ON THE MONOTONICITY OF THE PERMANENT

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Abstract. Let \( \Omega_n \) denote the set of all \( n \times n \) doubly stochastic matrices and let \( J_n = [1/n]_{n \times n} \). For \( A \in \Omega_n \), if \( f_A(t) = \text{per}((1-t)J_n + tA) \) is a nondecreasing function of \( t \) on \([0,1]\), we say that the monotonicity of permanent (abb. MP) holds for \( A \). Friedland and Minc [3] proved MP for \( (nJ_n - In)/(n - 1) \). In [6], Lih and Wang proposed a problem of determining whether MP holds for \( J_{n_1} \oplus \cdots \oplus J_{n_k} \), \( n_i > 0 \).

In this note, we prove MP for \( (mJ_m - In)/(n - 1) \), extending the result of Friedland and Minc, and give an affirmative answer to the Lih and Wang’s question.

1. Introduction

Let \( \Omega_n \) denote the set of all \( n \times n \) doubly stochastic matrices and let \( K_n \) denote the \( n \times n \) matrix of 1’s. As usual, let \( I_n \) denote the identity matrix of order \( n \) and let \( J_n = K_n/n \). For an \( n \times n \) matrix \( A \) and for a real number \( t \), let \( A_t = (1-t)J_n + tA \) and let \( f_A(t) = \text{per} A_t \), the permanent of \( A_t \).

In [3], Friedland and Minc proved that \( f_A(t) \) is a monotone increasing function of \( t \) on the closed unit interval \([0,1]\) if either \( A = I_n \) or \( A = (K_n - I_n)/(n - 1) \). The problem of finding matrices \( A \in \Omega_n \) for which \( f_A(t) \) is monotone increasing is found in [9, p. 158]. This property of \( f_A(t) \) now is called the monotonicity of permanent (abb. MP) for \( A \). In addition to those matrices which are permutation equivalent to \( I_n \) or \( (K_n - I_n)/(n - 1) \), several classes of matrices in \( \Omega_n \) have turned out to satisfy MP (see [5, 8] for example). In [6], Lih and Wang proved MP for \( J_s \oplus J_{n-s} \) and for

\[
\begin{bmatrix}
0 & Y \\
Y^T & Z
\end{bmatrix}
\]

where \( Y \) is an \( (n-t) \times t \) submatrix of \( J_t \) and \( Z = (2 - n/t)J_t \), and proposed a problem of determining whether MP holds for \( J_{n_1} \oplus \cdots \oplus J_{n_k} \) where \( n_i > 0 \), \( i = 1, \ldots, k \).

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In this note we prove MP for \(((K_m - P) \otimes K_s)/(m - 1)s\) for any \(m \times m\) permutation matrix \(P\) where \(\otimes\) stands for the Kronecker product and \(m \geq 2\), \(s \geq 1\), extending Friedland and Minc' result, and give an affirmative answer to Lih and Wang's problem.

For a matrix \(A\), let \(A(i|j)\) denote the matrix obtained from \(A\) by deleting the row \(i\) and column \(j\). Let \(D\) be a real valued function of \(\Omega_n\) defined by

\[
D(A) = \per A - \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \per A(i|j)
\]

for \(A \in \Omega_n\).

For \(A = [a_{ij}] \in \Omega_n\), we compute

\[
\frac{d}{dt} f_A(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(a_{ij} - \frac{1}{n}\right) \per A_i(i|j)
\]

and

\[
D(A_i) = \frac{t}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(a_{ij} - \frac{1}{n}\right) \per A_i(i|j)
\]

so that

\[
D(A_i) = \frac{t}{n} \frac{d}{dt} f_A(t).
\]

Thus we have

**Lemma 1.** MP holds for \(A \in \Omega_n\) if and only if \(D(A_i) \geq 0\) for all \(t, 0 \leq t \leq 1\).

2. MP for \(((K_m - I_m) \otimes K_s)/(m - 1)s\).

To prove MP for \(((K_m - I_m) \otimes P)/(m - 1)s\) for an \(m \times m\) permutation matrix \(P\), we may assume that \(P = I_m\). Let \(n = ms\) so that

\[
\frac{1}{(m - 1)s} (K_m - I_m) \otimes K_s = \frac{1}{n - s} (K_n - K_s^{(m)})
\]

where \(K_s^{(m)}\) denotes the \(m\)-fold direct sum of \(K_s\)'s. Note that, in the case of \(s = 1\),

\[
\frac{1}{n - 1} (K_m - I_m) \otimes K_s = \frac{1}{n - 1} (K_n - I_n),
\]

the matrix for which MP was proved by Friedland and Minc [3].

The following Lemma was conjectured by Sinkhorn [10] and proved by Bapat [1].

**Lemma 2 [Sinkhorn-Bapat].** Let \(A \in \Omega_n\) be such that \(\per A(i|j) = \per A\) for all \(i, j = 1, \ldots, n\); then either \(A = J_n\) or \(A = \frac{1}{2}(I_n + P_n)\), up to permutations of rows and columns where \(P_n\) stands for the \(n \times n\) full-cycle permutation matrix corresponding to the \(n\)-cycle \((1, 2, \ldots, n)\).

Let \(A\) be an \(n \times n\) real matrix. We say that the positions \((i, j)\) and \((k, l)\) of \(A\) are *equivalent* if there exist permutation matrices \(P, Q\) such that \(PAQ = A\).
and the transformation $X \rightarrow PXQ$ takes the $(i, j)$-entry of $X$ onto the $(k, l)$-entry of $PXQ$ for all $n \times n$ matrix $X$.

It is clear that, if the positions $(i, j)$ and $(k, l)$ of $A$ are equivalent then,

$$\text{per } A(i|j) = \text{per } A(k|l).$$

**Lemma 3.** Let $m \geq 2$, $s \geq 1$ be integers and $n = ms$. For $(K_m - I_m) \otimes K_s := [x_{ij}]$, let $Z = \{(i,j)|x_{ij} = 0\}$. Let $A = [a_{ij}]$ be an $n \times n$ matrix such that all $a_{ij}, (i,j) \in Z$, are the same and all $a_{kl}, (k,l) \notin Z$, are the same.

Then all the $Z$-positions of $A$ are equivalent and all the off $Z$-positions of $A$ are equivalent.

**Proof.** For $p, q = 1, \ldots, m$, let $T_{pq} = \{(i,j)|(p-1)s + 1 \leq i \leq ps, (q-1)s + 1 \leq j \leq qs\}$. Then $Z = \bigcup_{p=1}^{m} T_{pp}$. Clearly, all the positions of $A$ in a single $T_{pp}$ are equivalent.

To show the equivalence of the position $(1,s+1)$ and $(s+1,1)$ of $A$, we can use the transformation $X \rightarrow PXP$ where

$$P = \begin{bmatrix} 0 & I_s \\ I_s & 0 \end{bmatrix} \bigoplus I_{n-2s}.$$  

Thus we see that all the $T_{12} \cup T_{21}$-positions of $A$ are equivalent. In addition to that, this transformation tells us also that all the position of $A$ in $T_{11} \cup T_{12}$ are equivalent.

In the case that $m > 3$, we see that the transformation $X \rightarrow QXQ$ with

$$Q = I_s \bigoplus \begin{bmatrix} 0 & I_s \\ I_s & 0 \end{bmatrix} \bigoplus I_{n-3s}$$

assures the equivalence of all the $T_{12} \cup T_{13}$-positions of $A$. Repeating similar arguments a finite number of times, we see that all the $Z$-positions of $A$ are equivalent and all the off $Z$-positions of $A$ are equivalent. □

**Corollary.** Let $m \geq 2$, $s \geq 1$ be integers, $n = ms$, and let $A = (1/n-s)(K_m - I_m) \otimes K_s$. Then, for each $t \in [0, 1]$, there exist real numbers $\lambda_t$, $\mu_t$ such that

$$\text{per } A_t(i|j) = \text{per } A = \begin{cases} \lambda_t & \text{if } (i,j) \in Z \\ \mu_t & \text{otherwise}. \end{cases}$$

Expanding $\text{per } A_t$ along the first row, we get the following relationship between $\lambda_t$ and $\mu_t$:

$$s(1-t)\lambda_t + (n-s)(1-t+nt)\mu_t = 0.$$

Thus, if $t \neq 1$, we have

$$s(1-t)\lambda_t + (n-s+st)\mu_t = 0.$$  

**Lemma 4.** Let $m \geq 2$, $s \geq 1$ be integers, $n = ms$, and let $A = (1/n-s)(K_m - I_m) \otimes K_s$. Then, for each $t \in [0, 1]$,

$$D(A_t) = \frac{t}{1-t} \mu_t,$$
where \( \mu_t \) is the number defined in (1).

**Proof.** Taking account of (1), we have

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \text{per} A_t(i,j) = \left( \sum_{(i,j) \in \mathbb{Z}} + \sum_{(i,j) \notin \mathbb{Z}} \right) \text{per} A_t(i,j)
\]

\[
= n^2 \text{per} A_t + |Z| \lambda_t + (n^2 - |Z|) \mu_t
\]

\[
= n^2 \text{per} A_t + n \sigma \mu_t + n(n-s) \mu_t.
\]

Therefore, by (2),

\[
D(A_t) = \text{per} A_t - \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \text{per} A_t(i,j)
\]

\[
= -\frac{1}{n} (s \lambda_t + (n-s) \mu_t)
\]

\[
= \frac{t}{1-t} \mu_t. \quad \square
\]

Now we are ready to prove the following theorem which is an extended version of Friedland and Minc's result [3], of which we give here an elementary combinatorial proof.

**Theorem 1.** Let \( m \geq 2, \, s \geq 1 \) be integers, \( n = ms \). Then MP holds for \((1/n - s)(K_m - P) \otimes K_s\) for all \( m \times m \) permutation matrices \( P \).

**Proof.** Let \( A = (1/n - s)(K_m - I_m) \otimes K_s \). To prove the theorem, it suffices to prove the MP for \( A \) only. We need to show that \( D(A_t) \geq 0 \) for all \( t \in (0,1) \), or equivalently, that \( \mu_t \geq 0 \) for all \( t \in (0,1) \), by Lemmas 1 and 4.

Suppose that \( \mu_d < 0 \) for some \( d \in (0,1) \).

Since \( J_n \) is the unique matrix in \( \Omega_n \) with the minimum permanent [2], MP holds for matrices in a sufficiently small neighborhood of \( J_n \) in \( \Omega_n \). Hence we can take a \( c \in (0,d) \) such that \( D(A_c) > 0 \) so that \( \mu_c > 0 \). Then, by Mean Value Theorem, there exists a \( t \in (c,d) \) such that \( \mu_t = 0 \) and hence also that \( \lambda_t = 0 \) by (2). But then \( \text{per} A_t(i,j) = \text{per} A_t \) for all \( i,j = 1, \ldots, n \), even if \( A_t \neq J_n \) and \( A_t \neq \frac{1}{2}(I_n + P_n) \), contradicting Lemma 2.

Thus we have shown that \( \mu_t > 0 \) for all \( t \in (0,1) \) and the proof is completed. \( \square \)

### 3. MP for p.s.d. symmetric matrices in \( \Omega_n \)

In this section we give a very simple proof for MP for positive semi-definite (abbreviated p.s.d.) symmetric doubly stochastic matrices, which assures the MP for \( J_{n_1} \oplus \cdots \oplus J_{n_k} \) where \( n_i, \, i = 1, \ldots, k \), are positive integers.

The following Lemma can be found in [7].

**Lemma 5** [Marcus and Merris]. If \( A \in \Omega_n \) is p.s.d. symmetric, then \( D(A) \geq 0 \).

As a corollary of Lemma 5, we have the following
Theorem 2. MP holds for any p.s.d. symmetric doubly stochastic matrix.

Proof. If $A \in \Omega_n$ is p.s.d. symmetric, then so is $A_t$ for all $t \in [0,1]$, so that $D(A_t) \geq 0$ for all $t \in [0,1]$ by Lemma 5. Now MP for any p.s.d. symmetric doubly stochastic matrix follows from Lemma 1. □

Since $J_{n_1} \oplus \cdots \oplus J_{n_k}$, $n_i > 0$, is a p.s.d. symmetric matrix, we have the following result which is the answer to the Lih and Wang's problem in [6].

Corollary 1. MP holds for $J_{n_1} \oplus \cdots \oplus J_{n_k}$, $n_i > 0$.

Corollary 2 ([4]). Let $A_1, \ldots, A_k \in \Omega_2$. Then MP holds for $\oplus A_k$.

Proof. Every matrix in $\Omega_2$ is permutation equivalent to a p.s.d. symmetric matrix. Thus, for $A_1, \ldots, A_k \in \Omega_2$, $A_1 \oplus \cdots \oplus A_k$ is permutation equivalent to a p.s.d. symmetric matrix. □

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