provided every exact sequence $0 \to P \to G \to H \to 0$ splits, when P is a direct summand of A^I for some nonmeasurable (finite) set I, G is A-reflexive, and $R_A(H) = 0$.

Theorem 2.1. *The following conditions are equivalent for a slender abelian group A of nonmeasurable cardinality*:

a) *A has the dual Baer splitting property.*
b) *A submodule M of a free left $E(A)$-module of nonmeasurable cardinality is projective if M^* is isomorphic to a direct summand of A^I for some index-set such that $|I|$ is nonmeasruable.*

Proof. a) $\Rightarrow$ b). Let M be a submodule of a free left $E(A)$-module and choose an exact sequence $0 \to U \xrightarrow{\alpha} \oplus_I E(A) \xrightarrow{\beta} M \to 0$ of left $E(A)$-modules where $|I|$ is nonmeasurable. This sequence induces the exact sequence $0 \to M^* \xrightarrow{\beta^*} [\oplus_I E(A)]^* \xrightarrow{\alpha^*} G \to 0$ where $G = \operatorname{im} \alpha^* \subseteq U^*$ has zero A-radical. If M^* is a direct summand of A^J for some nonmeasurable J, then the last sequence splits because of a). Consequently, the top row of the commutative diagram

$$\begin{array}{ccccccccc} 0 \to & G^* & \xrightarrow{\alpha^{**}} & [\oplus_I E(A)]^{**} & \xrightarrow{\beta^{**}} & M^{**} & \to 0 \\ & \uparrow & & \uparrow \wr\, \psi_{\oplus_I E(A)} & & \uparrow \psi_M & \\ 0 \to & U & \xrightarrow{\alpha} & \oplus_I E(A) & \xrightarrow{\beta} & M & \longrightarrow 0 \end{array}$$

splits. Thus, the module M is projective if ψ_M is a monomorphism. But this is an immediate consequence of the fact that M is a submodule of a projective module.

b) $\Rightarrow$ a). We consider an exact sequence

$$0 \to P \xrightarrow{\alpha} G \xrightarrow{\beta} H \to 0 \tag{2.1}$$

where G is A-reflexive, $R_A(H) = 0$, and P is a direct summand of A^I for some index-set I of nonmeasurable cardinality. In induces the exact sequence

$$0 \to H^* \xrightarrow{\beta^*} G^* \xrightarrow{\alpha^*} M \to 0 \tag{2.2}$$

of left $E(A)$-modules, where $M = \operatorname{im} \alpha^*$ is a submodule of the projective module P^*. We obtain the following commutative diagram whose rows are exact:

$$\begin{array}{ccccccc} 0 \to & M^* & \xrightarrow{\alpha^{**}} & G^{**} & \xrightarrow{\beta^{**}} & H^{**} & \\ & \uparrow \psi & & \uparrow \wr\, \psi_G & & \uparrow \psi_H & \\ 0 \to & P & \xrightarrow{\alpha} & G & \xrightarrow{\beta} & H & \to 0 \end{array} \tag{2.3}$$

The induced map ψ is an isomorphism since ψ_H is one-to-one. Consequently, M^* is a direct summand of a product of copies of A. By b), M is projective. Since this results in the splitting of (2.2), the sequence in the top row of (2.3) splits too. But this is only possible if $\alpha(P)$ is a direct summand of G.

Corollary 2.2. *The following conditions are equivalent for a slender abelian group A of nonmeasurable cardinality*:

a) *A has the finite dual Baer splitting property.*
b) *A submodule M of a finitely generated, projective left $E(A)$-module is projective if M^* is A-projective of finite A-rank.*

Corollary 2.3. *Let A be a slender abelian group with a left hereditary endomorphism ring. Then, A has the dual Baer splitting property.*

However, $E(A)$ is not necessarily left hereditary if A has the dual Baer splitting property:

Example 2.4. Let A be a slender abelian group whose endomorphism ring is $\mathbf{Z}_2 + 2i\mathbf{Z}_2$. Then A has the finite dual Baer splitting property but $E(A)$ is not hereditary.

Proof. If M is a submodule of a finitely generated projective module, then $M = [\oplus_n E(A)] \oplus [\oplus_m J]$ where $J = 2\mathbf{Z}_2 + 2i\mathbf{Z}_2$ is the maximal ideal of the local ring $E(A)$ [B2, Theorem 1.7]. We will show if M^* is A-projective then $m = 0$ establishing our claim. Suppose, to the contrary, that $m > 0$ and M^* is A-projective. Then J^* is A-projective of finite A-rank.

The ideal J admits a projective resolution $0 \to J \xrightarrow{\alpha} P \xrightarrow{\beta} J \to 0$ where $P = E(A) \oplus E(A)$. This sequence induces the exact sequence

$$0 \to J^* \xrightarrow{\beta^*} P^* \xrightarrow{\alpha^*} G \to 0 \tag{2.4}$$

where $G = \operatorname{im} \alpha^* \subseteq J^*$ is torsion-free. From (2.4) we have the projective resolution

$$0 \to H_A(J^*) \xrightarrow{H_A(\beta^*)} H_A(P^*) \xrightarrow{H_A(\alpha^*)} N \to 0 \tag{2.5}$$

of the torsion-free, finitely generated right $E(A)$-module $N = \operatorname{im} H_A(\alpha^*)$. Since J^* is A-projective, N has projective dimension 1. By [B2], N is projective, and (2.5) splits. Consequently, the top row of the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & T_AH_A(J^*) \arrow[r, "T_AH_A(\beta^*)"] & T_AH_A(P^*) \arrow[r, "T_AH_A(\alpha^*)"] & T_A(N) \arrow[r] & 0 \\
0 \arrow[r] & J^* \arrow[u, "\wr\, \theta_{J^*}"'] \arrow[r, "\beta^*"] & P^* \arrow[u, "\wr\, \theta_{P^*}"'] \arrow[r, "\alpha^*"] & G \arrow[u] \arrow[r] & 0
\end{tikzcd}$$

splits, which results in the splitting of the bottom row. This induces the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & G^* \arrow[r, "\alpha^{**}"] & P^{**} \arrow[r, "\beta^{**}"] & J^{**} \arrow[r] & 0 \\
0 \arrow[r] & J \arrow[u] \arrow[r, "\alpha"] & P \arrow[u, "\wr\, \psi_P"'] \arrow[r, "\beta"] & J \arrow[u, "\psi_J"'] \arrow[r] & 0.
\end{tikzcd}$$

Hence, ψ_J is onto. Because $J \leq E(A)$, ψ_J is injective and the bottom sequence splits, a contradiction of the fact that J is not projective. Thus J^* is not A-projective, so $m = 0$ and M is projective.

Before we give an example of a slender group A which does not have the dual Baer splitting property, we prove

Corollary 2.5. *Let A be a slender abelian group which is flat as an $E(A)$-module with $E(A)$ left Noetherian. If A has the finite dual Baer splitting property, then $E(A)$ is left hereditary or has infinite global dimension.*

Proof. If $E(A)$ has global dimension n with $1 < n < \infty$, then there exists a finitely generated submodule M of a free module with p.d. $M = 1$ [Rt, Exercise 9.7]. We choose finitely generated projective modules $P \leq F$ such that $F/P \cong M$. If the sequence $0 \to T_A(P) \to T_A(F) \to T_A(M) \to 0$ splits, then M is projective in view of the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H_AT_A(P) \arrow[r] & H_AT_A(F) \arrow[r] & H_AT_A(M) \arrow[r] & 0 \\
0 \arrow[r] & P \arrow[u, "\wr\, \phi_P"'] \arrow[r] & F \arrow[u, "\wr\, \phi_A"'] \arrow[r] & M \arrow[u, "\wr\, \phi_M"'] \arrow[r] & 0,
\end{tikzcd}$$

a contradiction. Because $T_A(M)$ is a subgroup of an A-projective group, A does not have the dual Baer splitting property.

Example 2.6. Let A be a slender abelain group which is faithfully flat as an $E(A)$-module and satisfies $E(A) = \mathbf{Z}[X]$. Since $\mathbf{Z}[X]$ has global dimension 2, A does not have the finite dual Baer splitting property. On the other hand, A has the Baer splitting property by [A3, Theorem 2.1]. Such an A exists by [A5, Theorem 2.8].

Finally, it is not possible to remove the restriction in Corollary 2.2 that A is slender.

Example 2.7. Set $A = \prod_p \mathbf{Z}_p$ where $\mathbf{Z}_p$ denotes the localization of $\mathbf{Z}$ at the prime p. Note that $A = E(A)$ is not slender. By Corollary 3.4 in [A6], if G is A-reflexive then G is a subgroup of A^I for some nonmeasurable I such that $R_A(A^I/G) = 0$. Therefore, in order to show that A has the finite dual Baer splitting property, it suffices to show that every pure exact sequence $0 \to A^n \xrightarrow{\alpha} A^I$ splits whenever I is nonmeasurable. Denote the set of prime of $\mathbf{Z}$ by P.

Coordinate-wise, we obtain pure-exact sequences $0 \to \mathbf{Z}_p^n \xrightarrow{\alpha_p} \mathbf{Z}_p^I$ where $\alpha = (\alpha_p)_{p \in P}$. Since $\mathbf{Z}_p^I$ is a locally free $\mathbf{Z}_p$-module, each of these sequences splits [F, Proposition 87.2]. If β_p splits α_p then $\beta = (\beta_p)_{p \in P}$ satsfies $\beta\alpha = (\beta_p)_{p \in P}(\alpha_p)_{p \in P} = (\beta_p\alpha_p)_{p \in P} = (\mathrm{id}_{\mathbf{Z}_p^n})_{p \in P} = \mathrm{id}_A n$.

On the other hand, consider the ideal $U + J$ of $E(A)$ where $U = \oplus_p \mathbf{Z}_p$ and $J = p \prod \mathbf{Z}_p$. Suppose $\phi : U \to A$ is an $E(A)$-homomorphism. Clearly ϕ extends uniquely to a map $\hat{\phi} : E(A) \to A$. If we restrict $\hat{\phi}$ to $U + J$ we see that the natural map $\mathrm{Hom}_{E(A)}(U + J, A) \to \mathrm{Hom}_{E(A)}(U, A)$ is surjective. But $(U + J)/U = J/U \cap J$ is divisible so that $(U + J)^* \cong \mathrm{Hom}_{E(A)}(U + J, A) \cong \mathrm{Hom}_{E(A)}(U, A) \cong A$ is A-projective.

If $U + J$ is projective, then by a result of Sandomierski [CH, Proposition 8.24] $U + J$ is finitely generated, since $U + J$ contains the finitely generated essential submodule J. But $U + J$ is not finitely generated. Therefore $U + J$ is not projective, so A does not have property b) of Corollary 2.2.

3. Noetherian endomorphism rings

In this section, we give a treatment of a dual version for condition c) in Baer's Lemma. Although we restrict our attention to the case that $R_A(G) = 0$, it becomes apparent that some restrictions on G are necessary to ensure that every pure exact sequence $0 \to P \to G \to B \to 0$, where P is A-projective if finite A-rank, splits. Recall that G has a *strongly pure A-socle* if there is an embedding $\alpha : G \to A^I$, for some I, such that $\alpha(S_A(G))$ is pure in A^I.

The hypotheses of the next result are satisfied by all generalized rank 1 slender groups A such that $\mathbf{Q} \otimes_{\mathbf{Z}} E(A)$ is semi-simple Artinian, but we show that there exist groups without a hereditary endomorphism ring to which the result can be applied. Recall that a reduced torsion-free group A is called a generalized rank 1 group if $E(A)$ is two-sided Noetherian and hereditary. A right $E(A)$-module M is called nonsingular if $0 \neq x \in M$ implies $xI \neq 0$ for any essential right ideal I of $E(A)$. In the following, any $E(A)$-module which is torsion-free as a group is nonsingular.

Theorem 3.1. *Let A be a slender abelian group, which is flat as an $E(A)$-module, and whose endomorphism ring is two-sided Noetherian and has the property that $E(A)/I$ is bounded for all essential right ideals I of $E(A)$. The following conditions are equivalent:*

a) *Every pure-exact sequence $0 \to P \xrightarrow{\alpha} G \xrightarrow{\beta} B \to 0$, such that P is A-projective of finite A-rank, and G has a strongly pure A-socle, splits.*
b) *A has the finite dual Baer splitting property.*
c) *Every pure-exact sequence $0 \to P \xrightarrow{\alpha} F \xrightarrow{\beta} H \to 0$, where P and F are A-projective of finite A-rank, splits.*
d) *A submodule of a finitely generated free right $E(A)$-module is either projective or has infinite projective dimension.*

Proof. a) $\Rightarrow$ b). We consider an exact sequence $0 \to P \xrightarrow{\alpha} G \to H \to 0$ where $R_A(H) = 0$, G is A-reflexive and P is A-projective of finite A-rank. Since G is A-reflexive, there exists an A-cobalanced exact sequence $0 \to G \xrightarrow{\beta} A^I$ where $R_A(A^I/\beta(G)) = 0$ [A6, Corollary 3.4]. In particular, $0 \to P \xrightarrow{\beta\alpha} A^I$ is pure exact, and it suffices to show that $S_A(A^I)$ is pure in A^I. We choose a finite subset X of $S_A(A^I)$. As in [A2, Theorem 5.1], $S_A(A^I)$ is locally A-projective. There exists a subgroup U of $S_A(A^I)$ containing X and a cofinite subset I' of I with $S_A(A^I) = U \oplus S_A(A^{I'})$. Since $A^{I \setminus I'} \subseteq S_A(A^I)$, we obtain $A^I = U \oplus A^{I'}$. Consequently, $S_A(A^I)$ is pure in A^I.

b) $\Rightarrow$ c). Let $0 \to P \xrightarrow{\alpha} F \xrightarrow{\beta} H \to 0$ be a pure-exact sequence such that P and F are A-projective of fintie A-rank. It induces an exact sequence

$$0 \to H_A(P) \xrightarrow{H_A(\alpha)} H_A(F) \xrightarrow{H_A(\beta)} M \to 0 \tag{3.1}$$

of right $E(A)$-modules. Since M is non-singular and finitely generated, it is isomorphic to a submodule of a free module by [G, Theorem 5.17]. In particular, $R_A(T_A(M)) = 0$. Since $H \cong T_A(M)$, the sequence (3.1) splits.

c) $\Rightarrow$ d). If d) fails, then by [Rt, Exercise 9.7] and the fact that $E(A)$ is Noetherian, there exists a finitely generated submodule M of a free module which has projective dimension 1. We choose finitely generated projective modules $P \leq F$ with $F/P \cong M$. By c), the induced sequence $0 \to T_A(P) \to T_A(F) \to T_A(M) \to 0$ splits. Hence, $H_A T_A(M)$ is projective. Because $M = H_A T_A(M)$, M is projective, a contradiction.

d) $\Rightarrow$ a). We consider a pure exact sequence $0 \to P \xrightarrow{\alpha} G \xrightarrow{\beta} H \to 0$ as in a). There is a monomorphism $\delta : G \to A^I$ for some index-set I such that $\delta(S_A(G))$ is pure in A^I. Since it suffices to show the splitting of $\delta\alpha$, we may assume that $G = A^I$, and $\alpha(P)$ is pure in G.

A group A is said to be *discrete* in the *finite topology* if there is a finite subset $Y \subseteq A$ such that $\alpha(Y) = 0$ implies $\alpha = 0$ for all $\alpha \in E(A)$ [AM]. [A7, Theorem 5.1] can be used to show that, in this case, A is discrete in the finite topology. Hence there is a finite subset X of P such that $\mathrm{Hom}(P/\langle X\rangle, A) = 0$. As in a) $\Rightarrow$ b), there exists a direct summand U of A^I which contains $\alpha(\langle X\rangle)$ and a cofinite subset I' of I with $A^I = U \oplus A^{I'}$. If $\alpha(P)$ is not contained in U, then there is $i \in I'$ such that $\pi_i\delta\alpha(P) \neq 0$ where $\pi_j : A^{I'} \to A$ denotes the projection onto the jth coordinate, and $\delta : A^I \to A^{I'}$ is the projection with $\ker\delta = U$. Hence $0 \neq \pi_i\delta\alpha \in \mathrm{Hom}(P/\langle X\rangle, A) = 0$, a contradiction.

The pure exact sequence $0 \to P \xrightarrow{\alpha} U \xrightarrow{\pi} U/\alpha(P) \to 0$ induces

$$0 \to H_A(P) \xrightarrow{\alpha} H_A(U) \xrightarrow{H_A(\pi)} M \to 0 \tag{3.2}$$

where $M = \mathrm{im}\, H_A(\pi)$ is a finitely generated, nonsingular right $E(A)$-module of projective dimension at most 1. By [G, Theorem 5.17], M is contained in a free right $E(A)$-module. Because of d), M has to be projective. Thus, (3.2) splits. But this is only possible if $\alpha(P)$ is a direct summand of U.

The following was shown in [Go] using different techniques.

Corollary 3.2. *Let A be a subgroup of $\mathbf{Q}$, and G be a countable abelian group with $R_A(G) = 0$. Every pure A-projective subgroup of finite rank is a direct summand of G.*

Proof. By [Go, Proposition 2.1], there exists an embedding $\alpha : G \to A^\omega$ such that $\alpha(G)$ is p-pure in A^ω for all primes p of $\mathbf{Z}$ such that $A \neq pA$. Since $S_A(G)$ is pure in G and p-divisible for all primes p of $\mathbf{Z}$ with $pA = A$, we obtain $\alpha(S_A(G))$ is a pure subgroup of A^ω. Since $E(A)$ is a principal ideal domain, the corollary follows immediately from Theorem 3.1.

If A is a slender group which is flat over $E(A) = \mathbf{Z}_2 + 2i\mathbf{Z}_2$ (for example if A is constructed using Corner's theorem), then A satisfies the hypotheses of Theorem 3.1, but $E(A)$ is not hereditary.

The restriction on G in Theroem 3.1 and in the definition of the finite dual Baer splitting property is necessary in light of

Example 3.3. For every torsion-free reduced abelian group A of finite rank, whose endomorphism ring is not hereditary, there exists an exact sequence $0 \to A \xrightarrow{\alpha} G \xrightarrow{\beta} A \to 0$ with $R_A(G) = 0$ which does not split.

Proof. We will first show that $\mathrm{Ext}(A, A)$ is not torsion-free. If it were, then by [Ar2, Theorem 1.3 and Corollary 1.4], $E(A)$ is a product of maximal orders in

$\mathbf{Q}E(A)$. Since maximal orders are hereditary, so is $E(A)$ [Ar, Theorem 11.1], contrary to assumption. Hence there is an exact sequence

$$0 \to A \xrightarrow{\alpha} G \xrightarrow{\beta} A \to 0 \tag{3.3}$$

and a prime p of $\mathbf{Z}$ such that (3.3) represents an element of order p in $\operatorname{Ext}(A, A)$. Consequently, (3.3) does not split, and there is a subgroup U of G with $U \cong A$ such that $pG \subseteq \alpha(A) \oplus U \subseteq G$. This shows $R_A(G) = 0$.

We now turn to the case that $R_A(G) \neq 0$:

Example 3.4. For every cotorsion-free reduced group A, there exists a pure exact sequence $0 \to A \xrightarrow{\alpha} G \xrightarrow{\beta} B \to 0$ such that $\alpha(A) \cap R_A(G) = 0$ which does not split. If $r_0(A) < \infty$, then B can be chosen with $r_0(B) < \infty$.

Proof. Choose a prime p of $\mathbf{Z}$ with $A \neq pA$, and denote the p-adic integers by J_p. For $B = J_p$, we obtain the exact sequence $0 = \operatorname{Hom}(B, A) \to \operatorname{Hom}(B, A/pA) \to \operatorname{Ext}(B, A)[p]$. Since $\operatorname{Hom}(B, A/pA) \neq 0$, there exists an exact sequence $0 \to A \xrightarrow{\alpha} G \xrightarrow{\beta} B \to 0$ which represents an element of order p in $\operatorname{Ext}(B, A)$. We will show $\alpha(A) \cap R_A(G) = 0$. If $x \in \alpha(A) \cap R_A(G)$, then consider a map $\phi \in \operatorname{Hom}(G, A)$ such that $\phi\alpha = p(\mathrm{id}_A)$. Because of $px = \phi\alpha(x) = 0$, we have $x = 0$. If $r_0(A) = n$ then choose B to be a rank $n+1$ pure subgroup of J_p and repeat the above argument.

Proposition 3.5. *Let A be a subgroup of $\mathbf{Q}$. The following conditions are equivalent for an abelian group G with $R_A(G) = 0$:*

a) *Every pure exact sequence $0 \to P \to G$ such that P is a A-projective of finite A-rank splits.*
b) *G is isomorphic to a subgroup G' of A^I for some index-set I such that $S_A(G')$ is a pure A-subsocle of A^I.*

Proof. The implication b) $\Rightarrow$ a) is a direct consequence of Theroem 3.1.

a) $\Rightarrow$ b). Let τ be type of A. Then, $S_A(G) = \{g \in G \mid \operatorname{type}(g) \geq \tau\}$ is a pure subgroup of G. If U is a pure finite rank subgroup of $S_A(G)$, then U is homogeneous completely decomposable of type τ. Because of a), there is $\pi_U \in \operatorname{Hom}(G, U)$ with $\pi_U|_U = \mathrm{id}_U$.

Denote by $\mathscr{C}$ the collection of all rank 1 direct summands of type τ of G. We set $I = \operatorname{Hom}(G, A)$ and define a monomorphism $\phi : G \to A^I \oplus \prod\{U \mid U \in \mathscr{C}\}$ by $\phi(g) = ((\alpha(g))_{\alpha\in I}, \pi_U(g)_{u\in\mathscr{C}})$. Since every element of $S_A(G)$ is contained in some $U \in \mathscr{C}$, we obtain that $\phi(S_A(G))$ is pure in $A^I \oplus A^J$ where $|J| = |S_A(G)|$.

Example 3.6. Let G be a torsion-less group, which is not separable. Then, there exists a nonsplitting pure-exact sequence $0 \to F \to G \to H \to 0$ with F finite rank and free. Hence G is not a pure subgroup of $\mathbf{Z}^I$.

Proposition 3.7. *The following conditions are equivalent for a generalized rank 1 group A:*

a) *Every pure exact sequence $0 \to P \to F \to G \to 0$ such that P and F are A-projective of finite A-rank splits.*
b) *$(E(A)/I)^+$ is torsion for all essential right ideals I of $E(A)$.*

Proof. b) $\Rightarrow$ a) is an immediate consequence of Theroem 3.1.

a) $\Rightarrow$ b) Suppose to the contrary that I is an essential right ideal of $E(A)$ such that $(E(A)/I)^+$ is not torsion. Then, the $\mathbf{Z}$-purification I_* of I in $E(A)$ is a projective proper right ideal of $E(A)$ such that $E(A)/I_*$ is divisible [A3, Theorem 4.2]. Since A is flat as an $E(A)$-module, [A3, Proposition 2.2 and Theorem 5.1], the induced exact sequence $0 \to I_*A \to A \to T_A(E(A)/I_*) \to 0$ is pure exact.

On the other hand, it does not split since $T_A(E(A)/I_*)$ is a nonzero divisible group. But this yields a contradiction to condition a).

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