A SEMIGROUP TREATMENT OF A ONE DIMENSIONAL NONLINEAR PARABOLIC EQUATION

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ABSTRACT. In this paper we study the existence, uniqueness and differentiability of solutions (in a certain generalized sense) for the nonlinear parabolic equation

\[ u_t = u_{xx} - F(u, u_x) \quad (0 < x < 1, t > 0), \]

under the maximal monotone boundary conditions:

\[ (-1)^i u_x(t, i) \in \beta_i(u(t, i)), \quad t > 0, \quad i = 0, 1. \]

1. Introduction and summary of results

We consider the following initial boundary value problem:

\[
\begin{cases}
    u_t = u_{xx} - F(u, u_x), & t > 0, \quad 0 < x < 1 \\
    u_x(t, 0) \in \beta_0(u(t, 0)), & -u_x(t, 1) \in \beta_1(u(t, 1)), \quad t > 0 \\
    u(0, x) = u_0(x), & 0 \leq x \leq 1,
\end{cases}
\]

from the viewpoint of the theory of nonlinear semigroups.

Throughout this paper we assume that

(\beta) \quad \beta_i \text{ is a maximal monotone graph in } \mathbb{R} \times \mathbb{R} \text{ with } 0 \in \beta_i(0), \quad i = 0, 1,

(F0) \quad F: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \text{ is a continuous function such that } F(0, 0) = 0 \text{ and } F(\cdot, p): \mathbb{R} \to \mathbb{R} \text{ is nondecreasing for each } p \in \mathbb{R}.

Choose \( C[0, 1] \) as the Banach space associated with (P) and define an operator \( A \) in \( C[0, 1] \) by

\[
D(A) = \{ u \in C^2[0, 1]; \quad (-1)^i u'(i) \in \beta_i(u(i)), \quad i = 0, 1 \}
\]

\[ Au = u'' - F(u, u'), \quad u \in D(A).\]

Then \( A \) is dissipative in \( C[0, 1] \) and the problem (P) can be viewed as an evolution equation:

\[
\frac{du}{dt} = Au(t), \quad t > 0, \quad u(0) = u_0 \text{ in } C[0, 1].
\]
If $A$ is $m$-dissipative in $C[0,1]$, then for every $u_0 \in \overline{D(A)}^{C[0,1]}$ the Crandall–Liggett theorem [2] provides us with the unique solution (in a certain generalized sense) of (1.2) represented by the exponential formula:

$$(1.3) \quad u(t, \cdot) = \lim_{\lambda \to 0^+} (I - \lambda A)^{\frac{-t}{\lambda}} u_0(\equiv \exp(tA)u_0), \quad t \geq 0.$$ 

We shall call $u(t, \cdot)$ the semigroup solution of (P).

The purpose of this paper is to give conditions on $F(u, p)$ under which the operator $A$ defined by (1.1) is $m$-dissipative in $C[0,1]$.

**Theorem.** Let $(\beta)$ and $(FO)$ hold, and assume in addition that one of the following conditions holds:

1. \begin{equation}
|F(u, p)| \leq \zeta(u) \cdot \eta(|p|) \quad \text{for all } (u, p) \in R \times R, \text{ where } \zeta(\cdot) : R \to [0, \infty) \text{ is continuous, and } \eta(\cdot) : [0, \infty) \to (0, \infty) \text{ is continuous, nondecreasing and } \int_0^\infty \frac{r}{\eta(r)} dr = \infty.
\end{equation}

2. \begin{equation}
F : R \times R \to R \text{ is continuous and } F(0, p) = 0 \text{ for all } p \in R.
\end{equation}

Then the operator $A$ defined by (1.1) is $m$-dissipative in $C[0,1]$.

Our result is closely related to the work of Konishi [5], in which he treated the existence, uniqueness and differentiability of semigroup solutions for the equation $u_t = u_{xx} - F(u_x)$ under the periodic boundary conditions. By almost the same methods in [5], we can examine the differentiability of semigroup solutions of (P).

Let us define the operator $\tilde{A}$ in $L^\infty(0,1)$ by

$$(1.4) \quad D(\tilde{A}) = \{u \in C^1[0,1] ; u'' \in L^\infty(0,1), (-1)^i u'(i) \in \beta_i(u(i)), i = 0,1\},$$

$$\tilde{A} u = u'' - F(u, u'), u \in D(\tilde{A}).$$

**Proposition.** Let the hypothesis of the theorem be satisfied. For each $u_0 \in D(\tilde{A})$ let $u(t) = \exp(tA)u_0$, $t \geq 0$ be the semigroup solution of (P). Then we have the following:

(i) $u(t) \in D(\tilde{A})$ for $t \geq 0$.

(ii) $u \in C([0, \infty); C^1[0,1])$.

(iii) $u : [0, \infty) \to L^\infty(0,1) = L^1(0,1)^*$ is weakly $^*$ continuously differentiable and

$$w^* - (d/dt)u(t) = \tilde{A}u(t) \text{ in } L^\infty(0,1) \text{ for } t \geq 0.$$

2. Preliminaries

In what follows, by $C[0,1]$ we mean the Banach space of all real valued continuous functions on $[0,1]$ with the supremum norm $\| \|$ . $C[0,1]$ is a closed subspace of the Banach space $L^\infty(0,1)$ with the norm $\|u\|_{\infty} = \text{ess sup}\{|u(x)| ; x \in (0,1)\}$, and $\|u\|_{\infty} = \|u\|$ for all $u \in C[0,1]$. Similarly, let $C^1[0,1]$ denote the Banach space of all continuously differentiable functions on $[0,1]$ with the norm $\|u\|_1 = \max\{\|u\|, \|u'\|\}$. 

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An operator \( A : D(A) \subset X \to X \) in a Banach space \((X, \| \cdot \|)\) is called dissipative if
\[
\| u - v \| \leq \| u - v - \lambda (Au - Av) \| \quad \text{for all } u, v \in D(A)
\]
and \( \lambda > 0 \).

A dissipative operator \( A \) in \( X \) is said to be \( m \)-dissipative if \( R(I - \lambda A) = X \) for every, or equivalently, for some \( \lambda > 0 \); here \( R(I - \lambda A) \) denotes the range of \( I - \lambda A \).

A subset \( \beta \subset \mathbb{R} \times \mathbb{R} \) is called monotone if \((u_1 - u_2)(v_1 - v_2) \geq 0 \) whenever \( v_i \in \beta(u_i), \ i = 1, 2 \). A monotone set not properly contained in any other monotone set is called maximal monotone. For more details on this material see [1].

3. Proof of Theorem

We begin with the following lemmas.

**Lemma 1.** Let \((\beta)\) and \((\text{F0})\) hold. Then the operator \( A \) defined by (1.1) is dissipative in \( C[0,1] \).

**Proof.** Let \( u, v \in D(A) \) and let \( \| u - v \| = |u(x_0) - v(x_0)|, \ x_0 \in [0,1] \). Then we have
\[
(3.1) \quad u'(x_0) - v'(x_0) = 0, \quad (u(x_0) - v(x_0))(u''(x_0) - v''(x_0)) \leq 0,
\]
by maximum principle arguments and by the monotonicity of \( \beta_i, \ i = 0, 1 \).

From this and \((\text{F0})\) the assertion follows easily.

The following lemma is a slight modification of [4, Chapter 12, Lemma 5.1].

**Lemma 2.** Let \( \eta : [0, \infty) \to (0, \infty) \) be a continuous and nondecreasing function satisfying \( \int_0^\infty (r/\eta(r)) \, dr = \infty \), and let \( R \geq 0, \ K \geq 0, \ C \geq 0 \) be real numbers.

Then there exists a number \( M \) (depending only on \( \eta, R, K, C \)) with the following property:

If \( u \in C^2[0,1], \ |u(x)| \leq R \) and \( |u''(x)| \leq K + C\eta(|u'(x)|), \ x \in [0,1] \), then \( |u'(x)| \leq M \) for all \( x \in [0,1] \).

**Proof of Theorem.** By Lemma 1 it suffices to show \( R(I - A) = C[0,1] \). Define the operator \( A_0 \) in \( C[0,1] \) by \( A_0u = u'' \) for \( u \in D(A) \). Then \( A_0 \) is \( m \)-dissipative in \( C[0,1] \) and \((I - A_0)^{-1} : C[0,1] \to C^1[0,1] \) is continuous and compact. (See [3, Lemma 4.3] or [7, Proposition 1].)

Fix an arbitrary \( f \in C[0,1] \) and solve the equation
\[
(3.2) \quad u = (I - A_0)^{-1}(f - F(u, u')).
\]
Then the solution of (3.2) satisfies \( u - Au = f \).

For \( m = 1, 2, \ldots \) let
\[
B_m v = \begin{cases} 
  f - F(v, v') & \text{if } \|v\|_1 \leq m, \\
  f - F(mv/\|v\|_1, mv'/\|v\|_1) & \text{if } \|v\|_1 > m.
\end{cases}
\]
\[ B_m : C^1[0,1] \to C[0,1] \] is continuous and uniformly bounded. Consequently
\[ T_m = (I - A_0)^{-1} B_m : C^1[0,1] \to C^1[0,1] \] is continuous, compact and uniformly bounded. Hence \( T_m \) maps some closed ball in \( C^1[0,1] \) into itself, and so by the Schauder’s fixed point theorem we get a fixed point
\[(3.3) \quad u_m \in D(A) ; \quad u_m = (I - A_0)^{-1} B_m u_m, \quad m = 1, 2, \ldots.\]

If there is an \( m_0 \) such that \( \|u_{m_0}\|_1 \leq m_0 \) then \( u_{m_0} \) satisfies (3.2), and we complete the proof. Suppose, for contradiction, that \( \|u_m\|_1 > m \) for all \( m \).

Then (3.3) can be rewritten as
\[(3.4) \quad u_m - u_m'' + F(mu_m/\|u_m\|_1, mu_m'/\|u_m\|_1) = f, \quad m = 1, 2, \ldots.\]

Let \( x_0 \) be any point in \([0,1]\) satisfying \( ||u_m|| = |u_m(x_0)| \). Then, using (3.1) with \( u = u_m \) and \( v = 0 \) we get \( u_m'(x_0) = 0, \quad u_m(x_0)u_m''(x_0) < 0 \). Plugging this into (3.4) gives
\[ ||u_m||^2 + F(mu_m(x_0)/\|u_m\|_1, 0)u_m(x_0) \leq \|f\||u_m||. \]

From this and (F0) we obtain
\[(3.5) \quad ||u_m|| \leq \|f\| \quad \text{for all } m. \]

On the other hand, if \( F \) satisfies the condition (F1) then it follows from (3.4) and (3.5) that
\[ |u_m''(x)| \leq 2\|f\| + C\eta(|u_m'(x)|) \quad \text{for all } x \in [0,1], \]
where \( C = \max\{\zeta(u) ; |u| \leq \|f\|\} \). Hence, it follows from Lemma 2 that there exists a constant \( M \) such that
\[(3.6) \quad ||u_m'|| \leq M \quad \text{for all } m. \]

Thus, by (3.5) and (3.6), we have
\[ ||u_m||_1 \leq \max\{\|f\|, M\} \quad \text{for all } m, \]
which is the desired contradiction.

Next suppose that \( F \) satisfies the condition (F2). Multiplying (3.4) by \( u_m'' \) and integrating over \([0,1]\) we obtain
\[ -[u_m u_m'^1_0 + \int_0^1 u_m'^2 dx + \int_0^1 u_m''^2 dx] - \int_0^1 F(mu_m/\|u_m\|_1, mu_m'/\|u_m\|_1) u_m'' dx = - \int_0^1 f u_m'' dx. \]

Let \( G(u,p) = \int_0^p F(u,r) \, dr \). Then, by using (F0) and (F2), we can check easily that \( G(u,p)up \geq 0 \) and \( G_u(u,p)p \geq 0 \) for all \( u,p \in R \). Hence by (\( \beta \)) we
have
\[ \int_0^1 F(\mu_m'/\|\mu_m\|_1, \mu_m'/\|\mu_m\|_1)u_m'' \, dx = m^{-1}\|u_m\|_1 G(\mu_m'/\|\mu_m\|_1, \mu_m'/\|\mu_m\|_1) I_0 \]
\[ - \int_0^1 G_u(\mu_m'/\|\mu_m\|_1, \mu_m'/\|\mu_m\|_1)u_m' \, dx \leq 0. \]

From this and the monotonicity of \( \beta_i, i = 0, 1 \), we see that
\[ \int_0^1 u_m''^2 \, dx \leq - \int_0^1 f u_m'' \, dx \leq \frac{1}{2} \int_0^1 (f^2 + u_m''^2) \, dx, \]
and hence
\[ \int_0^1 u_m''^2 \, dx \leq \int_0^1 f^2 \, dx \leq \|f\|^2. \]

Since \( u_m'(x) = \int_x^1 u_m''(r) \, dr \), we can conclude that
\[ (3.7) \quad \|u_m'\| \leq \|f\|. \]

Thus, by (3.5) and (3.7), we have
\[ \|u_m\|_1 \leq \|f\| \quad \text{for all } m, \]
which is also the desired contradiction. This completes the proof.

**Remark.** The Theorem is true if we assume \((\beta), (F0)\) and one of the following conditions:

(F3) \( F_u: R \times R \to R \) is continuous, \( F(u, p) \geq 0 \) for all \( (u, p) \in R \times R \) and \( R(\beta_i) \subset [0, \infty), i = 0, 1 \).

(F4) \( F_u: R \times R \to R \) is continuous, \( F(u, p) \leq 0 \) for all \( (u, p) \in R \times R \) and \( R(\beta_i) \subset (-\infty, 0], i = 0, 1 \).

(F5) \( F_u: R \times R \to R \) is continuous and \( \beta_i = 0, i = 0, 1 \).

### 4. Proof of Proposition

In a similar way as in [5] we can prove the proposition. However, for completeness, we give here an outline of the proof.

The operator \( \tilde{A} \) defined by (1.4) is dissipative in \( L^\infty(0, 1) \) and satisfies the range condition:
\[ R(\lambda(\tilde{A})) \subset R(I - \lambda \tilde{A}) = C[0, 1] \supset \overline{D(\tilde{A})}^{C[0,1]} = \overline{D(\tilde{A})}^{L^\infty(0,1)} \]
\[ = \{u \in C[0, 1]; u(i) \in \overline{D(\beta_i)}; i = 0, 1\}(=D), \quad \lambda > 0. \]

Thus \( \tilde{A} \) generates a nonlinear contraction semigroup \( \exp(t\tilde{A}) \), \( t \geq 0 \) on \( D \) in the sense of Crandall–Liggett [2], which together with (1.3) implies
\[ \exp(t\tilde{A})u_0 = \exp(tA)u_0 \quad \text{for all } t \geq 0, \ u_0 \in D. \]
Let $u_0 \in D(\tilde{A})$. Then
\[
\|(I - \lambda A)^{-[t/\lambda]} u_0\| \leq \|u_0\| ,
\]
\[
\|\tilde{A}(I - \lambda A)^{-[t/\lambda]} u_0\|_\infty \leq \|\tilde{A} u_0\|_\infty \text{ for each } \lambda > 0 \text{ and } t \geq 0.
\]
Define the operator $\Lambda$ by
\[
D(\Lambda) = \{ u \in C[0, 1]; u' \in L^\infty(0, 1) \},
\]
\[
\Lambda u = u' \text{ for } u \in D(\Lambda).
\]
Then, by the same way as in the proofs of (3.6) and (3.7), we obtain the estimate:
\[
\|\Lambda (I - \lambda A)^{-[t/\lambda]} u_0\| \leq M,
\]
where $M$ is a constant independent of $\lambda > 0$ and $t \geq 0$. Hence it follows that
\[
\|\Lambda^2 (I - \lambda A)^{-[t/\lambda]} u_0\|_\infty \leq \|\tilde{A} u_0\|_\infty + \max\{|F(u, p)|; |u| \leq \|u_0\|, |p| \leq M\}.
\]
From these estimates we see that
\[
s - \lim_{\lambda \to 0^+} \Lambda (I - \lambda A)^{-[t/\lambda]} u_0 = \Lambda \exp(tA)u_0 \text{ in } C[0, 1],
\]
\[
w^* - \lim_{\lambda \to 0^+} \tilde{A}(I - \lambda A)^{-[t/\lambda]} u_0 = \tilde{A} \exp(tA)u_0 \text{ in } L^\infty(0, 1)
\]
for each $t \geq 0$.

Furthermore, we can check that
\[
\exp(\cdot)u_0 \in C([0, \infty); C^1[0, 1]) \text{ and the function}
\]
\[
t \in [0, \infty) \to \tilde{A} \exp(tA)u_0 \in L^\infty(0, 1)
\]
is weakly* continuous. Thus, by using the estimate due to Ōharu [6, (8)], we can conclude that
\[
\exp(tA)u_0 - u_0 = w^* - \int_0^t \tilde{A} \exp(sA)u_0 ds \text{ in } L^\infty(0, 1),
\]
$t \geq 0$, from which the assertion follows.

References

5. Y. Konishi, *On $u_t = u_{xx} - F(u_x)$ and the differentiability of the nonlinear semigroup associated with it*, Proc. Japan Acad. 48 (1972), 281–286.

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