PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 106, Number 1, May 1989

A SEMI-FREDHOLM PRINCIPLE FOR PERIODICALLY FORCED SYSTEMS WITH HOMOGENEOUS NONLINEARITIES

A. C. LAZER AND P. J. MCKENNA

(Communicated by Kenneth R. Meyer)

ABSTRACT. We show that if the potential in a second-order Newtonian system of differential equations is positively homogeneous of degree two and positive semidefinite, and if the unforced system has no nontrivial T-periodic solutions $(T > 0)$, then for any continuous T-periodic forcing, there is at least one T-periodic solution.

In this note we consider a system of second-order differential equations in which the nonlinear terms have one nice property shared by linear functions: they are positively homogeneous of degree one. One such system arises from an idealized model of a suspension bridge considered previously in [4 and 7].

Consider the partial differential equation

$$U_{tt} + a^2 U_{xxxx} + k(x)U^+ = f(x, t),$$

where $U = U(x, t)$, $0 \le x \le L$, with boundary conditions

$$U(0, t) = U(L, t) = U_{xx}(0, t) = U_{xx}(L, t) = 0.$$

If $k(x) \equiv 0$, we have the well-known equation for the transverse vibrations of a beam of length L, which is hinged at the endpoints, subject to an external force given by $f(x, t)$. If the beam is suspended from above by cables, whose strengths may vary from point to point, and the displacement U is measured in the downward direction, we add the term $k(x)U^+$, where $k(x) \ge 0$ and U^+ is the positive part of U. This accounts for the fact that cables exert no restoring force when slack.

When f is periodic in t it is natural to look for time-periodic solutions. We consider an approximate problem obtained by discretizing in the space variable x. If N is a large positive integer and we consider the mesh points $x_k = kL/N$, $k = 1, \dots, N-1$, we are lead to the system of ordinary differential equations

$$u''(t) + Au(t) + Du(t)^+ = p(t),$$

Received by the editors March 27, 1983 and, in revised form, May 31, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 34C25.
Key words and phrases. Leray-Schauder continuation method, periodic solution.
The first author was partially supported by NSF under Grant DMS9519882.
The second author was partially supported by NSF under Grant DMS8519726.

where $u(t)$ is the $(N-1)$-dimensional column vector whose k th component is $U(x_k, t)$, $p(t)$ is the vector with k th component $f(x_k, t)$, $u(t)^+$ has its k th component equal to $U(x_k, t)^+$, D is a diagonal matrix with nonnegative entries, and A is the symmetric, positive-definite matrix corresponding to the finite difference approximation to U_{xxxx}, taking into account the boundary conditions. We consider a more general class of systems.

Let $V \in C^1(\mathbf{R}^n, \mathbf{R})$ be positively homogeneous of degree two, i.e., $V(tx) = t^2 V(x)$ for $t \geq 0$ and $x \in \mathbf{R}^n$ $(n \geq 1)$. If, for brevity, we set $V'(x) = \operatorname{grad} V(x)$, then V' is positively homogeneous of degree one, i.e.,

$$V'(tx) = tV'(x) \tag{1.1}$$

for $t \geq 0$ and $x \in \mathbf{R}^n$. If $(\ ,\ )$ denotes the usual inner product on $\mathbf{R}^n$, then for Euler's identity we have

$$(x, V'(x)) = 2V(x) \tag{1.2}$$

for $x \in \mathbf{R}^n$.

Theorem 1. *Let $V \in C^1(\mathbf{R}^n, \mathbf{R})$ be positively homogeneous of degree two and positive semidefinite, i.e.,*

$$x \in \mathbf{R}^n \Rightarrow V(x) \geq 0. \tag{1.3}$$

Let $T > 0$. If the system

$$u''(t) + V'(u(t)) = 0 \tag{1.4}$$

has no T-periodic solution other than $u(t) \equiv 0$, then for any T-periodic function $p \in C^1(\mathbf{R}, \mathbf{R}^n)$ the system

$$u''(t) + V'(u(t)) = p(t) \tag{1.5}$$

has at least one T-periodic solution.

Proof. We first assume the stronger condition that V be *positive definite*, i.e.,

$$x \in \mathbf{R}^n, \qquad x \neq 0 \Rightarrow V(x) > 0. \tag{1.3*}$$

We first show that if p and V are as above, then for $\varepsilon > 0$ the system

$$u''(t) + \varepsilon u'(t) + V'(u(t)) = p(t) \tag{1.6}$$

has at least T-periodic solution. To do this we use the Leray-Schauder continuation method (see, for example, [1 or 6]). Let $|\ |$ denote the usual Euclidean norm on $\mathbf{R}^n$, and let C_T and C_T^1 denote the Banach spaces of T-periodic functions which are continuous and continuously differentiable respectively with norms

$$|v|_\infty = \sup_{[0,T]} |v(t)|, \qquad v \in C_T,$$

$$|v|_1 = |v|_\infty + |v'|_\infty, \qquad v \in C_T^1.$$

Since the second-order linear homogeneous differential system

$$u''(t) + \varepsilon u'(t) + u(t) = 0$$

($u \in \mathbf{R}^n$) has no T-periodic solution other than $u(t) \equiv 0$, it follows that, for every $f \in C_T$, there exists a unique T-periodic solution of

$$u''(t) + \varepsilon u'(t) + u(t) = f(t).$$

Moreover, if we denote the unique T-periodic solution of the last system by $K(f)$, then K may be viewed as a compact linear operator from C_T into itself. Let $N\colon C_T \to C_T$ be the completely continuous operator defined by $N(u) = K(u + p - V'(u))$. We claim that there exists a number $R > 0$ such that if $\lambda \in [0, 1]$ and $u \in C_T$, then

$$u = \lambda N(u) \tag{1.7}$$

implies that $|u|_\infty \le R$.

Since (1.7) holds if and only if

$$u''(t) + \varepsilon u'(t) + (1-\lambda)u(t) + \lambda V'(u(t)) = \lambda p(t), \tag{1.8}$$

and since $|u|_\infty \le |u|_1$ if $u \in C_T^1$, assuming that the above claim is false, we infer the existence of a sequence $\{u_m\}_1^\infty$ and a corresponding sequence of numbers $\{\lambda_m\}_1^\infty$ such that $u_m(t)$ is a solution of (1.8) when $\lambda = \lambda_m$ for $m = 1, 2, \dots, \lambda_m \in [0, 1]$, and

$$|u_m|_1 \to \infty \quad \text{as } m \to \infty. \tag{1.9}$$

Setting $w_m(t) = u_m(t)/|u_m|_1$ for $m = 1, 2, \dots$, it follows by homogeneity of V' that

$$w_m''(t) + \varepsilon w_m'(t) + (1-\lambda_m)w_m(t) + \lambda_m V'(w_m(t)) = \lambda_m p(t)/|u_m|_1 \tag{1.10}$$

for $m = 1, 2, \dots$. Since $|w_m|_1 = 1$ for $m \ge 1$, it follows from (1.10) that the sequence $\{|w_m''|_\infty\}_1^\infty$ is bounded. Therefore, both of the sequences $\{w_m(t)\}_1^\infty$ and $\{w_m'(t)\}_1^\infty$ are equicontinuous and uniformly bounded on $(-\infty, \infty)$ so, by Ascoli's lemma, there exists a subsequence $\{w_{m_k}(t)\}_{k=1}^\infty$ and a $w \in C_T^1$ with $|w|_1 = 1$ such that $w_{m_k}(t) \to w(t)$, $w_{m_k}'(t) \to w'(t)$ as $m \to \infty$, uniformly on $(-\infty, \infty)$. Since $0 \le \lambda_{m_k} \le 1$ for all $k \ge 1$, we may assume without loss of generality that $\lambda_{m_k} \to \lambda^* \in [0, 1]$ as $k \to \infty$. Therefore, from (1.10), it follows that the sequence $\{w_{m_k}''(t)\}_1^\infty$ converges uniformly on $(-\infty, \infty)$ so w is of class C^2 and

$$w''(t) + \varepsilon w'(t) + (1-\lambda^*)w(t) + \lambda^* V'(w(t)) = 0. \tag{1.11}$$

Taking the inner product of (1.11) with $w'(t)$ and observing that

$$\int_0^T (w'(t), w''(t))\,dt = \frac{1}{2}\int_0^T \frac{d}{dt}|w'(t)|^2\,dt = 0$$

and

$$\int_0^T (w'(t), (1-\lambda^*)w(t) + \lambda^* V'(w(t)))\,dt$$
$$= \int_0^T \frac{d}{dt}[(1-\lambda^*)|w(t)|^2/2 + \lambda^* V(w(t))]\,dt = 0,$$

we find that

$$\varepsilon \int_0^T |w'(t)|^2 \, dt = 0.$$

Hence, $w(t) = \xi =$ constant and according to (1.11)

$$(1 - \lambda^*)\xi + \lambda^* V'(\xi) = 0.$$

Taking the inner product of this last equation with ξ and using (1.2), we have

$$(1 - \lambda^*)|\xi|^2 + 2\lambda^* V(\xi) = 0.$$

Thus, since $0 \leq \lambda^* \leq 1$, it follows from $(1.3)^*$ that $\xi = 0$. Since, this contradicts the fact that $|w|_1 = 1$, the claim that there exists R independent of $u \in C_T$ and $\lambda \in [0, 1]$ such that (1.7) implies that $|u|_\infty < R$ has been established.

From the Leray-Schauder-Shaefer theorem, it follows that for each $\lambda \in [0, 1]$ there exists a $u \in C_T$ which satisfies (1.7). (See, for example, [9, 1, p. 61, or 6, p. 71]). In particular since (1.7) has a solution u when $\lambda = 1$, it follows that (1.6) has at least one T-periodic solution.

Let $\{\varepsilon_m\}_1^\infty$ be a sequence of positive numbers such that $\varepsilon_m \to 0$ as $m \to \infty$. By what has been shown for each $m = 1, 2, \ldots,$ there exists $u_m \in C_T$ such that u_m is a solution of (1.6) when $\varepsilon = \varepsilon_m$. We claim that the sequence $\{|u_m|_1\}_1^\infty$ is bounded. Assuming the contrary, we may suppose without loss of generality that $|u_m|_1 \to \infty$ as $m \to \infty$. Setting $z_m(t) = u_m(t)/|u_m|$, for $m \geq 1$, we have, by positive homogeneity of V',

$$z''_m(t) + \varepsilon_m z'_m(t) + V'(z_m(t)) = p((t)/|u_m|_1 \tag{1.12}$$

for $m = 1, 2, \ldots$. From this it follows that the sequences $\{z_m(t)\}_1^\infty$ and $\{z'_n(t)\}_1^\infty$ are equicontinuous and uniformly bounded on $(-\infty, \infty)$ so there exists $z \in C_T$ with $|z|_1 = 1$ and a subsequence $\{z_{m_k}(t)\}_1^\infty$ of $\{z_m\}_1^\infty$ such that $z_{m_k}(t) \to z(t)$ and $z'_{m_k}(t) \to z'(t)$ as $k \to \infty$ uniformly with respect to $t \in (-\infty, \infty)$. From (1.12), we infer that the sequence $\{z''_{m_k}(t)\}_1^\infty$ converges uniformly on $(-\infty, \infty)$. Hence z is of class C^2 and

$$z''(t) + V'(z(t)) = 0.$$

Since $|z|_1 = 1$, this contradicts the assumption that (1.4) has no nontrivial T-periodic solution, our claim that the sequence $\{|u_m|_1\}_1^\infty$ is bounded has been established.

From the differential equation

$$u''_m(t) + \varepsilon_m u'_m(t) + V'(u_m(t)) = p(t),$$

it follows that the sequence $\{u''_m(t)\}_1^\infty$ is also uniformly bounded on $(-\infty, \infty)$. Therefore, using the same type of argument used above, we infer the existence of a subsequence of $\{u_m(t)\}_1^\infty$ such that this subsequence, as well as the corresponding sequences of first and second derivatives converges uniformly on

$(-\infty, \infty)$. Since the limit of this subsequence is a T-periodic solution of (1.5), the proof of the theorem under assumption $(1.3)^*$ is complete.

To prove the theorem with $(1.3)^*$ replaced by (1.3), we observe that for all sufficiently small $\delta > 0$, the system

$$u''(t) + \delta u(t) + V'(u(t)) = 0$$

has no nontrivial T-periodic solution. Indeed, in the contrary case, for $\delta > 0$ arbitrarily small, we could find a solution with C_T^1-norm equal to 1. A compactness argument, similar to those used above, would give a nontrivial solution of (1.4), contradicting one of our hypotheses.

Therefore, since $\delta|x|^2/2 + V(x) > 0$ for $x \neq 0$, by what we have shown, for small $\delta > 0$ there exists a T-periodic solution of

$$u'' + \delta u + V'(u) = p(t).$$

The C_T^1-norms of these solutions are bounded as $\delta \to 0$ since (1.4) has no nontrivial T-periodic solution. Therefore, by the same type of compactness argument as used above, we obtain a T-periodic solution of (1.5).

Remark. It does not seem possible to prove the theorem more directly by connecting (1.5) rather than (1.6) to a linear equation by a homotopy.

Remark. The theorem remains true if the condition (1.3) is replaced by

$$(1.3)' \qquad x \in \mathbf{R}^N, \qquad x \neq 0 \Rightarrow V(x) < 0.$$

In the proof one would consider the parameter-dependent differential equation

$$u''(t) + \varepsilon u'(t) - (1-\lambda)u(t) + \lambda V'(u(t)) = \lambda p(t)$$

instead of (1.8) and use the fact that the linear differential equation

$$u''(t) + \varepsilon u'(t) - u(t) = f(t),$$

where $\varepsilon > 0$, has a unique T-periodic solution for any $f \in C_T$.

Examples 1. The homogeneous nonlinearity in the differential equation

$$(1.13) \qquad u''(t) + |u(t)| = p(t) \equiv p(t+T),$$

where $n = 1$, does not satisfy the condition (1.3), since in this case $V(x) = \frac{1}{2}(\operatorname{sgn} x)x^2$. If $p(t) \neq 0$ and $u(t)$ is a T-periodic solution of (1.13), then

$$\int_0^T |u(t)|\, dt = \int_0^T p(t)\, dt,$$

so, in order that (1.13) have a T-periodic solution, it is necessary that the mean value of p be positive. If $q(t)$ is a continuous T-periodic function with mean value zero, and $p(t) = c + q(t)$ where c is a constant, then by using the well-known upper and lower solution method for periodic solutions of second-order differential equations (see, for example, [10]) one can easily adapt the methods of [2] (which concerns a boundary value problem for a P.D.E.) to prove the existence of $\bar{c} = \bar{c}(q)$ such that (1.13) has a T-periodic solution if and only if $c \geq \bar{c}$.

2. Consider the differential equation

$$u'' + bu^+ - au^- = p(t) \equiv p(t+2\pi), \tag{1.14}$$

where $n = 1$, $a > 0$ and $b > 0$. If u is a nontrivial solution of

$$u'' + bu^+ - au^- = 0, \tag{1.15}$$

then the distance between two consecutive zeros of u which border an interval on which u is positive is $\pi/\sqrt{b}$, since $u'' + bu = 0$ on such an interval. Similarly the distance between two consecutive zeros of u which border an interval on which u is negative is $\pi/\sqrt{a}$. It follows that every nontrivial solution of (1.15) is periodic with least period $\pi/\sqrt{a} + \pi/\sqrt{b}$. Therefore, since the potential for the nonlinearity in (1.15), $V(x) = [b(x^+)^2 + a(x^-)^2]/2$, satisfies (1.3), it follows that if

$$\pi/\sqrt{b} + \pi/\sqrt{a} \neq 2\pi/m$$

for $m = 1, 2, \ldots$, then, for any continuous 2π-periodic function $p(t)$, (1.14) has a 2π-periodic solution.

3. Suppose that for some integer $m \geq 1$ we have

$$\pi/\sqrt{b} + \pi/\sqrt{a} = 2\pi/m,$$

and in addition that

$$(m-1)^2 < a < b < (m+1)^2. \tag{1.16}$$

Let H be the Hilbert space consisting of 2π-periodic functions defined on $(-\infty, \infty)$ whose restrictions to $[-\pi, \pi]$ belong to $L^2[-\pi, \pi]$ with the $L^2[-\pi, \pi]$ inner product. Let W be the two-dimensional subspace of H spanned by $\cos mt$ and $\sin mt$ and let $P: H \to W$ denote orthogonal projection. Since the spectrum of the linear operator $A: D(A) \subset H \to H$ defined by $Au = -u''$ is $\{k^2 | k = 0, 1, \ldots\}$ it follows from the Liapunov-Schmidt technique and (1.16) that for any $w_1 \in W$ there exists a *unique* $w_2 \in (I-P)W$ such that

$$w_2'' + (I-P)[b(w_1 + w_2)^+ - a(w_1 + w_2)^-] = 0 \tag{1.17}$$

and $w_2(t+2\pi) \equiv w_2(t)$. (See [5], or the proof of Proposition 2.1 of [3] for more details.) If u_0 is a nonzero solution of (1.15), then u_0 is $(2\pi/m)$-periodic and $Pu_0 \not\equiv 0$, for otherwise, since $w_2 \equiv 0$ solves (1.17) when $w_1 \equiv 0$, we would have $(I-P)u_0$, and hence u_0, identically zero.

We claim that if $c^2 + d^2 \neq 0$ then there exists no 2π-periodic solution of

$$u'' + bu^+ - au^- = c \cos mt + d \sin mt. \tag{1.18}$$

Assuming, on the contrary, that there exists a 2π-periodic solution u^* of (1.18), the reasoning used above shows that $Pu^* \not\equiv 0$. Therefore, since both Pu_0 and Pu^* are linear combinations of $\cos mt$ and $\sin mt$, both have the form

$r\sin(mt+\delta)$ for some $r > 0$. Therefore there exist numbers $\alpha > 0$ and γ such that $\hat{u}(t) = \alpha u_0(t+\gamma)$, then $P\hat{u} = Pu^*$. Therefore, since by homogeneity $\hat{u}$ is also a solution of (1.15), it follows that if $w_1 = P\hat{u}$, then both $w_2 = (I-P)\hat{u}$ and $w_2 = (I-P)u^*$ solve (1.17), so by uniqueness $(I-P)\hat{u} = (I-P)u^*$. Therefore we have the absurdity $\hat{u} = u^*$. This contradiction proves the claim.

This phenomenon is, of course, well known as *resonance* in the linear case $a = b = m^2$.

The work of Podolak [8] shows that if $n = 1$ and periodic boundary conditions are replaced by Dirichlet boundary conditions, then the statement of the theorem does not remain true.

References

1. K. Deimling, *Nonlinear functional analysis*, Springer-Verlag, Berlin, Heidelberg and New York, 1985.
2. J. L. Kazdan and F. W. Warner, *Remarks on some quasilinear elliptic equations*, Comm. Pure Appl. Math. **28** (1975), 567–597.
3. A. C. Lazer and P. J. McKenna, *Critical point theory and boundary value problems with nonlinearities crossing multiple eigenvalues*. II, Comm. in Partial Differential Equations **11** (1986), 1653–1676.
4. ——, *Large scale oscillatory behaviour in loaded asymmetric systems*, Ann. Inst. Henri Poincaré **4** (1987), 243–276.
5. ——, *A symmetry theorem and applications to nonlinear partial differential equations*, J. Differential Equations **72** (1988), 95–106.
6. N. G. Lloyd, *Degree theory*, Cambridge Univ. Press, London and New York, 1978.
7. P. J. McKenna and W. Walter, *Nonlinear oscillations in a suspension bridge*, Arch. Rat. Mech. Anal. **98** (1987), 168–177.
8. E. Podolak, *On the range of operator equations with an asymptotically nonlinear term*, Indiana Univ. Math. J. **25** (1976), 1127–1137.
9. H. Schaefer, *Uber die Methode der a priori Schranken*, Math. Ann. **29** (1955), 415–416.
10. K. Schmitt, *Periodic solutions of nonlinear second order ordinary differential equations*, Math. Z. **98** (1967), 200–207.

Added in Proof. The result in Example 2 is also in:

11. E. N. Dancer, *On the Dirichlet problem for weakly nonlinear elliptic partial differential equations*, Proc. Roy. Soc. Edinburgh, Sec. A 76 (1977), 283–300.

Department of Mathematics and Computer Science, University of Miami, Coral Gables, Florida 33124

Department of Mathematics, University of Connecticut, Storrs, Connecticut 06268