Abstract. We prove that if $E$ is a $BK$- $AK$-space whose dual $E'$ as well is $BK$- $AK$, then $\sigma(E', F)$ and $\sigma(E', F')$ have the same convergent sequences whenever $F$ is a subspace of $E''$ containing $\Phi$ and satisfying $F^\beta = E^\beta$. This extends a result due to Bennett [B 2] and the second author [S]. We provide new examples of $BK$-spaces having the Wilansky property. We show that the bidual $E''$ of a solid $BK$- $AK$-space $E$ whose dual as well is $BK$- $AK$ satisfies a separable version of the Wilansky property. This extends a theorem of Bennett and Kalton, who proved that $l^\infty$ has the separable Wilansky property.

Introduction

G. Bennett [B 2] and the second author [S] have independently obtained a positive answer to the following question of Wilansky: Is $c_0$ the only $FK$-space, densely containing $\Phi$, whose $\beta$-dual is $l^1$? Both approaches are essentially based on a characterization of the barrelledness of certain sequence spaces by means of their $\beta$-duals. In the present paper we extend the Bennett/Stadler result, providing more examples of $BK$-spaces having the Wilansky property (in the sense introduced in [B 2]).

Let us explain the situation by considering a typical example. The classical sliding hump argument (Toeplitz/Schur) asserts that $\sigma(l^1, c_0)$-bounded sets are $\| \cdot \|_1$-bounded. The Bennett/Stadler result generalizes this to the extent that still $\sigma(l^1, E)$-bounded sets are $\| \cdot \|_1$-bounded, when $\Phi \subset E \subset c_0$ and $E^\beta = l^1$. The latter may be expressed equivalently by saying that every subspace $E$ of $c_0$ containing $\Phi$ and having $E^\beta = l^1$ is barrelled. Finally, our present attempt shows that $\sigma(l^1, E)$-bounded sets are $\| \cdot \|_1$-bounded when $\Phi \subset E \subset l^\infty$ and $E^\beta = l^1$. Actually, we prove a little more. We show that $\sigma(l^1, E)$ and $\sigma(l^1, E')$ have the same convergent sequences in case $\Phi \subset E \subset l^\infty$ and $E^\beta = l^1$. This extension requires a modified technique, since both the approaches in [B 2] and
We obtain new classes of BK-spaces having the Wilansky property. For instance, we prove that every BK-AK-space $E$, such that $S(E')$ is separably complemented in $E'$, has the Wilansky property. Here $S(E')$ denotes the space of all $y \in E'$ which have sectional convergence with respect to the norm.

We prove that the bidual $E''$ of a solid BK-AK-space $E$ whose dual $E$ is as well BK-AK has the following separable Wilansky type property: If $D$ is a norm dense subspace of $E''$ containing $\Phi$ and having $D^\beta = E^\beta$ (= $E'$), then every separable FK-space $F$ containing $D$ must actually contain $E''$. When applied to the case $E = c_0$, this provides a result of Bennett and Kalton [BK$_1$, W, p. 259].

**Notation.** The sections of a sequence $x \in \omega$ are denoted by

$$P_n x = \sum_{i=1}^{n} x_i e^i,$$

where $e^i$ are the unit vectors. If $P_n x \to x$ ($n \to \infty$), then $x$ is said to have sectional convergence. If $E$ is a BK-space, then $S(E)$ denotes the space of all $x \in E$ having sectional convergence with respect to the norm on $E$.

Concerning all other notions from sequence space theory, we refer to the book [W].

**The main theorem**

This section presents our fundamental result.

**Theorem 1.** Let $E$ be a BK-AK-space such that $S(E')$ is complemented in $E'$ with separable complement $L$. Let $M = S(E')^\perp$ be the annihilator of $S(E')$ in $E''$. Let $F$ be any subspace of $E^\beta$ containing $\Phi$ and suppose $F^\beta = E^\beta$ (= $E'$). Then $\sigma(E', F + M)$ and $\sigma(E', \overline{F} + M)$ have the same convergent sequences.

**Proof.** We need some preparations. We may assume that $E$ has a monotone norm (see [W, p. 104]). Let $p_1: E' \to S(E')$, $p_2: E' \to L$ be the projection operators corresponding with the decomposition $E' = S(E') \oplus L$. Notice that $E'' = S(E')^\perp \oplus L^\perp$, $L' = S(E')^\perp = M$. We define norm continuous linear operators $Q_n: E' \to E'$, $n \in \mathbb{N}$, by

$$Q_n y = P_n \circ p_1 y + p_2 y.$$

Then we have

$$||y - Q_n y|| = ||p_1 y - P_n \circ p_1 y|| \to 0.$$

We have to prove that $\sigma(E', F + M)$-convergent sequences are $\sigma(E', \overline{F} + M)$-convergent. To this end, it suffices to prove that every $\sigma(E', F + M)$-null
sequence is bounded in norm. Indeed, suppose this has been proved for a
$\sigma(E', F + M)$-null sequence $(y^n)$, $\|y^n\| \leq K$, say. Then, for $x \in F$ fixed and $\varepsilon > 0$ choose $x \in F$ having $\|x - x\| < \varepsilon/K$. Then

$$\langle x, y^n \rangle \leq K\|x - x\| + \|\langle x, y^n \rangle \| < \varepsilon$$

for $n \geq n(\varepsilon)$.

Let $(y^n)$ be a $\sigma(E', F + M)$-null sequence and assume it is not bounded in
norm, $\|y^n\| \geq n2^n$, say. Let $v^n = y^n/n$.

I. There exist strictly increasing sequences $(k_j), (n_j)$ of integers such that
the following conditions (1) and (2) are satisfied:

(1) $\|Q_{k_1}v_{n_j}\| \leq 2^{-j}$, $j = 1, 2, \ldots$,

(2) $\|v^n - Q_{k_j}v_{n_j}\| \leq 2^{-j}$, $j = 1, 2, \ldots$.

Suppose $k_1, \ldots, k_j$ and $n_1, \ldots, n_j$ have already been defined in accor-
dance with (1) and (2). We claim that $\|Q_{k_j}v^n\| \to 0$ ($n \to \infty$). Since $(y^n)$ is
$\sigma(E', F + M)$-null, $(p_2y^n)$ is bounded for $\sigma(L, M)$, hence is norm bounded,
therefore $\|p_2v^n\| \to 0$. On the other hand, $y^n = p_1y^n + p_2y^n$ implies that $(p_1y^n)$
is $\sigma(E', F + M)$-bounded, hence $(p_1v^n)$ is $\sigma(E', F + M)$-null, hence is co-
ordinatewise null in view of $\Phi \subset F$. Clearly this implies $\|P_{k_j}v^n\| \to 0$, proving our claim. But now it is clear that a choice of $n_{j+1} > n_j$ satisfying (1)
is possible.

Next observe that $\|v^{n_{j+1}} - Q_{k_j}v^n_{n_{j+1}}\| \to 0$ ($k \to \infty$). This permits a choice of
$k_{j+1} > k_j$ in accordance with (2).

II. Let $z^j = Q_{k_j}v_{n_j} - Q_{k_{j-1}}v_{n_j} = P_{k_j}p_1v_{n_j} - P_{k_{j-1}}p_1v_{n_j}$, and let $\alpha_j = 1/\|z^j\|$. Then $(\alpha_j)$ is an $l^1$-sequence by (1), (2). Observe that $\alpha_jz^j \to 0$ with respect
to $\sigma(E', F + M)$, but $\|\alpha_jz^j\| = 1$. Therefore, a result of Pelczyński [P] guar-
antees the existence of a basic subsequence $(\alpha_jz^{j_h})$ of $(\alpha_jz^j)$. To simplify
the reasoning in the following, we assume that $(\alpha_jz^j)$ itself is a basic sequence
in $E'$.

III. We claim the existence of a null sequence $(\lambda_j)$ such that the sequence
$z$, defined by

$$(*) \quad z_k = \lambda_j\alpha_jz^j_k \quad \text{for } k_{j-1} < k \leq k_j,$$

is not an element of $S(E')$.

Let $G$ denote the subspace of $E'$ consisting of all sequences

$$z = \sum_{j=1}^{\infty} \lambda_j\alpha_jz^j,$$
where \((\lambda_j)\) is in \(c_0\) and the series converges in norm. Define a linear operator 
\(\varphi: G \to c_0\) by setting

\[
\varphi(z) = \varphi \left( \sum_{j=1}^{\infty} \lambda_j z^j \right) = (\lambda_j).
\]

\(\varphi\) is well defined since \((\alpha_j z^j)\) is a basic sequence by assumption. We prove
that \(\varphi\) is continuous. Let \(z \in G\), \(z = \sum \lambda_j z^j\). Then

\[
|\lambda_j| = ||\lambda_j \alpha_j z^j|| = \left\| \sum_{i=1}^{j} \lambda_i \alpha_i z^i - \sum_{i=1}^{j-1} \lambda_i \alpha_i z^i \right\| \\
= \left\| P_k z - P_{k-1} z \right\| \leq 2||z||,
\]

the latter in view of the monotonicity of the norm on \(E\) (and thus on \(E'\)). This proves
that \(\varphi\) is continuous.

Let \(\tilde{G}\) be the norm closure of \(G\) in \(E'\). Then \(\varphi\) extends to a continuous,
linear operator \(\tilde{\varphi}: \tilde{G} \to c_0\). Let \(z \in \tilde{G}\), then \(z = \sum \lambda_j \alpha_j z^j\) for some sequence
\((\lambda_j)\), since \((\alpha_j z^j)\) is a basic sequence. But notice that \(\tilde{\varphi}(z) = (\lambda_j)\) by a \(K\)-space
argument. So actually \((\lambda_j)\) is in \(c_0\), hence \(z \in G\), proving \(G = \tilde{G}\).

Notice that \(\varphi\) is a continuous injection. This proves that \(\varphi\) is not surjective.
For supposing it were, it would be a homeomorphism by the open mapping
theorem, i.e. we would have \(G \approx c_0\). But this is absurd, since no separable
dual space may contain a copy of \(c_0\). So \(\varphi\) is not surjective. Let \((\lambda_j)\) be any
null sequence which is not in the range of \(\varphi\). We prove that \(z\), defined by \((\ast)\),
is not in \(S(E')\). Indeed, \(z \in S(E')\) would imply \(||z - P_k z|| \to 0\) \((j \to \infty)\).
But note that

\[
P_k z = \sum_{i=1}^{j} \lambda_i \alpha_i z^i,
\]

hence \(z\) would be in \(G\), which was excluded. This ends step III.

IV. We prove that \((P_k z)\) is \(\sigma(E', F + M)\)-convergent with limit \(z\). Indeed,
let \(x \in F + M\), \(k \in \mathbb{N}\), \(k_{j-1} < k \leq k_j\). Then we have

\[
\langle x, P_k z \rangle = \sum_{i=1}^{j-1} \lambda_i \alpha_i \langle x, z^i \rangle + \lambda_j \alpha_j \langle x, P_k z^j \rangle.
\]

Here the first summand converges \((k \to \infty, k_{j-1} < k \leq k_j)\) since \(\langle x, z^i \rangle \to 0\)
and \((\alpha_j) \in l^1\). But the second summand converges as well in view of \(\lambda_j \to 0\)
\((k \to \infty, k_{j-1} < k \leq k_j)\) and

\[
|\alpha_j \langle x, P_k z^j \rangle| = ||P_k x, \alpha_j z^j|| \leq ||P_k x|| ||\alpha_j z^j|| \leq ||x||.
\]

In view of \(F^\beta = E'\) this implies \(z \in E'\) and so \(P_k z \to z\) in \(\sigma(E', F + M)\).
Now observe that the operators $Q_r$ are $\sigma(E', F+M)$-continuous, so $Q_r(P_k z) \to Q_r z$ ($k \to \infty$), proving $P_r z = Q_r z$, hence $z \in S(E')$. But this contradicts step III and therefore ends the proof. □

In the case where $S(E') = E'$, i.e., when $E'$ has sectional convergence, the proof may be simplified. Here we have $M = \{0\}$, $Q_n = P_n$. This yields the following.

**Corollary 1.** Let $E$ be a BK-AK-space such that $E'$ is as well BK-AK. Let $F$ be a subspace of $E''$ containing $\Phi$ and satisfying $F^\beta = E^\beta$ ($= E'$). Then $\sigma(E', F)$ and $\sigma(E', \overline{F})$ have the same convergent sequences. □

**Spaces with the Wilansky property**

An FK-space $E$ is said to have the Wilansky property if every subspace $F$ of $E$ satisfying $F^\beta = E^\beta$ is barrelled in $E$ (see [B2]). In [B2] and [S] it is proved that every BK-AK-space $E$ whose dual $E'$ is as well a BK-AK-space has the Wilansky property. Here we obtain:

**Theorem 2.** Let $E$ be a BK-AK-space such that $S(E')$ has a separable complement $L$ in $E'$. Let $G$ be any FK-space having $E \subset G \subset E^{\beta\beta}$. Then $G$ has the Wilansky property if and only if $E$ is of finite codimension in $G$.

*Proof. Necessity.* Suppose $E$ is of infinite codimension in $G$. Let $(y^n)$ be a linearly independent sequence in $G \setminus E$. Since $E^\beta = G^\beta$, $E$ is barrelled as a subspace of $G$, hence is closed in $G$. But now $F = E + \text{lin}\{y^n: n \in \mathbb{N}\}$ is a subspace of $G$ having $F^\beta = G^\beta$ which is not barrelled. Indeed, we may define a sequence $(f_n)$ in $G'$ such that $f_n$ is 0 on $E + \text{lin}\{y^1, \ldots, y^{n-1}\}$ and satisfies $f_n(y^n) = n|y^n|$ (for some continuous seminorm $| \cdot |$ on $G$). Then $f_n \to 0$, $\sigma(G', F)$, but $(f_n)$ is not bounded in $G'$.

*Sufficiency.* Let $F$ be a subspace of $G$ with $F^\beta = G^\beta$. We may assume that $F$ contains $\Phi$ (see [B2, Theorem 1]).

Let $U$ be a barrel in $F$. Since $M \cap E = \{0\}$, the space $F \cap M$ is finite dimensional. Let $S$ be some topological complement of $F \cap M$ in $M$. Let $B$ denote the unit ball in $S$. Note that $B$ is $\sigma(S'' , E')$-compact, since the unit ball in $M \approx L'$ is weak * compact and $S$ is of finite codimension in $M$. Now let $V = U + B$. Then $V^0$, the $\langle E'' , E' \rangle$-polar of $V$, is $\sigma(E', F + M)$-bounded, since $V$ spans $F + M$. By Theorem 1, $\sigma(E', F + M)$-bounded sets are norm bounded in $E'$, so that $V^0$ is norm bounded in $E'$. Hence $V^{00}$ is a norm neighbourhood of 0 in $E''$, hence $V^{00} \cap F$ is a norm neighbourhood of 0 in $F$, since $G$ (and hence $F$) must have the topology induced by $E''$. We end the proof by showing $V^{00} \cap F \subset U$. By the definition of $V$, we have $V^{00} = \overline{U} + B$, the closure being taken in $\sigma(E'', E')$, since $B$ is $\sigma(E'', E')$-compact. But $V^{00} \cap F = \overline{U} \cap F$ in view of $B \cap F = \{0\}$. Since $F$ has only finitely many dimensions "outside $E"$, we deduce that $\overline{U} \cap F = U$, which ends the proof of Theorem 2. □
Corollary 2. [B₂, § 6]. \( c \) and \( cs \) have the Wilansky property. □

More generally, a \( BK\text{-}AK \)-space \( E \) has the Wilansky property if \( S(E') \) is of finite codimension in \( E' \), and the same is true for any \( G \) having \( E \subset G \subset E^{\beta\beta} \) such that \( E \) is of finite codimension in \( G \). In a forthcoming paper [NS], we use this fact to prove that for every invertible permanent triangular matrix \( A \) whose inverse \( A^{-1} \) is a bidiagonal matrix, the convergence domain \( c_A \) has the Wilansky property.

**Remark.** In Theorems 1, 2, the assumption that \( E \) has separable dual may be replaced by any condition ensuring that \( c_0 \) does not embed into \( E' \). See for instance [Kw].

**Separable Wilansky property**

It is clear from Theorem 2 that the bidual \( E'' \) of a \( BK\text{-}AK \)-space \( E \) whose dual \( E' \) is as well \( BK\text{-}AK \) does not have the Wilansky property unless \( E \) has finite codimension in \( E'' \). Nevertheless, the bidual space \( E'' \) satisfies some weaker Wilansky type property, which might be called the separable Wilansky property.

**Theorem 3.** Let \( E \) be a solid \( BK\text{-}AK \)-space whose dual \( E' \) is as well \( BK\text{-}AK \). Let \( D \) be a norm dense subspace of \( E'' \) containing \( \Phi \) and satisfying \( D^\beta = E^\beta \) (= \( E' \)). Then every separable FK-space \( F \) which contains \( D \), actually contains \( E'' \).

**Proof.** Let \( x \in E'' \) be fixed. Since \( D \) is a norm dense in \( E'' \), it is also \( \tau(E'', E') \)-sequentially dense in \( E'' \), i.e. there exists a sequence \( (x^n) \) in \( D \) which converges to \( x \) in \( \tau(E'', E') \). We claim that \( \tau(E'', E')|D = \tau(D, E') \).

Indeed, by Theorem 1, \( \sigma(E', D) \) and \( \sigma(E', E'') \) have the same convergent sequences, hence the same compact sets [W, p. 252]. This implies \( \tau(E'', E')|D = \tau(D, E') \).

Consequently, the sequence \( (x^n) \) is Cauchy in \( (D, \tau(D, E')) \). We prove that the inclusion mapping \( i: (D, \tau(D, E')) \rightarrow F \) is continuous. This is a consequence of Kalton’s closed graph theorem (see [BK₂, Theorem 5]), for \( \sigma(E', D) \) is sequentially complete. Indeed, since \( \sigma(E', D) \) and \( \sigma(E', E'') \) have the same convergent sequences, they also have the same Cauchy sequences. But \( \sigma(E', E'') \) is sequentially complete as a consequence of the fact that \( E \), and hence \( E'' = E'' \), is solid. This proves that \( \sigma(E', D) \) is sequentially complete.

Since \( i: (D, \tau(D, D^\beta)) \rightarrow F \) is continuous, the sequence \( (x^n) \) is Cauchy in \( F \), and hence converges to some \( \overline{x} \in F \). From K-space reasons, we have \( x = \overline{x} \), proving \( x \in F \). □

Certainly, in Theorem 3, the solidity of the space \( E \) may be replaced by the condition that \( \sigma(E', E'') \) is sequentially complete.

**Corollary 3.** (Compare [BK₁, Theorem 3].) Let \( F \) be a separable FK-space containing \( \Phi \) and suppose \( F \cap l^\infty \) is norm dense in \( l^\infty \). Then \( l^\infty \subset F \).
Proof. This follows from Theorem 3 and the fact that every norm dense subspace $D$ of $l^\infty$ satisfies $D^\beta = l^1$ (see [W, Lemma 16.3.3]). \qed

The result of Bennett and Kalton has been generalized by Snyder [Sn] to a nonseparable version. He proves that every FK-space $F$ containing $\Phi$ and satisfying $F + c_0 = l^\infty$ must have $F = l^\infty$.

**Scarce copies**

The concept of scarce copies of sequence spaces has been introduced by Bennett [B1]. He proves that every scarce copy of $\omega$ and $l^1$ is barrelled, but that all other standard sequence spaces do not have this property. For instance, $c_0$ does not have any barrelled scarce copy at all (see [B1] for details). Here we obtain another negative result on the barrelledness of scarce copies.

**Theorem 4.** Let $E$ be a FK-AB-space contained in $l^\infty$ such that $E^\gamma \subset bs$. Then $E$ does not have any barrelled scarce copy.

**Proof.** Suppose $\Sigma(E, r)$ is a barrelled scarce copy of $E$. This implies $\Sigma(E, r)^\beta \subset E^\gamma = E^\gamma$, the latter since $E$ has AB (see [W, p. 167]). Therefore $\Sigma(E, r)^\beta \subset bs$.

We prove that $\Sigma(c_0, r)$ is a barrelled scarce copy of $c_0$, thus obtaining a contradiction, since $c_0$ has no barrelled scarce copies. Since $c_0$ has the Wilansky property, barrelledness of $\Sigma(c_0, r)$ will be a consequence of $\Sigma(c_0, r)^\beta \subset l^1$. So let $y \notin l^1$. Since $c_0^\gamma = l^1$, there exists $x \in c_0$ such that $xy \notin bs$, hence $xy \notin \Sigma(E, r)^\beta$. Let $z \in \Sigma(E, r)$ be chosen with $xyz \notin cs$. By the definition of $\Sigma(E, r)$, there exist $z^1, \ldots, z^n \in \sigma(E, r)$ having $z = z^1 + \cdots + z^n$. This implies $xyz^i \notin cs$ for some $i$. We claim that $xz^i \in \sigma(c_0, r) \subset \Sigma(c_0, r)$. Since $z^i \in E \subset l^\infty$, we have $xz^i \in c_0$. On the other hand,

$$c_n(xz^i) \leq c_n(z^i) \leq r_n$$

for every $n$ implies $xz^i \in \sigma(c_0, r)$. This proves $y \notin \Sigma(c_0, r)^\beta$. \qed

**References**


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