A SEMI-FREDHOLM PRINCIPLE
FOR PERIODICALLY FORCED SYSTEMS
WITH HOMOGENEOUS NONLINEARITIES

A. C. LAZER and P. J. MCKENNA

Abstract. We show that if the potential in a second-order Newtonian system of differential equations is positively homogeneous of degree two and positive semidefinite, and if the unforced system has no nontrivial \( T \)-periodic solutions \((T > 0)\), then for any continuous \( T \)-periodic forcing, there is at least one \( T \)-periodic solution.

In this note we consider a system of second-order differential equations in which the nonlinear terms have one nice property shared by linear functions: they are positively homogeneous of degree one. One such system arises from an idealized model of a suspension bridge considered previously in [4 and 7].

Consider the partial differential equation
\[
U_{tt} + a^2 U_{xxxx} + k(x) U^+ = f(x, t),
\]
where \( U = U(x, t), \ 0 \leq x \leq L \), with boundary conditions
\[
U(0, t) = U(L, t) = U_{xx}(0, t) = U_{xx}(L, t) = 0.
\]
If \( k(x) \equiv 0 \), we have the well-known equation for the transverse vibrations of a beam of length \( L \), which is hinged at the endpoints, subject to an external force given by \( f(x, t) \). If the beam is suspended from above by cables, whose strengths may vary from point to point, and the displacement \( U \) is measured in the downward direction, we add the term \( k(x) U^+ \), where \( k(x) \geq 0 \) and \( U^+ \) is the positive part of \( U \). This accounts for the fact that cables exert no restoring force when slack.

When \( f \) is periodic in \( t \) it is natural to look for time-periodic solutions. We consider an approximate problem obtained by discretizing in the space variable \( x \). If \( N \) is a large positive integer and we consider the mesh points \( x_k = kL/N, \ k = 1, \ldots, N - 1 \), we are lead to the system of ordinary differential equations
\[
u''(t) + Au(t) + Du(t)^+ = p(t),
\]
Received by the editors March 27, 1983 and, in revised form, May 31, 1988. 1980 Mathematics Subject Classification (1985 Revision). Primary 34C25. Key words and phrases. Leray-Schauder continuation method, periodic solution. The first author was partially supported by NSF under Grant DMS9519882. The second author was partially supported by NSF under Grant DMS8519726.

©1989 American Mathematical Society 0002-9939/89 $1.00 + $.25 per page

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

119
where \( u(t) \) is the \((N-1)\)-dimensional column vector whose \( k \) th component is \( U(x_k, t) \), \( p(t) \) is the vector with \( k \) th component \( f(x_k, t) \), \( u(t)^+ \) has its \( k \) th component equal to \( U(x_k, t)^+ \), \( D \) is a diagonal matrix with nonnegative entries, and \( A \) is the symmetric, positive-definite matrix corresponding to the finite difference approximation to \( U_{xxxx} \), taking into account the boundary conditions. We consider a more general class of systems.

Let \( V \in C^1(\mathbb{R}^n, \mathbb{R}) \) be positively homogeneous of degree two, i.e., \( V(tx) = t^2V(x) \) for \( t \geq 0 \) and \( x \in \mathbb{R}^n \) \((n \geq 1)\). If, for brevity, we set \( V'(x) = \text{grad} \ V(x) \), then \( V' \) is positively homogeneous of degree one, i.e.,

\[
V'(tx) = tV'(x)
\]
for \( t \geq 0 \) and \( x \in \mathbb{R}^n \). If \( (\ , \ ) \) denotes the usual inner product on \( \mathbb{R}^n \), then for Euler's identity we have

\[
(x, V'(x)) = 2V(x)
\]

for \( x \in \mathbb{R}^n \).

**Theorem 1.** Let \( V \in C^1(\mathbb{R}^n, \mathbb{R}) \) be positively homogeneous of degree two and positive semidefinite, i.e.,

\[
x \in \mathbb{R}^n \Rightarrow V(x) \geq 0.
\]

Let \( T > 0 \). If the system

\[
u''(t) + V'(u(t)) = 0
\]

has no \( T \)-periodic solution other than \( u(t) = 0 \), then for any \( T \)-periodic function \( p \in C^1(\mathbb{R}, \mathbb{R}^n) \) the system

\[
u''(t) + V'(u(t)) = p(t)
\]
has at least one \( T \)-periodic solution.

**Proof.** We first assume the stronger condition that \( V \) be positive definite, i.e.,

\[
x \in \mathbb{R}^n, \quad x \neq 0 \Rightarrow V(x) > 0.
\]

We first show that if \( p \) and \( V \) are as above, then for \( \varepsilon > 0 \) the system

\[
u''(t) + \varepsilon u'(t) + V'(u(t)) = p(t)
\]
has at least \( T \)-periodic solution. To do this we use the Leray-Schauder continuation method (see, for example, [1 or 6]). Let \( || \) denote the usual Euclidean norm on \( \mathbb{R}^n \), and let \( C_T \) and \( C^1_T \) denote the Banach spaces of \( T \)-periodic functions which are continuous and continuously differentiable respectively with norms

\[
|v|_\infty = \sup_{[0,T]} |v(t)|, \quad v \in C_T,
\]
\[
|v|_1 = |v|_\infty + |v'|_\infty, \quad v \in C^1_T.
\]

Since the second-order linear homogeneous differential system

\[
u''(t) + \varepsilon u'(t) + u(t) = 0
\]
A SEMI-FREDHOLM PRINCIPLE

121

(u \in \mathbb{R}^n) has no \( T \)-periodic solution other than \( u(t) \equiv 0 \), it follows that, for every \( f \in C_T \), there exists a unique \( T \)-periodic solution of

\[ u''(t) + \varepsilon u'(t) + u(t) = f(t). \]

Moreover, if we denote the unique \( T \)-periodic solution of the last system by \( K(f) \), then \( K \) may be viewed as a compact linear operator from \( C_T \) into itself. Let \( N: C_T \to C_T \) be the completely continuous operator defined by

\[ N(u) = K(u + p - V'(u)). \]

We claim that there exists a number \( R > 0 \) such that if \( \lambda \in [0,1] \) and \( u \in C_T \), then

\[ u = \lambda N(u) \]

implies that \( |u|_\infty \leq R \).

Since (1.7) holds if and only if

\[ u''(t) + \varepsilon u'(t) + (1 - \lambda)u(t) + \lambda V'(u(t)) = \lambda p(t), \]

and since \( |u|_\infty \leq |u|_1 \) if \( u \in C_T^1 \), assuming that the above claim is false, we infer the existence of a sequence \( \{u_m\}_{m=1}^\infty \) and a corresponding sequence of numbers \( \{\lambda_m\}_{m=1}^\infty \) such that \( u_m(t) \) is a solution of (1.8) when \( \lambda = \lambda_m \) for \( m = 1, 2, \ldots \), \( \lambda_m \in [0, 1] \), and

\[ |u_m|_1 \to \infty \quad \text{as} \quad m \to \infty. \]

Setting \( w_m(t) = u_m(t)/|u_m|_1 \) for \( m = 1, 2, \ldots \), it follows by homogeneity of \( V' \) that

\[ w_m''(t) + \varepsilon w_m'(t) + (1 - \lambda_m)w_m(t) + \lambda_m V'(w_m(t)) = \lambda_m p(t)/|u_m|_1 \]

for \( m = 1, 2, \ldots \). Since \( |w_m|_1 = 1 \) for \( m \geq 1 \), it follows from (1.10) that the sequence \( \{|w_m|_1\}_{m=1}^\infty \) is bounded. Therefore, both of the sequences \( \{w_m(t)\}_{m=1}^\infty \) and \( \{w'_m(t)\}_{m=1}^\infty \) are equicontinuous and uniformly bounded on \(( -\infty, \infty )\) so, by Ascoli’s lemma, there exists a subsequence \( \{w_{mk} \}_{k=1}^\infty \) and a \( w \in C_T \) with \( |w|_1 = 1 \) such that \( w_{mk}(t) \to w(t) \), \( w'_{mk}(t) \to w'(t) \) as \( m \to \infty \), uniformly on \(( -\infty, \infty )\). Since \( 0 \leq \lambda_{mk} \leq 1 \) for all \( k \geq 1 \), we may assume without loss of generality that \( \lambda_{mk} \to \lambda^* \in [0, 1] \) as \( k \to \infty \). Therefore, from (1.10), it follows that the sequence \( \{w_{mk}(t)\}_{k=1}^\infty \) converges uniformly on \(( -\infty, \infty )\) so \( w \) is of class \( C^2 \) and

\[ w''(t) + \varepsilon w'(t) + (1 - \lambda^*)w(t) + \lambda^* V'(w(t)) = 0. \]

Taking the inner product of (1.11) with \( w'(t) \) and observing that

\[ \int_0^T \langle w'(t), w''(t) \rangle \, dt = \frac{1}{2} \int_0^T \frac{d}{dt} |w'(t)|^2 \, dt = 0 \]

and

\[ \int_0^T \langle w'(t), (1 - \lambda^*)w(t) + \lambda^* V'(w(t)) \rangle \, dt \]

\[ = \int_0^T \frac{d}{dt} \left[ (1 - \lambda^*)|w(t)|^2 / 2 + \lambda^* V(w(t)) \right] \, dt = 0, \]
we find that
\[ \varepsilon \int_0^T |w'(t)|^2 \, dt = 0. \]
Hence, \( w(t) = \xi = \) constant and according to (1.11)
\[ (1 - \lambda^*) \xi + \lambda^* V'(\xi) = 0. \]
Taking the inner product of this last equation with \( \xi \) and using (1.2), we have
\[ (1 - \lambda^*)|\xi|^2 + 2\lambda^* V(\xi) = 0. \]
Thus, since \( 0 \leq \lambda^* \leq 1 \), it follows from (1.3) that \( \xi = 0 \). Since, this
contradicts the fact that \( |w|_1 = 1 \), the claim that there exists \( R \) independent
of \( u \in C_T \) and \( \lambda \in [0, 1] \) such that (1.7) implies that \( |u|_\infty < R \) has
been established.

From the Leray-Schauder-Shaefer theorem, it follows that for each \( \lambda \in [0, 1] \)
there exists a \( u \in C_T \) which satisfies (1.7). (See, for example, [9, 1, p. 61, or 6,
p. 71]). In particular since (1.7) has a solution \( u \) when \( \lambda = 1 \), it follows that
(1.6) has at least one \( T \)-periodic solution.

Let \( \{ \varepsilon_m \}_{m=1}^\infty \) be a sequence of positive numbers such that \( \varepsilon_m \to 0 \) as \( m \to \infty \). By what has been shown for each \( m = 1, 2, \ldots \)
there exists \( u_m \in C_T \) such that \( u_m \) is a solution of (1.6) when \( \varepsilon = \varepsilon_m \). We claim that the sequence
\( \{|u_m|_1\}_{m=1}^\infty \) is bounded. Assuming the contrary, we may suppose without loss
of generality that \( |u_m|_1 \to \infty \) as \( m \to \infty \). Setting \( z_m(t) = u_m(t)/|u_m|_1 \), for
\( m \geq 1 \), we have, by positive homogeneity of \( V' \),
\[ z_m''(t) + \varepsilon_m z_m'(t) + V'(z_m(t)) = p(t)/|u_m|_1 \]
for \( m = 1, 2, \ldots \). From this it follows that the sequences \( \{z_m(t)\}_{m=1}^\infty \) and
\( \{z_m'(t)\}_{m=1}^\infty \) are equicontinuous and uniformly bounded on \( (-\infty, \infty) \) so there
exists \( z \in C_T \) with \( |z|_1 = 1 \) and a subsequence \( \{z_{m_k}(t)\}_{k=1}^\infty \) of \( \{z_m\}_{m=1}^\infty \) such
that \( z_{m_k}(t) \to z(t) \) and \( z_{m_k}'(t) \to z'(t) \) as \( k \to \infty \) uniformly with respect to \( t \in (-\infty, \infty) \). From (1.12), we infer that the sequence \( \{z''_{m_k}(t)\}_{k=1}^\infty \) converges
uniformly on \( (-\infty, \infty) \). Hence \( z \) is of class \( C^2 \) and
\[ z''(t) + V'(z(t)) = 0. \]
Since \( |z|_1 = 1 \), this contradicts the assumption that (1.4) has no nontrivial
\( T \)-periodic solution, our claim that the sequence \( \{|u_m|_1\}_{m=1}^\infty \) is bounded has
been established.

From the differential equation
\[ u_m''(t) + \varepsilon_m u_m'(t) + V'(u_m(t)) = p(t), \]
it follows that the sequence \( \{u_m''(t)\}_{m=1}^\infty \) is also uniformly bounded on \( (-\infty, \infty) \). Therefore, using the same type of argument used above, we infer the existence
of a subsequence of \( \{u_m(t)\}_{m=1}^\infty \) such that this subsequence, as well as the corresponding sequences of first and second derivatives converges uniformly on
Since the limit of this subsequence is a $T$-periodic solution of (1.5), the proof of the theorem under assumption (1.3)$^*$ is complete.

To prove the theorem with (1.3)$^*$ replaced by (1.3), we observe that for all sufficiently small $\delta > 0$, the system

$$u''(t) + \delta u(t) + V'(u(t)) = 0$$

has no nontrivial $T$-periodic solution. Indeed, in the contrary case, for $\delta > 0$ arbitrarily small, we could find a solution with $C^1_T$-norm equal to 1. A compactness argument, similar to those used above, would give a nontrivial solution of (1.4), contradicting one of our hypotheses.

Therefore, since $\delta |x|^2/2 + V(x) > 0$ for $x \neq 0$, by what we have shown, for small $\delta > 0$ there exists a $T$-periodic solution of

$$u'' + \delta u + V'(u) = p(t).$$

The $C^1_T$-norms of these solutions are bounded as $\delta \to 0$ since (1.4) has no nontrivial $T$-periodic solution. Therefore, by the same type of compactness argument as used above, we obtain a $T$-periodic solution of (1.5).

Remark. It does not seem possible to prove the theorem more directly by connecting (1.5) rather than (1.6) to a linear equation by a homotopy.

Remark. The theorem remains true if the condition (1.3) is replaced by

(1.3)$'$

$$x \in \mathbb{R}^N, \quad x \neq 0 \Rightarrow V(x) < 0.$$

In the proof one would consider the parameter-dependent differential equation

$$u''(t) + \varepsilon u'(t) - (1 - \lambda)u(t) + \lambda V'(u(t)) = \lambda p(t)$$

instead of (1.8) and use the fact that the linear differential equation

$$u''(t) + \varepsilon u'(t) - u(t) = f(t),$$

where $\varepsilon > 0$, has a unique $T$-periodic solution for any $f \in C_T$.

Examples 1. The homogeneous nonlinearity in the differential equation

(1.13)

$$u''(t) + |u(t)| = p(t) \equiv p(t + T),$$

where $n = 1$, does not satisfy the condition (1.3), since in this case $V(x) = \frac{1}{2} |\text{sgn } x| x^2$. If $p(t) \neq 0$ and $u(t)$ is a $T$-periodic solution of (1.13), then

$$\int_0^T |u(t)| \, dt = \int_0^T p(t) \, dt,$$

so, in order that (1.13) have a $T$-periodic solution, it is necessary that the mean value of $p$ be positive. If $q(t)$ is a continuous $T$-periodic function with mean value zero, and $p(t) = c + q(t)$ where $c$ is a constant, then by using the well-known upper and lower solution method for periodic solutions of second-order differential equations (see, for example, [10]) one can easily adapt the methods of [2] (which concerns a boundary value problem for a P.D.E.) to prove the
existence of \( \bar{c} = \bar{c}(q) \) such that (1.13) has a \( T \)-periodic solution if and only if

\[ c \geq \bar{c}. \]

2. Consider the differential equation

\[ u^{\prime\prime} + bu^+ - au^- = p(t) \equiv p(t + 2\pi), \]

where \( n = 1, \ a > 0 \) and \( b > 0 \). If \( u \) is a nontrivial solution of

\[ u^{\prime\prime} + bu^+ - au^- = 0, \]

then the distance between two consecutive zeros of \( u \) which border an interval on which \( u \) is positive is \( \pi/\sqrt{b} \), since \( u^{\prime\prime} + bu = 0 \) on such an interval. Similarly the distance between two consecutive zeros of \( u \) which border an interval on which \( u \) is negative is \( \pi/\sqrt{a} \). It follows that every nontrivial solution of (1.15) is periodic with least period \( \pi/\sqrt{a} + \pi/\sqrt{b} \). Therefore, since the potential for the nonlinearity in (1.15), \( V(x) = \frac{b(x^+)^2 + a(x^-)^2}{2} \), satisfies (1.3), it follows that if

\[ \frac{\pi}{\sqrt{b}} + \frac{\pi}{\sqrt{a}} \neq 2\pi/m \]

for \( m = 1, 2, \ldots \), then, for any continuous \( 2\pi \)-periodic function \( p(t) \), (1.14) has a \( 2\pi \)-periodic solution.

3. Suppose that for some integer \( m \geq 1 \) we have

\[ \frac{\pi}{\sqrt{b}} + \frac{\pi}{\sqrt{a}} = \frac{2\pi}{m}, \]

and in addition that

\[ (m - 1)^2 < a < b < (m + 1)^2. \]

Let \( H \) be the Hilbert space consisting of \( 2\pi \)-periodic functions defined on \( (-\infty, \infty) \) whose restrictions to \( [-\pi, \pi] \) belong to \( L^2[-\pi, \pi] \) with the \( L^2[-\pi, \pi] \) inner product. Let \( W \) be the two-dimensional subspace of \( W \) spanned by \( \cos mt \) and \( \sin mt \) and let \( P: H \rightarrow W \) denote orthogonal projection. Since the spectrum of the linear operator \( A: D(A) \subset H \rightarrow H \) defined by \( Au = -u^{\prime\prime} \) is \( \{k^2|k = 0, 1, \ldots\} \) it follows from the Liapunov-Schmidt technique and (1.16) that for any \( w_1 \in W \) there exists a unique \( w_2 \in (I - P)W \) such that

\[ w_2^{\prime\prime} + (I - P)[b(w_1 + w_2)^+ - a(w_1 + w_2)^-] = 0 \]

and \( w_2(t + 2\pi) \equiv w_2(t) \). (See [5], or the proof of Proposition 2.1 of [3] for more details.) If \( u_0 \) is a nonzero solution of (1.15), then \( u_0 \) is \( (2\pi/m) \)-periodic and \( Pu_0 \neq 0 \), for otherwise, since \( w_2 \equiv 0 \) solves (1.17) when \( w_1 \equiv 0 \), we would have \( (I - P)u_0 \), and hence \( u_0 \), identically zero.

We claim that if \( c^2 + d^2 \neq 0 \) then there exists no \( 2\pi \)-periodic solution of

\[ u^{\prime\prime} + bu^+ - au^- = c \cos mt + d \sin mt. \]

Assuming, on the contrary, that there exists a \( 2\pi \)-periodic solution \( u^* \) of (1.18), the reasoning used above shows that \( Pu^* \neq 0 \). Therefore, since both \( Pu_0 \) and \( Pu^* \) are linear combinations of \( \cos mt \) and \( \sin mt \), both have the form
$r \sin(mt + \delta)$ for some $r > 0$. Therefore there exist numbers $\alpha > 0$ and $\gamma$ such that $\dot{u}(t) = \alpha u(t + \gamma)$, then $P\dot{u} = Pu^\ast$. Therefore, since by homogeneity $\dot{u}$ is also a solution of (1.15), it follows that if $w_1 = P\dot{u}$, then both $w_2 = (I - P)\dot{u}$ and $w_2 = (I - P)u^\ast$ solve (1.17), so by uniqueness $(I - P)\dot{u} = (I - P)u^\ast$. Therefore we have the absurdity $\dot{u} = u^\ast$. This contradiction proves the claim.

This phenomenon is, of course, well known as resonance in the linear case $a = b = m^2$.

The work of Podolak [8] shows that if $n = 1$ and periodic boundary conditions are replaced by Dirichlet boundary conditions, then the statement of the theorem does not remain true.

**REFERENCES**


*Added in Proof.* The result in Example 2 is also in:


DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF MIAMI, CORAL GABLES, FLORIDA 33124

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CONNECTICUT, STORRS, CONNECTICUT 06268

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use