ON $q$-DERANGEMENT NUMBERS

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Abstract. We derive a $q$-analogue of the classical formula for the number of derangements of an $n$ element set. Our derivation is entirely analogous to the classical derivation, but relies on a descent set preserving bijection between the set of permutations with a given derangement part and the set of shuffles of two permutations.

A classical application of binomial inversion (more generally the principle of inclusion–exclusion) is the derivation of the formula for the number of derangements (permutations with no fixed points) of an $n$ element set:

$$d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

This is obtained by counting permutations according to their number of fixed points and then inverting the resulting equation.

In this note we shall derive a formula of I. Gessel [G] for $q$-counting derangements by the major index statistic in a way entirely analogous to the classical $q = 1$ case. That is, we shall $q$-count permutations with $k$ fixed points and then use Gauss inversion ($q$-binomial inversion or more generally Möbius inversion on the lattice of subspaces of a vector space) to derive the following formula for $q$-derangement numbers:

$$d_n(q) = [n!] \sum_{k=0}^{n} \frac{(-1)^k}{[k]!} q \binom{k}{2}.$$

A key step in our derivation and an interesting result in its own right is a descent-preserving bijection between the set of permutations with a given derangement part and the set of shuffles of two permutations. This bijection enables us to use a formula of A. Garsia and I. Gessel for $q$-counting shuffles.

Gessel [G] obtained the formula for $q$-derangement numbers as a corollary of an Eulerian generating function formula for counting permutations by descents, major index, and cycle structure, which is proved via a correspondence

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between partitions and permutations. \( q \)-Derangement numbers have also been interpreted combinatorially on sets of permutations bijectively associated with derangements by A. Garsia and J. Remmel [GR] using the inversion index statistic and by J. Désarménien [D 2 ] (see [D 1 ] and [DW]) using the major index and inversion index statistics.

We shall briefly review some permutation statistic notation and terminology. For each integer \( n \geq 1 \), let \([n]\) denote the polynomial \( 1 + q + q^2 + \cdots + q^{n-1} \) and let \([n]!\) denote the polynomial \([n][n-1] \cdots [1]\). Also \([0]!\) is taken to be 1. The \( q \)-binomial coefficients are given by

\[
\binom{n}{k} = \frac{[n]!}{[k]![n-k]!}
\]

for \( 0 \leq k \leq n \).

For any positive integer \( n \), let \( \langle n \rangle \) denote the set \( \{1, 2, \ldots, n\} \). We shall think of permutations in the symmetric group \( S_n \) as words with \( n \) distinct letters in \( \langle n \rangle \). More generally, for a set \( A \) of \( n \) positive integers, \( S_A \) denotes the set of permutations of \( A \) or words with \( n \) distinct letters in \( A \). The descent set of a permutation \( \sigma = \sigma_1, \sigma_2, \ldots, \sigma_n \) is \( \text{des}(\sigma) = \{i \in \langle n-1\rangle | \sigma_i > \sigma_{i+1}\} \). The major index of \( \sigma \) is \( \text{maj}(\sigma) = \sum_{i \in \text{des}(\sigma)} i \). Let us recall MacMahon's [M] formula for \( \text{maj} \)-counting permutations in \( S_n \):

\[
\sum_{\sigma \in S_n} q^{\text{maj}(\sigma)} = [n]!.
\]

A letter \( i \in A \) is said to be a fixed point of \( \sigma \in S_A \) if \( \sigma(i) = i \). A permutation with no fixed points is called a derangement. Let \( D_n \) denote the set of all derangements in \( S_n \). The \( q \)-derangement numbers are defined by

\[
d_n(q) = \sum_{\sigma \in D_n} q^{\text{maj}(\sigma)}.
\]

It will be convenient to view the empty word \( \Lambda \) as a derangement and to define \( D_0 \) to be the set \( \{\Lambda\} \). We also let \( \text{maj}(\Lambda) = 0 \) and \( d_0(q) = 1 \).

For any permutation \( \alpha \in S_A \), where \( A = \{a_1 < a_2 < \cdots < a_k\} \), define the reduction of \( \alpha \) to be the permutation in \( S_k \) obtained from \( \alpha \) by replacing each letter \( a_i \) by \( i \), \( i = 1, 2, \ldots, k \). The derangement part of a permutation \( \sigma \in S_n \), denoted \( dp(\sigma) \), is the reduction of the subword of nonfixed points of \( \sigma \). For example, \( dp(5,3,1,4,7,6,2) = 4,3,1,5,2 \). We shall use the convention that the derangement part of the identity permutation is the empty word \( \Lambda \). Note that the derangement part of a permutation is a derangement, and that conversely, any derangement in \( D_k \) and \( k \) element subset of \( \langle n \rangle \) determines a permutation in \( S_n \) with \( n-k \) fixed points. Hence, the number of permutations in \( S_n \) with a given derangement part in \( D_k \) is \( \binom{n}{k} \). Our goal is to \( q \)-count permutations with a given derangement part.

Let \( \alpha \in D_k \). There is an obvious bijection between the set \( \{\sigma \in S_n | dp(\sigma) = \alpha\} \) and the set \( \text{Sh}(\alpha, \beta) \) of all shuffles of \( \alpha \) and \( \beta = k + 1, k + 2, \ldots, n, \)
i.e. permutations in $\mathcal{S}_n$ which contain $\alpha$ and $\beta$ as complementary subwords. Indeed, for each permutation $\sigma$ in the former set, replace the subword of non-fixed points of $\sigma$ by $\alpha$ and the complementary subword of fixed points by $\beta$. A very useful result of Garsia and Gessel [GG, Theorem 3.1] allows us to $q$-count the latter set. Unfortunately, since the above-mentioned bijection does not preserve the major index, it does not help us in $q$-counting the former set. However, we shall show that there is another bijection between these two sets of permutations which, in fact, preserves descent sets.

Define a letter $\sigma_i$ of $\sigma = \sigma_1, \sigma_2, \ldots, \sigma_n \in \mathcal{S}_n$ to be an excédant of $\sigma$ if $\sigma_i > i$ and a subcedant of $\sigma$ if $\sigma_i < i$. Let $s(\sigma)$ and $e(\sigma)$ be the number of subcedants and excédants, respectively, of $\sigma$. We now fix $n$ and let $k \leq n$. For $\sigma \in \mathcal{S}_k$, let $\tilde{\sigma}$ be the permutation of $k$ letters obtained from $\sigma$ by replacing its $i$th smallest subcedant by $i$, $i = 1, 2, \ldots, s(\sigma)$, its $i$th smallest fixed point by $s(\sigma) + i$, $i = 1, 2, \ldots, k - s(\sigma) - e(\sigma)$, and its $i$th largest excédant by $n - i + 1$, $i = 1, 2, \ldots, e(\sigma)$. Note that $\tilde{\sigma}$ depends on $n$ as well as $\sigma$. For example, if $\sigma = 32654\underline{1}J$ (with subcedants underlined and excédants overlined) and $n = 8$ then $\tilde{\sigma} = 638721$. If $k = n$ then $\tilde{\sigma} \in \mathcal{S}_n$. If $\sigma$ is a derangement then $\tilde{\sigma} \in \mathcal{S}_A$, where $A = \{1, 2, \ldots, s(\sigma)\} \cup \{n - e(\sigma) + 1, n - e(\sigma) + 2, \ldots, n\}$.

Lemma 1. Let $\sigma \in \mathcal{S}_k$, $k \leq n$. Then $\text{des}(\sigma) = \text{des}(\tilde{\sigma})$.

Proof. Suppose $\sigma = \sigma_1, \sigma_2, \ldots, \sigma_k$ and $\tilde{\sigma} = \tilde{\sigma}_1, \tilde{\sigma}_2, \ldots, \tilde{\sigma}_k$. For each $i \in (k - 1)$, we shall show $i \in \text{des}(\sigma)$ if and only if $i \in \text{des}(\tilde{\sigma})$, by considering the nine possible designations of subcedant $(s)$, excédant $(e)$, and fixed point $(f)$ to $\sigma_i$ and $\sigma_{i+1}$. First note that if $\sigma_i$ is a subcedant of $\sigma$ then $\tilde{\sigma}_i \leq \sigma_i$ and if $\sigma_i$ is an excédant of $\sigma$ then $\tilde{\sigma}_i \geq \sigma_i$.

Cases 1–3. Suppose $(\sigma_i, \sigma_{i+1})$ is an $(s, s)$, $(e, e)$, or $(f, f)$ pair. It is then clear that $\sigma_i < \sigma_{i+1}$ if and only if $\tilde{\sigma}_i < \tilde{\sigma}_{i+1}$.

Case 4. Suppose $(\sigma_i, \sigma_{i+1})$ is a $(s, e)$ pair. Then we have

$$\tilde{\sigma}_i \leq \sigma_i < i < i + 1 < \sigma_{i+1} < \tilde{\sigma}_{i+1},$$

which shows that $i \notin \text{des}(\sigma)$ and $i \notin \text{des}(\tilde{\sigma})$.

Case 5. Suppose $(\sigma_i, \sigma_{i+1})$ is a $(s, f)$ pair. Now we have

$$\sigma_i < i < i + 1 = \sigma_{i+1} \quad \text{and} \quad \tilde{\sigma}_i \leq s(\sigma) < \tilde{\sigma}_{i+1},$$

which shows that $i \notin \text{des}(\sigma)$ and $i \notin \text{des}(\tilde{\sigma})$.

Case 6. Suppose $(\sigma_i, \sigma_{i+1})$ is a $(f, s)$ pair. Then since $\sigma_{i+1} < i + 1$ and $\sigma_i = i$, we have

$$\sigma_{i+1} < \sigma_i \quad \text{and} \quad \tilde{\sigma}_{i+1} \leq s(\sigma) < \tilde{\sigma}_i.$$

This shows that $i \in \text{des}(\sigma)$ and $i \in \text{des}(\tilde{\sigma})$.

Cases 7–9. The remaining three cases are that $(\sigma_i, \sigma_{i+1})$ is a $(f, e)$, $(e, s)$, or $(e, f)$ pair. These cases are handled similarly to the previous three cases and are left to the reader. □
Theorem 2. Let \( \alpha \in D_k, \ k \leq n, \) and \( \gamma = s(\alpha) + 1, s(\alpha) + 2, \ldots, n - e(\alpha). \) Then the map \( \varphi: \{ \sigma \in \mathcal{S}_n \mid dp(\sigma) = \alpha \} \to \text{Sh}(\tilde{\alpha}, \gamma) \) defined by \( \varphi(\sigma) = \tilde{\sigma} \) is a bijection which preserves descent sets, i.e. \( \text{des}(\sigma) = \text{des}(\varphi(\sigma)) \). Consequently, for all \( J \subseteq (n - 1) \),

\[
|\{ \sigma \in \mathcal{S}_n \mid dp(\sigma) = \alpha, \ \text{des}(\sigma) = J \}| = |\{ \sigma \in \text{Sh}(\tilde{\alpha}, \gamma) \mid \text{des}(\sigma) = J \}|.
\]

Proof. In view of Lemma 1, we need only show that \( \varphi \) is an invertible map whose image is \( \text{Sh}(\tilde{\alpha}, \gamma) \). First, we claim that if \( dp(\alpha) = \alpha \) then \( \tilde{\sigma} \) is obtained from \( \sigma \) by replacing the subword of nonfixed points of \( \sigma \) by \( \tilde{\alpha} \) and the subword of fixed points of \( \sigma \) by \( \gamma \). Indeed, the subword of fixed points of \( \sigma \) is replaced by the word \( s(\sigma) + 1, s(\sigma) + 2, \ldots, n - e(\sigma), \) which is precisely \( \gamma \) since \( s(\sigma) = s(\alpha) \) and \( e(\sigma) = e(\alpha). \) Also since \( \alpha \) is the reduction of the subword of nonfixed points of \( \sigma \), the position of the \( i \)th smallest subcedant of \( \alpha \) is the same as the position of the \( i \)th smallest subcedant of \( \sigma \) in the subword of nonfixed points. The same is true for the \( i \)th smallest excédant. Hence each subcedant and excédant of \( \sigma \) is replaced by the same letter that replaces the corresponding subcedant or excédant of \( \alpha \). This means that the subword of subcedants and excédants of \( \sigma \) is replaced by \( \tilde{\alpha} \). We may now conclude that \( \tilde{\sigma} \in \text{Sh}(\tilde{\alpha}, \gamma) \).

The above description of \( \tilde{\sigma} \) as a shuffle of \( \tilde{\alpha} \) and \( \gamma \) also implies that \( \varphi \) is invertible. Indeed, if we replace the \( \tilde{\alpha} \) subword of any \( \tau \in \text{Sh}(\tilde{\alpha}, \gamma) \) by the permutation, of the subword positions, whose reduction is \( \alpha \), and the letters of the \( \gamma \) subword by their positions, we obtain a unique permutation \( \sigma \in \mathcal{S}_n \) such that \( dp(\sigma) = \alpha \) and \( \varphi(\sigma) = \tau. \)

Remark. Although a descent set preserving bijection between \( \{ \sigma \in \mathcal{S}_n \mid dp(\sigma) = \alpha \} \) and \( \text{Sh}(\alpha, \beta) \), where \( \beta = k + 1, k + 2, \ldots, n \), will not be needed in the sequel, we should point out here that one can be constructed by composing the bijection \( \varphi \) with a bijection between \( \text{Sh}(\alpha, \beta) \) and \( \text{Sh}(\tilde{\alpha}, \gamma) \) described in [BW, Proof of Proposition 4.1].

Corollary 3. Let \( \alpha \in D_k \) and \( k \leq n. \) Then

\[
\sum_{\substack{\alpha \in \mathcal{S}_n \mid dp(\sigma) = \alpha \}} q^{\text{maj}(\sigma)} = q^{\text{maj}(\alpha)} \binom{n}{k}.
\]

Proof. Since \( \text{maj}(\sigma) \) depends only on \( \text{des}(\sigma), \) it follows from Theorem 2 that

\[
\sum_{dp(\sigma) = \alpha} q^{\text{maj}(\sigma)} = \sum_{\sigma \in \text{Sh}(\tilde{\alpha}, \gamma)} q^{\text{maj}(\sigma)} = q^{\text{maj}(\tilde{\alpha})} \binom{n}{k},
\]

with the last step following from Garsia–Gessel [GG, Theorem 3.1]. (For a bijective alternative proof and generalization of the Garsia–Gessel result, see [BW].) By Lemma 1, \( \text{maj}(\tilde{\alpha}) = \text{maj}(\alpha), \) which completes the proof.

Theorem 4. For all \( n \geq 0, \)

\[
d_n(q) = [n!] \sum_{k=0}^{n} \frac{(-1)^k}{[k]!} q^{\binom{k}{2}}.
\]
Proof. By \textit{maj}-\textit{q}-counting the permutations in $\mathcal{S}_n$ according to derangement part and applying Corollary 3, we obtain

\[
[n]! = \sum_{\sigma \in \mathcal{S}_n} q^{\text{maj}(\sigma)}
= \sum_{k=0}^{n} \sum_{\alpha \in D_k} \sum_{d_p(\sigma) = \alpha} q^{\text{maj}(\sigma)}
= \sum_{k=0}^{n} \sum_{\alpha \in D_k} q^{\text{maj}(\alpha)} \binom{n}{k}
= \sum_{k=0}^{n} \binom{n}{k} d_k(q).
\]

Gauss inversion \cite[A, p. 96]{A} on the resulting equation yields,

\[
d_n(q) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} q^{\binom{n-k}{2}} [k]!
= \sum_{k=0}^{n} \frac{[n]!}{[n-k]!} (-1)^{n-k} q^{\binom{n-k}{2}},
\]

which is equivalent to the desired formula. \hfill \Box

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