REFINABLE MAPS AND $\theta_n$-CONTINUA

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Abstract. Many properties of continua, e.g., irreducibility, span zero, and various kinds of aposyndesis and unicoherence, are preserved by refinable maps or their inverses. The purpose of this paper is to consider the extent to which the property of being a $\theta_n$-continuum is preserved by refinable maps or by inverses of refinable maps. One consequence of this study is to generalize results of the first author where the domain or range is a graph. Several of his results are corollaries of ours.

Definitions and preliminaries

A continuum is a compact, connected, metric space and a $\theta_n$-continuum [1, [4, p. 261] is one where the complement of any subcontinuum has at most $n$ components. A $\theta'_n$-continuum [5, p. 56] is a $\theta_n$-continuum which admits a unique, monotone, upper semi-continuous decomposition, called the canonical decomposition, whose elements have void interior and whose quotient space is a graph. The elements of the decomposition are called tranches. The order of a tranche is its order considered as an element of the quotient space. If $A$ is a subset of a continuum $X$, then $K(A) = \cap\{Q|Q$ is a subcontinuum of $X$ that contains $A$ in its interior} ([7, Theorem 2, p. 404]), and $T(A) = \{x \in X|$ no subcontinuum in $X\setminus A$ contains $x$ in its interior} [4, pp. 261–262]. It is easily seen that $K(H) = T(H)$ for each subcontinuum $H$ of a $\theta_n$-continuum. It is also easily seen that $K(H)\setminus H$ has void interior for any subcontinuum $H$ of a $\theta'_n$-continuum. From this and [4, Theorem 1, p. 263], it follows that a continuum $X$ is a $\theta'_n$-continuum if and only if $X$ is a $\theta_n$-continuum for which $K(H)\setminus H$ has void interior for each subcontinuum $H$ of $X$. Let $f: X \to Y$ mean $f$ is a function from $X$ onto $Y$. Suppose $X$ and $Y$ are continua, $\epsilon > 0$, and $f: X \to Y$. Then $f$ is an $\epsilon$-map if $f$ is a map, i.e., a continuous function, and $f^{-1}(y)$ has diameter less than $\epsilon$ for every $y \in Y$. A map $g: X \to Y$ is an...
\(\varepsilon\)-approximation of \(f\), if \(d(f(x), g(x)) < \varepsilon\) for every \(x \in X\). If \(g\) is both an \(\varepsilon\)-map and an \(\varepsilon\)-approximation of \(f\), then \(g\) is an \(\varepsilon\)-refinement of \(f\). A map \(f : X \to Y\) is refinable [2, p. 263], if for each \(\varepsilon > 0\) there is an \(\varepsilon\)-refinement of \(f\). If the \(\varepsilon\)-refinements of \(f\) can be taken to be homeomorphisms, then \(f\) is a near homeomorphism.

Let \(f : X \to Y\) be a refinable map, and \(B\) be a closed subset of \(Y\). For each positive integer \(n\), let \(f_n\) be a \(\frac{1}{n}\)-refinement of \(f\) such that \(\{f_n^{-1}[B]\}\) converges. Then \(B'\) denotes \(\lim_{n \to \infty} f_n^{-1}[B]\). Observe that \(B'\) is not uniquely determined by \(f\), since it depends on the choice of \(\{f_n\}\). Heath (Ford) and Rogers [2, Theorem 1, p. 264] have proved that, for each subcontinuum \(B\) of \(Y\), every \(B'\) is a subcontinuum of \(X\) such that \(f(B') = B\) and \(f^{-1}[\text{Int}(B)] \subseteq B'\), where \(\text{Int}(B)\) is the interior of \(B\). We use both of these results frequently. To avoid notational difficulties we occasionally use \(B^0\) for \(\text{Int}(B)\). If \(f : X \to Y\) and decomposition spaces of \(X\) and \(Y\) are also being considered we use \(f[A]\) for \(\{f(x) : x \in A\}\), if \(A \subseteq X\), and we use \(d[A, B]\) for \(\inf\{d(x_1, x_2) : x_1 \in A\) and \(x_2 \in B\}\), if \(A \subseteq X\) and \(B \subseteq X\).

1. Focus on the Domain

In this section we consider refinable maps whose domains are \(\theta_n\)-continua or \(\theta'_n\)-continua. First we establish a lemma.

**Lemma 1.** Suppose \(f : X \to Y\) is refinable and \(B\) is a closed subset of \(Y\). Then \([K(B)]' \subseteq K(B')\), if these sets are determined by the same sequence of \(\frac{1}{n}\)-refinements of \(f\).

**Proof.** For \(i = 1, 2, \ldots\), let \(f_i\) be a \(\frac{1}{i}\)-refinement of \(f\) such that \(f_1^{-1}(B), f_2^{-1}(B), \ldots, f^{-1}(K(B)), f_2^{-1}(K(B)), \ldots\), converge to \(B'\) and \([K(B)]'\), respectively. We wish to show that \(x \notin [K(B)]'\) if \(x \notin K(B')\). Suppose \(x \notin K(B')\) and let \(H\) be a subcontinuum of \(X\) such that \(B' \subseteq H' \subseteq H \subseteq X\setminus\{x\}\). Let \(\varepsilon > 0\) be a positive number such that \(d(x, H) > 2\varepsilon\) and \(N_{2\varepsilon}(B') \subseteq H\). Let \(m > \frac{1}{\varepsilon}\) be such that \(f_m^{-1}(B) \subseteq N_{\varepsilon}(B')\). Then \(B = f_m(f_m^{-1}(B)) \subseteq Y\setminus f_m(X\setminus H') \subseteq \{f_m(H)\} \subseteq f_m(H),\) since \(d(f_m^{-1}(B), X\setminus H') > \varepsilon\) and \(f_m\) is an \(\varepsilon\)-map. Hence \(K(B) \subseteq f_m(H)\). Since \(f_m\) is an \(\varepsilon\)-map and \(d[H, N_{\varepsilon}(x)] > \varepsilon\), we have \(f_m^{-1}(K(B)) \subseteq f_m^{-1}(f_m(H)) \subseteq X\setminus N_{\varepsilon}(x),\) so \(x \notin [K(B)]'\).

**Corollary 2.** If the continuum \(X\) has the property that \(K(H\setminus H)\) has void interior for every subcontinuum \(H\) of \(X\), and \(f : X \to Y\) is a refinable map, then \(Y\) has this property for \(K\) also.

**Proof.** Let \(H\) be a subcontinuum of \(Y\). Then \(f^{-1}([K(H)]' \subseteq [K(H)]' \subseteq K(H')\) by Lemma 1, and therefore, since \(K(H')\setminus H'\) has void interior, \(f^{-1}(\{K(H)\}) \subseteq H'\). So \([K(H)]' = f(f^{-1}(\{K(H)\})' \subseteq f(H') = H'.\) Hence \([K(H\setminus H)]' = \emptyset\).

The next two theorems show that the image of a \(\theta'_n\)-continuum under a refinable map is a \(\theta_n\)-continuum and, although domain and image need not
be homeomorphic [3, Example 2, p. 148], the canonical decompositions into graphs are homeomorphic.

**Theorem 3.** If $X$ is a $\theta'_n$-continuum and $f: X \rightarrow Y$ is a refinable map, then $Y$ is a $\theta'_n$-continuum, and, if $L$ is a tranche in $X$, then $f[L]$ is contained in a tranche in $Y$. Hence, if $X_D$ and $Y_D$ are the canonical decompositions of $X$ and $Y$, respectively, then $f$ induces a map from $X_D$ onto $Y_D$.

**Proof.** From Corollary 2, it follows that $Y$ has the property that $K(H) \backslash H$ has void interior for every subcontinuum $H$ of $Y$, since $X$, as a $\theta'_n$-continuum, has this property. So, to see that $Y$ is a $\theta'_n$-continuum, it suffices to show that $Y$ is a $\theta'_n$-continuum. Suppose $Y$ contains a subcontinuum $H$ such that $Y \backslash H = \bigcup_{i=1}^{n+1} A_i$, where the $A_i$'s are mutually separated sets. Since $K(H) \backslash H$ has void interior, there exists a continuum $R$ such that $H \subseteq \text{Int}(R) \subseteq R$ and $A_i \backslash R \neq \emptyset$, for $i = 1, \ldots, n+1$. Let $A^*_i = A_i \backslash R$ for $i = 1, \ldots, n+1$.

For $k = 1, \ldots$, let $f_k$ be a $\frac{1}{k}$-refinement of $f$ such that $H' = \lim_{k \to \infty} f_k^{-1}[H]$ and $R' = \lim_{k \to \infty} f_k^{-1}[R]$ exist. Since $X$ is a $\theta'_n$-continuum, $X \backslash R' = \bigcup_{m=1}^{n+1} B_m$, where each $B_m$ is a component of $X \backslash R'$ and $j \leq n$. Let $k_0$ be chosen so that $d(A^*_i) \backslash f_{k_0}[R'] \neq \emptyset$, for $i = 1, \ldots, n+1$, and $d[f_{k_0}^{-1}(H), \bigcup_{m=1}^{n+1} B_m] < 1/k_0$. Then $\bigcup_{i=1}^{n+1} \text{cl}(A^*_i) \backslash f_{k_0}[R'] \subseteq \bigcup_{m=1}^{n+1} f_{k_0}[B_m]$, and $f_{k_0}[B_m] \cap H = \emptyset$, for $m = 1, \ldots, j$. Also, since each $B_m$ is connected, $f_{k_0}[B_m]$ must be contained in one member of the collection of mutually separated sets $\{A_i|i = 1, \ldots, n+1\}$. But $j < n+1$ so it follows that there exists an integer $n_0$ such that $A^*_n \backslash f_{k_0}[X] \neq \emptyset$, a contradiction, since $f_{k_0}$ is surjective. It follows that $Y$ is a $\theta'_n$-continuum.

Let $L$ be a tranche in $X$. Then by [4, Lemma 4, pp. 268, 269], and the equality of $K(A)$ and $T(A)$ in $\theta'_n$-continua, there exists a point $p \in L$ such that $K^n(p) = L$, where $K^1(p) = K(\{p\})$ and $K^m(p) = K(K^{m-1}(p))$ for $m = 2, 3, \ldots$. Hosokawa has proved [6, Theorem 1", p. 368] that $f[K(p)] \subseteq K(f(p))$, and his argument can be extended to subsets $A$ of $X$ so that $f[K(A)] \subseteq K(f(A))$. Then $f[L] = f[K^n(p)] \subseteq K^n(f(p))$ by $n$ applications of the extension of Hosokawa's result. Since $Y$ is a $\theta'_n$-continuum, $K(A)$ has void interior if $A$ is a subcontinuum of $Y$ with void interior. By $n$ applications of this fact, $K^n(f(p))$ has void interior and hence must be contained in the tranche of $Y$ containing $f(p)$. So $f[L]$ is also contained in this tranche. Thus the map $f_D = \eta_Y \circ f \circ \eta_X^{-1}$ is a map from $X_D$ onto $Y_D$, where $\eta_X$ and $\eta_Y$ are the quotient maps from $X$ onto $X_D$ and $Y$ onto $Y_D$, respectively.

**Theorem 4.** The induced map $f_D$, of Theorem 3, from $X_D$ onto $Y_D$ is monotone and has the property that for each $y_D$ in $Y_D$, $f_D^{-1}(y_D)$ contains no simple closed curve and contains at most one point not of order two. Hence, $f_D$ is a near homeomorphism.

**Proof.** Let $y_D \in Y_D$, and let $C_1, C_2, \ldots$ be continua in $Y_D$ such that $\{y_D\} = \bigcap_{i=1}^{\infty} C_i = \bigcap_{i=1}^{\infty} C_0^i$, and $C_1 \supseteq C_0^i \supseteq C_2 \supseteq C_0^2 \supseteq C_3 \supseteq \cdots$. Then $\eta_Y^{-1}(y_D) =
\[ \bigcap_{i=1}^{\infty} \eta_{i}^{-1}[\mathcal{E}_{i}^{0}] = \bigcap_{i=1}^{\infty} \eta_{i}^{-1}[\mathcal{E}_{i}^{0}]. \]

It follows that \( f^{-1}[\eta_{Y}^{-1}(y_{D})] = f^{-1}\left[\bigcap_{i=1}^{\infty} \eta_{i}^{-1}(\mathcal{E}_{i}^{0})\right] \subseteq \bigcap_{i=1}^{\infty} f^{-1}[\eta_{i}^{-1}(\mathcal{E}_{i}^{0})] \subseteq \bigcap_{i=1}^{\infty} \bigcap_{i=1}^{\infty} \eta_{Y}^{-1}(\mathcal{E}_{i}) \] = \( f^{-1}[\eta_{Y}^{-1}(y_{D})] \). So all of these sets are the same, in particular \( f^{-1}[\eta_{Y}^{-1}(y_{D})] = \bigcap_{i=1}^{\infty} \eta_{i}^{-1}[\mathcal{E}_{i}^{0}] \). Since the latter set is the intersection of a nested sequence of continua (if the \( \frac{1}{i} \)-refinements are chosen correctly), \( f^{-1}[\eta_{Y}^{-1}(y_{D})] \) is connected and, consequently, so is \( \eta_{X}[f^{-1}[\eta_{Y}^{-1}(y_{D})]] = (\eta_{Y} \circ f \circ \eta_{X})^{-1}(y_{D}) = f_{D}^{-1}(y_{D}) \). This shows that \( \eta_{Y} \circ f \) and \( f_{D} \) are monotone.

Now let us suppose that for some point \( T \) of \( Y_{D} \), the continuum \( f_{D}^{-1}(T) \) contains two points of order greater than 2 in the graph \( X_{D} \). This implies that there are two tranches \( T_{1} \) and \( T_{2} \) of order greater than 2 as points of \( X_{D} \) and a subcontinuum \( I_{0} \) of \( X \) irreducible from \( T_{1} \) to \( T_{2} \) such that \( \eta_{Y}(T) \supseteq f[T_{1} \cup T_{2} \cup I_{0}] \) and all of the tranches in \( I_{0} \setminus (T_{1} \cup T_{2}) \) are of order 2 in \( X_{D} \).

Let \( P_{2} \) be the set of all points of \( X_{D} \) that are of order 2 in \( X_{D} \). Let \( \mathcal{Z} \) be a subset of \( X_{D} \) (i.e., a collection of tranches in \( X \)) such that (1) \( \mathcal{Z} \supseteq X_{D} \setminus P_{2} \); (2) if \( \mathcal{E} \) is a component of \( (X_{D} \setminus \eta_{X}(I_{0})) \cap P_{2} \) such that \( \text{cl}(\mathcal{E}) \) is an arc, then \( \mathcal{Z} \) contains one and only one point of \( \mathcal{E} \); (3) if \( \mathcal{E} \) is a component of \( P_{2} \) such that \( \text{cl}(\mathcal{E}) \) is a simple closed curve, then \( \mathcal{Z} \) contains two and only two points of \( \mathcal{E} \); and (4) \( T_{1} \) and \( T_{2} \) are the only members of \( \mathcal{Z} \) that intersect \( I_{0} \). Now, in addition to (4), each point of \( X_{D} \setminus \mathcal{Z} \) has order 2 in \( X_{D} \), and the closures of any two components of \( X_{D} \setminus \mathcal{Z} \) are arcs that have no more than one common end point.

Let \( L = (X \setminus I_{0}) \setminus (\bigcup \mathcal{Y}) \). Let \( \{I_{1}, \ldots, I_{a}\} \) be the collection of closures of all of those components of \( L \) that have a limit point in \( T_{1} \). Let \( \{J_{1}, \ldots, J_{b}\} \) be the collection of closures of all of those components of \( L \) that have a limit point in \( T_{2} \).

A construction similar to the construction of \( \mathcal{Z} \) can be used to select a finite collection \( \mathcal{Y} \) of points of \( Y_{D} \) such that \( T \in \mathcal{Y} \), each point of \( Y_{D} \setminus \mathcal{Y} \) has order 2 in \( Y_{D} \), and the closure of each component of \( Y_{D} \setminus \mathcal{Y} \) is an arc.

In \( X \), let \( \delta = \frac{1}{2} \min\{d[U_{1}, U_{2}]: U_{1} \in \mathcal{Z}, U_{2} \in \mathcal{Z}, \text{ and } U_{1} \neq U_{2}\} \). In \( Y \), let \( \gamma = \) a positive number such that \( N_{2}\delta[T] \) is contained in the component of \( Y \setminus (\bigcup \mathcal{Y}) \) that contains \( T \). Let \( D_{1} \) and \( D_{2} \) be connected open subsets of \( X \) such that \( T_{1} \subseteq D_{1} \subseteq N_{\delta}[T_{1}], T_{2} \subseteq D_{2} \subseteq N_{\delta}[T_{2}], \) and \( f[D_{1} \cup D_{2} \cup I_{0}] \subseteq N_{\gamma}[T] \). Note that \( 5\delta \leq d[T_{1}, T_{2}] \), so there is a point \( p \) in \( I_{0} \setminus \text{cl}(N_{2\delta}[T_{1} \cup T_{2}]) \) and \( d[p, \text{cl}(D_{1} \cup D_{2})] > \delta \). Let \( e_{1} = d[\text{cl}(D_{1}), \text{cl}((X \setminus I_{0}) \setminus \bigcup_{i=1}^{a} I_{i})] \) and \( e_{2} = d[\text{cl}(D_{2}), \text{cl}((X \setminus I_{0}) \setminus \bigcup_{i=1}^{b} J_{i})] \). Let \( e_{3} = \min\{d[I_{i}, D_{1}, J_{j}]: 1 \leq i \neq j \leq a\} \) and \( e_{4} = \min\{d[J_{j}, D_{2}, I_{i}]: 1 \leq i \neq j \leq b\} \). Let \( e_{5} = d[I_{0} \setminus (D_{1} \cup D_{2}), \text{cl}(X \setminus I_{0})] \). Let \( \varepsilon = \min\{\delta, \gamma, e_{1}, e_{2}, e_{3}, e_{4}, e_{5}\} \). Let \( f_{\varepsilon} \) be an \( \varepsilon \)-refinement of \( f \). Then \( f_{\varepsilon}[\text{cl}(D_{1})] \cap f_{\varepsilon}[\text{cl}(D_{2})] = \emptyset \), since \( d[\text{cl}(D_{1}), \text{cl}(D_{2})] \geq 3\delta \geq \varepsilon \).

It follows that either \( f_{\varepsilon}[\text{cl}(D_{1})] \) or \( f_{\varepsilon}[\text{cl}(D_{2})] \) does not intersect \( T \) in \( Y \). To see this, assume both do intersect \( T \). Since \( d[I_{0} \setminus (D_{1} \cup D_{2}), \text{cl}(X \setminus I_{0})] = \varepsilon_{5} \geq \varepsilon \) and \( d[p, \text{cl}(D_{1} \cup D_{2})] > \delta \geq \varepsilon \), it follows that \( f_{\varepsilon}(p) \notin f_{\varepsilon}[I_{0} \setminus (D_{1} \cup D_{2})] \subseteq f_{\varepsilon}[I_{0} \setminus (D_{1} \cup D_{2})] \subseteq f_{\varepsilon}[I_{0}] \) and \( f_{\varepsilon}[I_{0} \setminus (D_{1} \cup D_{2})] \cap f_{\varepsilon}[\text{cl}(X \setminus I_{0})] = \emptyset \). Hence,
$f_e[I_0 \setminus (D_1 \cup D_2)]$ is a nonvoid subset of $\text{Int}(f_e[I_0])$, and $f_e[I_0 \setminus [f_e[\text{cl}(D_1 \cup D_2)]]$ is a nonvoid open set. Since $T$ has void interior, $f_e[I_0 \setminus [f_e[\text{cl}(D_1 \cup D_2)]]$ intersects some component $C$, of $Y \setminus (U \cup \mathbb{Z})$, whose closure intersects $T$. Since $\text{cl}(C)$ is irreducible from $T$ to some member $V$ of $\mathbb{Z}$, $f_e[\text{cl}(D_1)]$ and $f_e[\text{cl}(D_2)]$ are continua that intersect $T$, and $(f_e[I_0 \setminus [f_e[\text{cl}(D_1 \cup D_2)]] \cap C$ is a nonvoid open set whose closure does not intersect $V$, it follows that $f_e[I_0 \cup [\text{cl}(D_1 \cup D_2)] \cap C$ is an open and closed (relative to $C$) proper subset of $C$. That is not possible, since $C$ is connected, so either $f_e[\text{cl}(D_1)]$ or $f_e[\text{cl}(D_2)]$ does not intersect $T$.

Since at least one of $f_e[\text{cl}(D_1)]$ and $f_e[\text{cl}(D_2)]$ does not intersect $T$, at least one of them neither intersects $T$ nor cuts the other from $T$ in the component of $(Y \setminus (U \cup \mathbb{Z})) \cup T$ that contains $T$. Assume without loss of generality that $f_e[\text{cl}(D_1)]$ does not intersect $T$ or cut $f_e[\text{cl}(D_2)]$ from $T$ in that set. Then there is a tranche $V$ in $\mathbb{Z}$ and a component $\mathcal{C}$ of $Y \setminus V$ containing $\eta^{-1}f_e[\text{cl}(D_1)]$ such that $\text{cl}(\mathcal{C})$ is an arc in $Y$ with end points $T$ and $V$. Now $f_e[I_0] \cap V = \emptyset$, since $d[T, V] \geq 2\gamma \geq \varepsilon$. As we saw above, no point of $f_e[I_0 \setminus (D_1 \cup D_2)]$ is a limit point of $f_e[I_0 \setminus \text{cl}(D_1)]$. Hence the closed set $f_e[I_0 \setminus (D_1 \cup D_2)]$ does not intersect the component of $\eta^{-1}f_e[\text{cl}(D_1)]$ that contains $V$. Now $f_e[\text{cl}(D_1) \cup (I_0 \setminus D_2)]$ is a continuum containing, but not contained in, $f_e[\text{cl}(D_1)]$, so any subcontinuum of $\eta^{-1}f_e[\text{cl}(D_1)]$ that intersects $f_e[\text{cl}(D_1)]$ and the component of $\eta^{-1}f_e[\text{cl}(D_1)]$ that contains $T$ also intersects $f_e[\text{cl}(D_1) \cup (I_0 \setminus D_2)] \setminus f_e[\text{cl}(D_1)] \subseteq f_e[I_0 \setminus (D_1 \cup D_2)]$. But $f_e[I_i] \setminus f_e[\text{cl}(D_1)] \neq \emptyset$, since $I_i \setminus N_e[\text{cl}(D_1)] \neq \emptyset$, and $f_e[I_i] \cap f_e[I_0 \setminus (D_1 \cup D_2)] = \emptyset$, for $i = 1, \ldots, a$. Hence $f_e[I_1]$ and $f_e[I_2]$ both intersect $f_e[\text{cl}(D_1)]$ and the component of $\eta^{-1}f_e[\text{cl}(D_1)]$ that contains $V$. For $i = 1, 2$, $f_e[I_i] \cap f_e[I_0 \setminus \text{cl}(D_1)] = \emptyset$, since $d[I_i, \text{cl}(D_1), I_0] \geq \varepsilon$. Therefore, $f_e[I_1 \setminus \text{cl}(D_1)]$ and $f_e[I_2 \setminus \text{cl}(D_1)]$ both intersect the component of $\eta^{-1}f_e[\text{cl}(D_1)]$ that contains $V$ and intersect each other. But $d[I_1 \setminus D_1, I_2 \setminus D_2] \geq \varepsilon$, a contradiction. It follows that $f_D$ does not map two points of order greater than 2 into the same point.

The case where $f_D^{-1}[T]$ contains one point of order greater than 2 and one point of order 1 and the case where $f_D^{-1}[T]$ contains a simple closed curve are proved with arguments similar to those of the proof above and are omitted. Since $f_D^{-1}[T]$ contains no simple closed curve and contains at most one point not of order 2, it follows from [3, Theorem 2, p. 142] that $f_D$ is a near homeomorphism.

**Corollary 5** [3, Theorem 1, p. 141]. If $X$ is a graph and $f: X \to Y$ is a refinable map, then $Y$ is a graph, $X$ and $Y$ are homeomorphic, and $f$ is a near homeomorphism.

**Proof.** The corollary follows from Theorem 4 and the observation that $Y$ is locally connected and therefore $X$, $Y$, and $f$ are essentially the same as $X_D$, $Y_D$, and $f_D$, respectively.
2. Focus on the Range

Now we switch our attention to the range $Y$ of a refinable map, give it some structure, and investigate what implications this has for the domain $X$. The main result (Theorem 9) shows that $X$ is a $\theta_n$-continuum if $Y$ is a $\theta'_n$-continuum, but a simple example is given to show that $X$ need not be a $\theta'_n$-continuum.

**Theorem 6.** If $Y$ is a $\theta_n$-continuum, $X$ is $Y$-like and has the property that $K(H) \setminus H$ has void interior for each subcontinuum $H$ of $X$, then $X$ is a $\theta'_n$-continuum.

**Proof.** Because of the hypothesis concerning the set function $K$ it suffices to show that $X$ is a $\theta_n$-continuum. Suppose that $X$ is not a $\theta_n$-continuum. Let $H$ be a subcontinuum of $X$ such that $X \setminus H = \bigcup_{i=1}^{n+1} A_i$ where $\{A_i | i = 1, \ldots, n+1\}$ is a collection of mutually separated sets. Since $K(H) \setminus H$ has void interior, there exists a subcontinuum $R$ such that $H \subseteq \text{Int}(R) \subseteq R$ and $A_i \cap R \neq \emptyset$ for $i = 1, \ldots, n+1$. Observe that $\{\text{cl}(A_i^*) | i = 1, \ldots, n+1\}$ is a collection of nonempty mutually separated closed sets. For each $i = 1, \ldots, n+1$, choose $x_i \in A_i^*$ and let $e_i = \min \{d(x_i, R) | i = 1, \ldots, n+1\}$. Let $e_2 = \min \{d(\text{cl}(A_i^*), \text{cl}(A_j^*)) | 1 \leq i < j \leq n+1\}$, and take $e = \min \{e_1, e_2\}$.

Now let $f : X \to Y$ be an $e$-map and note that $f(R)$ is a continuum with $Y \setminus f(R) \neq \emptyset$. Since $Y$ is a $\theta_n$-continuum, $Y \setminus f(R) = \bigcup_{j=1}^{j} B_j$ where $j \leq n$ and $B_k$ is a connected set for $k = 1, \ldots, j$. Since $x_i \in A_i^*$ and $d(x_i, R) \geq e$, $f(A_i^*) \cap \bigcup_{j=1}^{j} B_j \neq \emptyset$ for $i = 1, \ldots, n+1$. Because $j < n+1$, it follows that there exists an integer $j_0$ such that $B_{j_0}$ intersects at least two members of the collection $\{f(\text{cl}(A_i^*)) | i = 1, \ldots, n+1\}$. But this collection consists of mutually separated sets since $e \leq e_2$ and $f$ is an $e$-map. Thus the connected set $B_{j_0}$ can be written as the union of a finite number (greater than 1) of mutually separated sets, a contradiction.

**Corollary 7.** If $Y$ is a $\theta_n$-continuum, $X$ has the property that $K(H) \setminus H$ has void interior for each subcontinuum $H$ of $X$, and $f : X \to Y$ is a refinable map, then $X$ is a $\theta'_n$-continuum and so is $Y$.

**Proof.** Since $f$ is a refinable map, $X$ is $Y$-like. Hence, by Theorem 6, $X$ is a $\theta'_n$-continuum, and by Theorem 3, $Y$ is a $\theta'_n$-continuum.

**Corollary 8** [3, Corollary 2, p. 145]. If $Y$ is a graph, $f : X \to Y$ is a refinable map, and $X$ is locally connected, then $X$ is a graph, $X$ and $Y$ are homeomorphic, and $f$ is a near homeomorphism.

**Proof.** Since a graph is a $\theta_n$-continuum for some $n$ and any locally connected continuum has the property concerning the set function $K$ in the last corollary, it follows from that corollary that $X$ is a graph (a locally connected $\theta_n$-continuum). Then by Corollary 5, $X$ and $Y$ are homeomorphic, and $f$ is a near homeomorphism.
There exists a refinable map from the sin $\frac{1}{x}$ continuum onto an interval $Y$. This shows that the locally connected condition in Corollary 8 cannot be replaced by the property of the set function $K$ in Corollary 7, even when $Y$ is very nice. Theorem 3 shows that the property of being a $\theta_n'$-continuum held by the domain is preserved by a refinable map. If we reverse the role of the domain and range, we can show that the domain is a $\theta_n$-continuum (but not necessarily a $\theta_n'$-continuum) if the range is a $\theta_n'$-continuum. The next theorem and example show these facts.

**Theorem 9.** If $Y$ is a $\theta_n'$-continuum and $f: X \to Y$ is a refinable map, then $X$ is a $\theta_n$-continuum.

**Proof.** Let $Y_D$ be the canonical decomposition of $Y$ and let $\eta_Y$ be the quotient map from $Y$ onto $Y_D$. Since $\eta_Y \circ f$ is monotone (established in the first paragraph of the proof of Theorem 4), $f^{-1}[\eta_Y^{-1}[L]]$ is a subcontinuum of $X$ if $L$ is a subcontinuum of $Y_D$. Such subcontinua of $X$ cannot separate $X$ into more than $n$ components, since $Y$ is a $\theta_n$-continuum. To see this, assume that $L$ is a subcontinuum of $Y_D$ and $X \setminus f^{-1}[\eta_Y^{-1}[L]]$ has more than $n$ components. Then the images under $f$ of some two of them, $C_1$ and $C_2$, must lie in the same component of $Y \setminus \eta_Y^{-1}[L]$, since the latter set has at most $n$ components. So there exists a continuum $L_1$ in $Y_D \setminus L$ such that $\eta_Y^{-1}[L_1] \cap f[C_1]$ and $\eta_Y^{-1}[L_1] \cap f[C_2]$ are nonvoid. Hence $f^{-1}[\eta_Y^{-1}[L_1]]$ is not connected and the monotonicity of $\eta_Y \circ f$ is contradicted.

Let $K$ be a subcontinuum of $X$ and let $B = \{y_D \in Y_D \mid K$ intersects, but does not contain, $f^{-1}[\eta_Y^{-1}(y_D)]\}$. We wish to show that $B$ is finite. Since $Y_D$ contains only a finite number of points of order different from 2, it is sufficient to show that $B$ contains only a finite number of points of order 2. The graph $Y_D$ is the union of a finite collection $\{A_1, \ldots, A_k\}$ of arcs whose non-end points are of order 2 in $Y_D$. Hence, it is sufficient to show that none of these arcs contains 3 points of $B$ that are not end points of the arc. Suppose one of the arcs, without loss of generality $A_1$, contains 3 such points $a$, $b$, and $c$ where $b$ separates $A_1$ between $a$ and $c$. Since $\eta_Y[f[K]]$ is connected and $\{a, b, c\} \subseteq \eta_Y[f[K]]$, without loss of generality there is a point $p$ of $\eta_Y[f[K]]$ between $a$ and $b$ in $A_1$. Let $a'$, $p''$, $b'$, and $c''$ be points of $f^{-1}[\eta_Y^{-1}(a)] \setminus K$, $f^{-1}[\eta_Y^{-1}(p)] \cap K$, $f^{-1}[\eta_Y^{-1}(b)] \setminus K$, and $f^{-1}[\eta_Y^{-1}(c)] \cap K$, respectively.

If $f_\epsilon$ is an $\epsilon$-refinement of $f$, where $\epsilon < d[\{a', b'\}, K]$ and is small enough otherwise, then $f_\epsilon[\{a', b'\}] \cap f_\epsilon[K] = \emptyset$ and the points $\eta_Y(f_\epsilon(a'))$, $\eta_Y(f_\epsilon(p''))$, $\eta_Y(f_\epsilon(b'))$, and $\eta_Y(f_\epsilon(c''))$ lie along Int($A_1$) in the same order as $a$, $p$, $b$, and $c$. But this is impossible because of the irreducibility properties of $\eta_Y^{-1}[A_1]$, since $f_\epsilon[K]$ is a continuum. Therefore, $B$ is a finite set.

Suppose $X$ is not a $\theta_n$-continuum. Then there is a subcontinuum $K$ of $X$ such that $X \setminus K$ has at least $n+1$ components. But $\eta_Y[f[K]]$ is a continuum in
Y_D so \( X \setminus f^{-1}[\eta^{-1}_Y[f(K)]] \) has at most \( n \) components. Hence, some component \( C \) of \( X \setminus K \) is contained in \( f^{-1}[\eta^{-1}_Y[f(K)]] \). Since \( B \) is finite and \( C \) is connected, there is a \( y_D \) and \( Y_D \) such that \( f^{-1}[\eta^{-1}_Y(y_D)] \) contains \( C \). Let \( m \) be the order of \( y_D \) in \( Y_D \), and let \([a_i, y_D]\), \( i = 1, \ldots, m \), be disjoint (except for \( y_D \)) arcs in \( Y_D \) such that, for \( i = 1, \ldots, m \), \([a_i, y_D] = [a_i, y_D] \setminus \{y_D\} \) contains only points of order 2. For \( i = 1, \ldots, m \), there is a point \( b_i \) in the open arc \((a_i, y_D)\) such that either \( K \) contains \( f^{-1}[\eta^{-1}_Y([b_i, y_D])] \) or \( K \cap f^{-1}[\eta^{-1}_Y([b_i, y_D])] = \emptyset \). This follows from the finiteness of \( B \) and the fact that \( cl(\eta^{-1}_Y([a_i, y_D])) \) is an irreducible subcontinuum of a \( \theta'_n \)-continuum. For \( i = 1, \ldots, m \), if \( K \) does not intersect \( f^{-1}[\eta^{-1}_Y([b_i, y_D])] \), let \( K_i \) be the closure of the component of \( f^{-1}[\eta^{-1}_Y([b_i, y_D])] \) that contains \( f^{-1}[\eta^{-1}_Y([b_i, y_D])] \). On the other hand, if \( K \) contains \( f^{-1}[\eta^{-1}_Y([b_i, y_D])] \) let \( K_i \) be the component of \( K \cap f^{-1}[\eta^{-1}_Y([b_i, y_D])] \) that contains \( f^{-1}[\eta^{-1}_Y([b_i, y_D])] \).

Clearly, \( \bigcup_{i=1}^m K_i \cup K \) is a continuum that does not intersect \( C \). We now show that this leads to a contradiction. Let \( x \in C \). Note that \( f[\bigcup_{i=1}^m K_i \cup K] \) contains the neighborhood \( \bigcup_{i=1}^m \eta^{-1}_Y([b_i, y_D]) \) of \( \eta^{-1}_Y(y_D) \), since \( f[\bigcup_{i=1}^m K_i \cup K] \) is closed and \( \eta^{-1}_Y(y_D) \) has void interior. If \( \varepsilon < d(x, \bigcup_{i=1}^m K_i \cup K) \) and \( N_{\varepsilon}[\eta^{-1}_Y(b_i)] \subseteq \eta^{-1}_Y([a_i, y_D]), \) for \( i = 1, \ldots, m \), then any \( \varepsilon \)-refinement of \( f \) maps \( \bigcup_{i=1}^m K_i \cup K \) onto a connected set that must be a neighborhood of \( \eta^{-1}_Y(y_D) \) and maps \( x \) into that neighborhood, a contradiction. Since the assumption that \( K \) separates \( X \) into more than \( n \) components leads to a contradiction, \( X \) is a \( \theta_n \)-continuum.

**Example 10.** A \( \theta'_2 \)-continuum \( Y \), a \( \theta_2 \)-continuum \( X \) that is not a \( \theta'_2 \)-continuum and a refinable map \( f: X \to Y \).

Let \( X \) be the union of Knaster's planar indecomposable “S” continuum \( T \) with two end points \( p, q \) and an arc \( [q, r] \) such that \( T \cap [q, r] = \{q\} \). Let \( Y = [0, 1] \) and let \( f: X \to Y \) be a map that sends \( T \) into 0 and \( [q, r] \) homeomorphically onto \([0, 1]\). If \( 0 < \varepsilon < 1 \) let \( C_\varepsilon \) be an \( \varepsilon \)-chain from \( p \) to \( q \) in \( T \). Define \( f_\varepsilon \) such that \( f_\varepsilon(q) = \frac{\varepsilon}{2} \), \( f_\varepsilon \) maps \([q, r]\) homeomorphically onto \([\frac{\varepsilon}{2}, 1]\) in such a way that \( d(f_\varepsilon(t), f(t)) < \varepsilon \) for \( t \in [q, r] \), and \( f_\varepsilon \) maps \( T \) onto \([0, \frac{\varepsilon}{2}]\) by “following” the chain \( C_\varepsilon \) from \( p \) to \( q \). Clearly \( f_\varepsilon \) is an \( \varepsilon \)-refinement of \( f \). Thus \( f \) is refinable.

**References**

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