DEGREES OF IRREDUCIBLE CHARACTERS
AND NORMAL $p$-COMPLEMENTS

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Abstract. John Tate [1] proved that if $P \in \text{Syl}_p(G)$, $H$ is a normal subgroup
of a finite group $G$ and $P \cap H \leq \Phi(P)$ ($\Phi(G)$ is the Frattini subgroup of $G$)
then $H$ has a normal $p$-complement. We prove in this note that Tate's theorem
has nice character-theoretic applications.

Theorem. Let $B$ be the intersection of the kernels of all nonlinear irreducible
characters of $G$ with $p'$-degree. Then $B \cap G' \cap P \leq P'$ where $P \in \text{Syl}_p(G)$.
Also, $B$ has a normal $p$-complement.

Proof. We suppose that $P_0 = B \cap G' \cap P \not\leq P'$. Let $\text{Lin}(P)$ be the set of
all linear characters of $P$, and let $\lambda \in \text{Lin}(P)$ satisfy $P_0 \not\leq \ker \lambda$. Then the
induced character $\lambda^G$ has degree $|G: P| \equiv 0 \pmod{p}$. Let $\chi$ be an irreducible
component of $\lambda^G$. Then $P_0 \not\leq \ker \chi$ by Frobenius reciprocity. So $p$
divides $\chi(1)$ for all nonlinear irreducible components $\chi$ of $\lambda^G$. Since $p$
does not divide $\lambda^G(1)$, the character $\lambda^G$ has a linear component $\lambda^0$. Then $\lambda^0_p = \lambda$.
Thus
$$P \cap \ker \lambda^0 = \ker \lambda \not\leq P_0 \Rightarrow P_0 \not\leq \ker \lambda^0.$$ 
Since $G' \leq \ker \lambda^0$, we have
$$B \cap G' \cap P = P_0 \not\leq G',$$
which is a contradiction.

The last assertion follows from

Lemma. Let $P \in \text{Syl}_p(G)$ and let $H \leq G$. If $H \cap G' \cap P \leq P'$, then $H$ has a
normal $p$-complement.

Proof. Let $O^p(G)$ be the intersection of all $N \leq G$ such that $G/N$ is a $p$-group.
Then $O^p(G)$ is characteristic in $G$. So $O^p(H) \leq G$ and
$$O^p(H) \cap G' \cap P \leq H \cap G' \cap P \leq P'.$$

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Since $O_p^\circ(H)$ has no normal subgroup of index $p$, we have $O_p^\circ(H) \cap P \leq G'$. Hence

$$O_p^\circ(H) \cap G' \cap P = O_p^\circ(H) \cap P$$

and $H$ has a normal $p$-complement by Tate's theorem.

**Corollary** (J. G. Thompson [2]). Suppose that a prime $p$ divides $\chi(1)$ for all nonlinear irreducible characters $\chi$ of $G$. Then $G$ has a normal $p$-complement.

This follows from Theorem, since $B = G$, where $B$ is defined in the theorem.

**Remark.** We prove that Tate’s theorem for $p > 2$ is a corollary to the following well-known result of J. G. Thompson [3]:

Let $p > 2$, let $P \in \text{Syl}_p(G)$, and, for every characteristic subgroup $P_0$ of $P$, $P_0 \neq 1$, the normalizer $N_G(P_0)$ has a normal $p$-complement. Then $G$ has a normal $p$-complement.

Suppose that $H \leq G$, $p > 2$, $P \in \text{Syl}_p(G)$, and $P_1 = H \cap P \leq \Phi(P)$. Suppose that $H$ has no normal $p$-complement. By Thompson’s theorem, there exists a characteristic subgroup $P_0$ of $P_1$, $P_0 \neq 1$, such that $N_H(P_0)$ has no normal $p$-complement, and let $P_0$ have a maximal order among all subgroups with this property. Since $P_1 \leq P$, we have $P_0 \leq P$. So $P < N_G(P_0)$. Since $N_H(P_0) \leq N_G(P_0)$, the subgroup $N_G(P_0)$ has no normal $p$-complement. Without loss of generality we may assume that $PH = G$. Then

$$N_G(P_0) = P(H \cap N_G(P_0)) = PN_{H}(P_0) = N_{H}(P_0)P$$

by modular law. Since $N_G(P_0)$ has no normal $p$-complement we may assume without loss that $N_G(P_0) = G$. So $P_0 \leq G$. Suppose that $P_0 \notin \Phi(G)$. Then there exists such a maximal subgroup $M$ of $G$ that $P_0M = G$. Then $P = P_0(P \cap M)$ by modular law. So $P \cap M = P$ (since $P_0 \leq \Phi(P)$), and $P_0 \leq M$, $M = P_0M = G$, a contradiction. Hence $P_0 \leq \Phi(G)$. By Thompson’s theorem $G/P_0$ has a normal $p$-complement $T/P_0$ by virtue of a maximal choice of $P_0$. If $K$ is a $p'$-Hall subgroup of $T$ (Schur–Zassenhaus), then

$$G = N_G(K)T = N_G(K)KP_0 = N_G(K)P_0$$

(Schur–Zassenhaus and Frattini). Since $P_0 \leq \Phi(G)$, we have $N_G(K) = G$ and $K \leq G$. Obviously, $K$ is a normal $p$-complement of $G$.

Further applications of a generalization of Tate’s theorem (Roquette’s theorem [4]) can be found in Chapter 6 of the book [5].

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**References**


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