A GENERALIZATION OF THE WEDDERBURN-ARTIN THEOREM

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Abstract. The structure of rings such that each of its homomorphic images has the property that each cyclic right module over it is essentially embeddable in a direct summand is determined. Such rings are precisely (i) right uniserial rings, (ii) $n \times n$ matrix rings over two-sided uniserial rings with $n > 1$, or (iii) sums of rings of the types (i) and (ii).

1. Introduction

In this paper we study rings $R$ with the following property (P): For all homomorphic images $\overline{R}$ of $R$, every cyclic right $\overline{R}$-module is essentially embeddable in a direct summand of $\overline{R}$. Our results generalize the celebrated Wedderburn-Artin theorem which characterizes rings $R$ such that over all the homomorphic images $\overline{R}$ the cyclic modules are isomorphic to direct summands of $\overline{R}$. Examples of rings satisfying (P) include semisimple artinian rings and right uniserial rings. Indeed we show that a ring $R$ has property (P) if and only if $R$ is a direct sum of right uniserial rings and matrix rings over right self-injective right uniserial rings if and only if $R$ is a semiperfect ring whose cyclic right modules are essentially embeddable in direct summands (Theorem 3.5). Throughout this paper, all rings have 1 and all modules are right unital, unless otherwise stated. By a right (left) uniserial ring, we mean a ring having a unique composition series of right (left) ideals. A ring which is both right and left uniserial will simply be called uniserial. A right uniserial ring is uniserial if it is right self-injective. For any module $M$, $E(M)$, $\text{Soc}(M)$ and $J(M)$ will denote, respectively, the injective hull, the socle, and the Jacobson radical of $M$.

2. Preliminary results

Throughout this section, we assume that $R$ is a ring satisfying property (P).

2.1. Lemma. $R$ is a semiperfect ring.
Proof. Let $N$ = prime radical of $R$ under our hypothesis, each right ideal of $R/N$ is an annihilator right ideal and hence $R$ is semiperfect [3, p. 204, Exercise 24.3(d)-(e)]. □

Since $R$ is semiperfect, $R$ has a complete orthogonal set $e_1, \ldots, e_n$ of idempotents such that, for all $i$, $e_i R e_i$ is a local ring. In the lemmas which follow the decomposition $R = e_1 R \oplus \cdots \oplus e_n R$ will be frequently used. For $R$ modules $A$ and $B$, the notation $A \hookrightarrow B$ shall mean that $A$ is essentially embeddable in $B$.

2.2. Lemma. For $R = e_1 R \oplus \cdots \oplus e_n R$, the following are true:

(i) $e_i R$ is uniform for all $i$,
(ii) $\text{Soc } R$ is essential in $R$, and
(iii) $R$ has Goldie dimension $n$.

Proof. Let $S = \{S_1, \ldots, S_k\}$ be an irredundant set of representatives for the simple $R$-modules and let $P = \{e_1 R, \ldots, e_k R\}$ be a complete set of representatives for the projective indecomposable $R$ modules.

Since every simple module $S$ is cyclic, it is essentially embeddable in $e R$ for some idempotent $e \in R$. Clearly $e R$ is indecomposable. Thus we can define a function $f: S \to P$ by $f(S_i) = e_i R$ where $S_i \hookrightarrow e_i R$. The function $f$ must be one to one, hence onto. It easily follows that each $e_j R$ ($j = 1, \ldots, n$) contains an essential simple submodule $T_j$ and, therefore, each $e_j R$ is uniform. Also, $T_1 \oplus \cdots \oplus T_n = \text{Soc } R$ is essential in $R$. Thus $R$ has Goldie dimension $n$. □

2.3. Lemma. $R$ is right artinian.

Proof. Clearly each cyclic $R$-module has nonzero socle. Thus, $R$ is left perfect because $R$ is semiperfect [2]. Furthermore, since $J(R)/(J(R))^2$ is completely reducible, $J(R)/(J(R))^2$ is embeddable in $\text{Soc } R$. This yields $J(R)/(J(R))^2$ is finitely generated and so $R$ is right artinian [1, p. 322]. □

2.4. Lemma. For $i \neq j$, let $e_i R$ and $e_j R$ be indecomposable summands of $R$. Then, either $e_i R$ is isomorphic to $e_j R$ or $\text{Hom}_R(e_i R, e_j R) = 0$.

Proof. Suppose $\sigma: e_i R \to e_j R$ is not zero, then $e_i R/\text{Ker } \sigma$ is embeddable in $e_j R$. Since $e_j R$ is uniform (Lemma 2.2), such an embedding must be essential. This implies $E(e_i R/\text{Ker } \sigma) \cong E(e_j R)$. Also, since $R$ satisfies property (P) and it has Goldie dimension $n$, $E(R/\text{Ker } \sigma) \cong E(R)$. Let $R = e_1 R \oplus \cdots \oplus e_n R$. Then

$$R/\text{Ker } \sigma \cong e_1 R \oplus \cdots \oplus e_i R/\text{Ker } \sigma \oplus \cdots \oplus e_j R \oplus \cdots \oplus e_n R,$$

which yields

\[
\begin{align*}
E(e_1 R) & \oplus \cdots \oplus E(e_j R) \oplus \cdots \oplus E(e_i R) \\
& \cong E(R/\text{Ker } \sigma) \cong E(R) \cong E(e_1 R) \oplus \cdots \oplus E(e_j R) \oplus \cdots \oplus E(e_i R) \oplus \cdots \oplus E(e_n R).
\end{align*}
\]
Since $e_k R$ is uniform for all $k$, $E(e_k R)$ has local endomorphism ring. Hence from (1) $E(e_i R) \cong E(e_j R)$. But this implies that $E(e_i R)$ and $E(e_j R)$ contain isomorphic copies of the same simple submodule $S$ and, therefore, $e_i R$ and $e_j R$ both contain essentially a copy of $S$. This implies that $e_i R$ is isomorphic to $e_j R$. □

2.5. Lemma. $R$ is a direct sum of matrix rings over local rings.

Proof. Let $[e_i R] = \sum e_j R$, where the $\sum$ runs over all $j$ for which $e_j R \cong e_i R$. Renumbering if necessary we may write

$$R = [e_1 R] \oplus \cdots \oplus [e_k R]$$

where $k \leq n$. By Lemma 2.4, $[e_i R]$ is an ideal in $R$ and so

$$R \cong M_{n_1}(e_1 R e_1) \oplus \cdots \oplus M_{n_k}(e_k R e_k)$$

where $n_i$ is the number of summands in $[e_i R]$. □

Next we proceed to show that each local ring $e_i R e_i$ is indeed right uniserial.

2.6. Lemma. If $R = S_n$ is the $n \times n$ matrix ring over a local ring $S$, then $S$ is right uniserial.

Proof. Write $R = e_{11} R \oplus \cdots \oplus e_{nn} R$, where $e_{11}, e_{22}, \ldots, e_{nn}$ are the usual matrix units. Notice that each $e_{ii} R$ is indecomposable since $S$ is local.

Consider $I \subset e_{11} R$. Then $R/I \cong e_{11} R/I \times e_{22} R \times \cdots \times e_{nn} R$ is essentially embeddable in $R$ because the Goldie dimension of $R$ is $n$. Thus

$$E(R/I) \cong E(R)$$

and so

$$E(e_{11} R/I) \times E(e_{22} R) \times \cdots \times E(e_{nn} R) \cong E(e_{11} R) \times E(e_{22} R) \times \cdots \times E(e_{nn} R).$$

Since $e_{ii} R$ is uniform (Lemma 2.2), $E(e_{ii} R)$ is also uniform. Therefore, by Azumaya diagram, $E(e_{11} R/I) \cong E(e_{11} R)$. This implies $e_{11} R/I$ is uniform. It follows that the submodules of $e_{11} R$ are linearly ordered. We show now that $S \cong e_{11} R e_{11}$ is right uniserial. Let $A, B$ be right ideals of $e_{11} R e_{11}$. Then $A e_{11} R e_{11} e_{11} R$ and $B e_{11} R e_{11} R$ and so either $A e_{11} R \subset B e_{11} R$ or $B e_{11} R \subset A e_{11} R$. But then either $A = A e_{11} R e_{11} e_{11} R e_{11} R = B$ or $B = B e_{11} R e_{11} e_{11} R e_{11} = A$, proving our assertion. □

In the next section we shall obtain a characterization of rings with property (P).

2.7. Remark. Note that in the proof of Lemmas 2.2–2.6 we have only used that $R$ is a semiperfect ring each of whose cyclic $R$-modules is essentially embeddable in a direct summand of $R$.
3. Main results

We begin with

3.1. Theorem. Let $R$ be a ring with property $(P)$. Then $R$ is a direct sum of matrix rings over right uniserial rings.

Proof. The proof follows from Lemmas 2.5, 2.6, 2.7 and the fact that ring direct summands of a ring with property $(P)$ inherit the property $(P)$. ∎

It is obvious that right uniserial rings have property $(P)$. In what follows we will concentrate on showing that for a right uniserial ring $S$, the matrix ring $R = S_n$ $(n > 1)$ satisfies property $(P)$ if and only if $S$ is right self-injective. For the sake of our discussion we define property $(Q)$ for modules. We say that an $R$-module $M$ has property $(Q)$ if each factor of $M$ is essentially embeddable in a direct summand of $M$.

3.2. Lemma. The $n \times n$ matrix ring over $R$ has property $(Q)$ as a module over itself if and only if the $R$-module $R^{(n)}$ has property $(Q)$.

Proof. Given a category isomorphism $F = \mathcal{M}_S \rightarrow \mathcal{M}_T$ between the categories of right modules of two rings $S$ and $T$, it is obvious that a module $M \in \mathcal{M}_S$ satisfies $(Q)$ if and only if $F(M) \in \mathcal{M}_T$ satisfies $(Q)$. Our lemma follows from the fact that if $e_{11} \in R_n$ is the usual matrix unit then $R^{(n)} \in \mathcal{M}_R$ corresponds to $R_n \in \mathcal{M}_{R_n}$ under the category isomorphism.

\[ - \otimes_{R_n} R_n e_{11} : \mathcal{M}_{R_n} \rightarrow \mathcal{M}_R. \] ∎

3.3. Lemma. If the $R$-module $R^{(n)}$ has property $(Q)$ where $R$ is right uniserial and $n > 1$, then $R$ is right self-injective.

Proof. Let $R$ be a right uniserial ring which is not right self-injective. Then there exists $s \in R$ such that $xs \notin Rx$. Without loss of generality, we may assume that $s$ is invertible. Define $I = (x, -xs, 0, 0, \ldots, 0)R \subseteq R^{(n)}$. We claim that $R^{(n)}/I$ is not embeddable in $R^{(n)}$. Notice that both $e_1R$ and $e_2R$ are isomorphic to $R$ as $R$-modules, where $e_1 = (1, 0, 0, \ldots, 0)$ and $e_2 = (0, 1, 0, \ldots, 0)$. Also, since $e_1R \cap e_2R = e_1xR_1 = e_2xR$. If $\psi : R^{(n)}/I \rightarrow R^{(n)}$ were an embedding of $R^{(n)}/I$ into $R^{(n)}$, and if $\psi(e_1) = (a_1, a_2, \ldots, a_n)$ and $\psi(e_2) = (b_1, b_2, \ldots, b_n)$, then there must exist $i, j$ such that $a_i$ invertible and $b_j$ invertible. However, $\psi(e_1x) = (a_1x, a_2x, \ldots, a_nx)$ and $\psi(e_2xs) = (b_1xs, b_2xs, \ldots, b_nxs)$, which implies that $a_i x = b_j xs$. Hence $b_j^{-1}a_j x = xs$, contradicting our choice of $s$. So we have shown that the $R$-module $R^{(n)}$ does not satisfy $(Q)$. ∎

3.4. Lemma. If $R$ is a right self-injective right uniserial ring, then $R_n$ satisfies property $(P)$.

Proof. Since $R$ is self-injective, it follows that $R_n$ is also self-injective. Therefore, $R_n$ satisfies property $(Q)$ as a module over itself if and only if the injective hull of any cyclic $R_n$-module is embeddable in $R_n$. Let $e_{11} \in R_n$ be
the usual matrix unit and let $I$ be a right ideal of $R_n$. Since $R_n \to R_n/I \to 0$ is exact, $(R_n \otimes_{R_n} R_ne_{11})_R \to (R_n/I \otimes_{R_n} R_ne_{11})_R \to 0$ is also exact. But $(R_n \otimes_{R_n} R_ne_{11})_R \cong (R_ne_{11})_R \cong R^{(n)}$. Therefore, $N = R_n/I \otimes_{R_n} R_ne_{11}$ is a homomorphic image of $R^{(n)}$. Thus $N$ is an extension of a sum of $k$ cyclic $R$-modules, $(k \leq n)$ [5, Lemma 1.16]. But then, since $e_{11}R_n$ corresponds to $R$ under Hom$_R(R_ne_{11}, -)$, the inverse of $(\_ \otimes_{R_n} R_ne_{11})$, it follows that there exist $k$ quotients $Q_1, \ldots, Q_k$, of $e_{11}R_n$ such that $Q_1 \oplus \cdots \oplus Q_k \hookrightarrow R_n/I$. Now, $E(Q_i) \hookrightarrow\' e_{11}R_n$ for all $i$. Hence $E(R_n/I) = E(Q_1) \oplus \cdots \oplus E(Q_k) \hookrightarrow\' (e_{11}R_n)^{(k)} \hookrightarrow R_n$, proving that $E(R_n/I)$ is embeddable in $R_n$. Since each homomorphic image of $R$ is again right self-injective right uniserial, it follows that $R_n$ satisfies property $(P)$. □

Our results are summarized in the following theorem.

3.5. **Theorem.** A ring $R$ satisfies $(P)$ if and only if $R$ is a direct sum of right uniserial rings and matrix rings over right self-injective right uniserial rings if and only if $R$ is a semiperfect ring whose cyclics are essentially embeddable in a direct summand of $R$.

**Proof.** The proof follows from Theorem 3.1 and Lemmas 3.2, 3.3 and 3.4 and Remark 2.7. □

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**References**


