THE COEFFICIENTS OF NEVANLINNA'S PARAMETRIZATION ARE NOT IN $H^p$

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Abstract. We construct an example of a Pick-Nevanlinna interpolation problem such that the coefficients of its Nevanlinna's parametrization are not in $H^p$, for $p > 0$.

1. INTRODUCTION

Let $D$ be the unit disc in the complex plane and let $H^p(D)$, $0 < p \leq \infty$, be the usual Hardy spaces on $D$.

We consider the following classical Pick-Nevanlinna interpolation problem: Given two sequences of numbers $\{z_n\}, \{w_n\}$ in $D$, find all analytic functions $f \in H^\infty(D)$ satisfying

\[ \|f\|_\infty = \sup\{|f(z)| : z \in D\} \leq 1 \quad \text{and} \quad f(z_n) = w_n, \quad n = 1, 2, \ldots. \]

(\star) $\|f\|_\infty \leq \sup\{|f(z)| : z \in D\} \leq 1$ and $f(z_n) = w_n$, $\quad n = 1, 2, \ldots.$

Pick and Nevanlinna found necessary and sufficient conditions in order that such an analytic function exists. If $E$ denotes the set of all analytic functions on $D$ satisfying (\star), Nevanlinna showed that in the case where $E$ consists of more than one element, there is a parametrization of the form:

\[ E = \left\{ f \in H^\infty(D) : f = \frac{p\varphi + q}{r\varphi + s}, \varphi \in H^\infty(D), \|\varphi\|_\infty \leq 1 \right\} \]

where $p, q, r, s$ are certain analytic functions on $D$ depending on $\{z_n\}$ and $\{w_n\}$. It is known that $p, q, r, s$ are in the Smirnov class $N^+(D)$. Furthermore, $p, q, r, s$ belong to $H^p(D)$ if and only if $s$ is in $H^p(D)$.

For details and proofs of results above, see [1, pp. 50, 165] and [3, p. 491]. In [2, p. 205] it is claimed that $s$ belongs to $H^2(D)$. Recently, Stray [3] asked for a complex analytic proof of this result. In this note we show that this result is false. Indeed, we will give an example of a Pick-Nevanlinna interpolation...
problem such that the function $s$ appearing in its Nevanlinna's parametrization belongs to no $H^p(D)$ for $p > 0$.

In a private communication, D. Sarason told us that he already knew the fact that $s$ belongs to $H^2(D)$ was false.

2. CONSTRUCTION OF THE EXAMPLE

Let us choose \{\{c_n\}\} a sequence of positive numbers such that \(\sum_{n=1}^{\infty} c_n \log(1/c_n) < +\infty\) and \(\sum_{n=1}^{\infty} c_n^q = +\infty\) for each $q < 1$ (for instance, $c_n = n^{-1}(\log(n))^{-3}$ satisfies these conditions).

Take a sequence of points $e^{i\theta_n}$ converging to 1, so that the arcs $I_n = \{e^{it}: \theta_n - c_n/2 < t < \theta_n + c_n/2\}$ will be pairwise disjoint and consecutive. Put $z_n = (1 - c_n)e^{i\theta_n}$.

Claim. There exists $h \in H^\infty(D)$, $||h||_\infty \leq 1$ so that $\int_0^{2\pi} \log(1 - |h(e^{i\theta})|) d\theta > -\infty$ and $1 - |h(z_n)| \leq C(\alpha)(1 - |z_n|)^\alpha$ for each $\alpha < 1$, where $C(\alpha)$ is a constant depending on $\alpha$.

Proof of the claim. Put $A = \{e^{i\theta_n}, 1\}$ and $g(e^{i\theta}) = \text{dist}(e^{i\theta}, A)$. Write $u(z) = P_z(g)$, the Poisson integral of $g$, and let $v(z)$ be the harmonic conjugate of $u(z)$.

Take $h(z) = \exp(-u(z) - iv(z))$. Then $h$ is analytic on $D$ and, since $g$ is positive, one has $||h||_\infty \leq 1$.

Also,

$$\int_0^{2\pi} \log(1 - |h(e^{i\theta})|) d\theta = \int_0^{2\pi} \log(1 - e^{-g(e^{i\theta})}) d\theta \geq C_2 + C_1 \sum_{n=1}^{\infty} c_n (\log(c_n) - 1) > -\infty,$$

$C_1$ and $C_2$ some constants, because $\sum_{n=1}^{\infty} c_n \log(1/c_n) < +\infty$.

Furthermore:

$$1 - |h(z_n)| = 1 - \exp(-P_{z_n}(g)) \leq P_{z_n}(g) = [P_{z_n}(g) - g(e^{i\theta_n})] \leq C(\alpha)|z_n - e^{i\theta_n}|^\alpha = C(\alpha)(1 - |z_n|)^\alpha$$

for each $\alpha < 1$, because the Poisson integral of a $\text{Lip}_\alpha$ function on the unit circle is in $\text{Lip}_\alpha$ of the closed unit disc, for $0 < \alpha < 1$. So we have proved the claim.

Let $h$ be a function satisfying the conditions of the claim. Put $w_n = h(z_n)$, $n = 1, 2, \ldots$ and consider the following Pick-Nevanlinna interpolation problem:

(*) Find all analytic functions $f \in H^\infty(D)$ satisfying $||f||_\infty \leq 1$ and $f(z_n) = w_n$, $n = 1, 2, \ldots$. 
Since \( h \) solves \((*)\) and \( \int_0^{2\pi} \log(1 - |h(e^{i\theta})|) \, d\theta > -\infty \), the function
\[
h(z) + B(z) \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log(1 - |h(e^{i\theta})|) \, d\theta \right),
\]
where \( B \) is the Blaschke product with zeros \( \{z_n\} \), also solves \((*)\). Therefore, \((*)\) has more than one solution. Then, the set \( E \) of all solutions of \((*)\) can be parametrized as:
\[
E = \left\{ f \in H^\infty(D) : f = \frac{p\varphi + q}{r\varphi + s}, \, \varphi \in H^\infty(D) \text{ and } \|\varphi\|_\infty \leq 1 \right\}.
\]
Suppose now that \( s \in H^p(D) \) for some \( p > 0 \), and let us arrive at a contradiction.

Choosing \( \varphi \equiv 0 \) in the parametrization, one has \( q/s \in H^\infty(D) \) and \( q/s(z_n) = w_n, \, n = 1, 2, \ldots \). It is well known that \( 1/|s(z)|^2 \leq 1 - |q/s(z)|^2 \) for \( z \in D \) (see Lemma 3 in \([3]\)). So
\[
\frac{1}{|s(z_n)|^2} \leq 1 - \left| \frac{q}{s}(z_n) \right|^2 = 1 - |w_n|^2
\]
(1)
\[
= 1 - |h(z_n)|^2 \leq 2C(\alpha)(1 - |z_n|^\alpha) \quad \text{for each } \alpha < 1.
\]

Since the arcs \( \{I_n\} \) are pairwise disjoint, the sequence \( \{z_n\} \) is an interpolating sequence of \( H^\infty(D) \) (see \([4, \text{p. 77}]\)). Applying Carleson’s theorem (see \([1, \text{p. 63}]\)), one gets
\[
\sum_{n=1}^\infty (1 - |z_n|^\alpha)|s(z_n)|^p < +\infty.
\]
But using (1) for any fixed \( \alpha < 1 \),
\[
\sum_{n=1}^\infty (1 - |z_n|^\alpha)|s(z_n)|^p \geq 2^{-p/2}C(\alpha)^{-p/2} \sum_{n=1}^\infty (1 - |z_n|)^{1-p\alpha/2}
\]
\[
= 2^{-p/2}C(\alpha)^{-p/2}\sum_{n=1}^\infty c_n^{1-p\alpha/2} = +\infty
\]
because \( \sum_{n=1}^\infty c_n^q = +\infty \) for each \( q < 1 \). This gives us the contradiction. Therefore, \( s \notin H^p(D) \) for each \( p > 0 \).

**References**


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