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APPROXIMATION BY POLYNOMIALS 
WITH LOCALLY GEOMETRIC RATES

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Abstract. In contrast to the behavior of best uniform polynomial approximants on [0,1] we show that if $f \in C[0,1]$ there exists a sequence of polynomials $\{P_n\}$ of respective degree $\leq n$ which converges uniformly to $f$ on [0,1] and geometrically fast at each point of [0,1] where $f$ is analytic. Moreover we describe the best possible rates of convergence at all regular points for such a sequence.

1. Introduction

Our paper is related to the fact that best polynomial approximants are very far from giving good approximation on subsets of the original set. In fact, let $\| \cdot \|_{[0,1]}$ denote the sup norm on $[0,1]$, let $f$ be continuous and real-valued on $[0,1]$, and $Q_n = Q_n(f)$ be the best uniform approximant to $f$ out of $\Pi_n$, the set of polynomials of degree at most $n$. A celebrated result of Kadec [2] says that the extremal points of $\{|f - Q_n(f)|\}_{n=0}^\infty$ are dense on $[0,1]$, and so on any subinterval $I \subseteq [0,1]$ the approximation given by $\{Q_n(f)\}_{n=0}^\infty$ (considering the whole sequence) is not better than on the whole interval $[0,1]$, no matter how smooth $f$ is on $I$.

In [3] it was shown that the situation radically changes if one considers near best approximants instead of best ones. For example, when $f$ is piecewise analytic on $[0,1]$ and otherwise $k$-times continuously differentiable at the...
non-regular points it was shown that for each \( \beta > 1 \) there are constants \( C \), 
\( c > 0 \) and polynomials \( p_n \in \Pi_n \), \( n = 1, 2, \ldots \), such that

\[
|f(x) - p_n(x)| \leq \frac{C}{n^{k+1}} \exp(-cn[d(x)]^\beta), \quad x \in [0, 1],
\]

where \( d(x) \) measures the distance from \( x \) to the nearest non-regular point of \( f \). It was also shown that a similar estimate with \( \beta = 1 \) is, in general, impossible. These polynomials \( p_n \), unlike the polynomials of best approximation, yield geometric convergence on (closed) intervals of analyticity even though \( \{E_n(f)\} \), 
\( E_n(f) := \|f - Q_n(f)\|_{[0,1]} \), has order only \( n^{-k-1} \).

The problem whether similar results hold for more general sets of functions (not just for piecewise analytic ones) has however remained open. To be more precise we ask the following: let \( f \in C[0, 1] \) be analytic on the (relative to \([0, 1]\)) open subset \( D \) of \([0, 1] \). Is it possible to find polynomials \( P_n \in \Pi_n \), 
\( n = 0, 1, \ldots \), such that

\[
\|f - P_n\|_{[0,1]} \to 0 \quad \text{as} \quad n \to \infty,
\]

and at every point of \( D \) we have geometric convergence, i.e.

\[
\lim_{n \to \infty} |f(x) - P_n(x)|^{1/n} < 1, \quad x \in D.
\]

We will show that this is always possible and describe the behavior of the left-hand side of (1) which is, in a certain sense, best possible.

We will assume that \( D \) is the exact set of analyticity, i.e. \( D \) contains every regular point of \( f \). For \( x \in [0, 1] \) let \( d(x) \) be the distance from \( x \) to the nearest singularity of \( f \), where \( f \) is considered to be extended to the complex plane and we also count the singularities outside \([0, 1]\). In other words, \( d(x) \) is the largest radius such that the Taylor expansion of \( f \) about \( x \) converges in \( \{z \in \mathbb{C}: |z - x| < d(x)\} \). Of course, if \( x \) is not a regular point of \( f \), then \( d(x) = 0 \).

If \( d(x) > 0 \) for every \( x \in [0, 1] \), i.e. if \( f \) is analytic on \([0, 1]\), then the best uniform approximants converge geometrically to \( f \) and so in what follows we assume that \( f \) has a singularity somewhere on \([0, 1]\).

**Theorem 1.** Suppose that \( \beta > 1 \) and \( f \in C[0, 1] \) has a singularity on \([0, 1]\). There are polynomials \( P_n \in \Pi_n \), \( n = 0, 1, \ldots \), such that

\[
\|f - P_n\|_{[0,1]} \to 0 \quad \text{as} \quad n \to \infty,
\]

and for \( x \in [0, 1] \)

\[
|f(x) - P_n(x)| \leq C_{f,x} \exp(-cn[d(x)]^\beta),
\]

where \( c > 0 \) is an absolute constant and the constant \( C_{f,x} \) is bounded for \( x \) in any compact subset of \( D \).

Next we show that Theorem 1 is best possible in the sense that (3) with \( \beta = 1 \) is, in general, impossible.
Theorem 2. There are no positive constants $C_x$, $x \in [-1, 1]$, and $c > 0$ such that $C_x$, $x \in D$, are bounded for every compact subset $D$ of $[-1, 1]$ not containing the origin and for every $n$ there are polynomials $P_n \in \Pi_n$ with

$$|x - P_n(x)| \leq C_x \exp(-cn|x|), \quad x \in [-1, 1].$$

For convenience here the basic interval $[0, 1]$ has been replaced by $[-1, 1]$. Note that then for $f(x) = |x|$ we have $d(x) = |x|$.

Finally, we also show that Theorem 1 cannot be sharpened by putting a constant $C_f$ into (3)—the constant must depend on $x$, even allowing $c$ and $\beta$ to depend on $f$.

Theorem 3. There exists $f \in C[0, 1]$ such that for no constants $\beta$, $C$, $c > 0$ can one find polynomials $P_n \in \Pi_n$, $n = 1, 2, \ldots$, with the property

$$(4) \quad |f(x) - P_n(x)| \leq C \exp(-cn[d(x)]^\beta), \quad n = 1, 2, \ldots, \quad x \in [0, 1].$$

2. Proof of Theorem 1

It is enough to prove the theorem for $\beta \in (1, 3/2]$ because $d(x) \leq 1$. Also it is enough to prove the theorem for $n \geq N_0$ because afterward we can increase $C_{f,x}$ so that (3) will be fulfilled for any natural number $n$. In the beginning $n$ is arbitrary and only when necessary we place restrictions on $N_0$.

For fixed $n \geq 1$ we set $\tilde{d}(x) := d(x) + n^{-1/\beta}$ and define the points $0 = x_0 < x_1 < \cdots < x_m < x_{m+1} = 1$ by $x_0 := 0$, $x_{k+1} := x_k + (1/40)\tilde{d}(x_k)$ whenever this defines a number $x_{k+1}$ with $x_{k+1} + (1/40)\tilde{d}(x_{k+1}) \leq 1$; in the opposite case, which occurs, say, for $k = m$, we set $x_{k+1} = x_{m+1} := 1$.

From the definition of $d(x)$ we have

$$|\tilde{d}(x) - \tilde{d}(x_k)| = |d(x) - d(x_k)| \leq |x - x_k|,$$

and so for $x \in [x_k, x_k + (1/40)\tilde{d}(x_k)]$, $0 \leq k \leq m$,

$$\frac{39}{40} \tilde{d}(x_k) \leq \tilde{d}(x) \leq \frac{41}{40} \tilde{d}(x_k).$$

This easily implies that for each $k = 0, \ldots, m$,

$$|x_{k+1} - x_k| \leq \frac{1}{19} \tilde{d}(x_k),$$

and thus

$$(6) \quad \frac{18}{19} \tilde{d}(x_k) \leq \tilde{d}(x) \leq \frac{20}{19} \tilde{d}(x_k), \quad x \in [x_k, x_{k+1}].$$

We claim that

$$(7) \quad [x_{k-2}, x_{k+3}] \subseteq \left[ x - \frac{\tilde{d}(x)}{4}, x + \frac{\tilde{d}(x)}{4} \right], \quad x \in [x_k, x_{k+1}].$$
(for definiteness set \( x_{-2} = x_{-1} = 0 \) and \( x_{m+2} = x_{m+3} = 1 \)). In fact, we get from (5) and (6) that
\[
x - \frac{\tilde{d}(x)}{4} \leq x_k + \frac{1}{19} \tilde{d}(x_k) - \frac{1}{4} \cdot \frac{18}{19} \tilde{d}(x_k)
\]
\[
\leq [x_{k-1} + \frac{1}{19} \tilde{d}(x_{k-1})] - \frac{7}{38} \cdot \frac{18}{19} \tilde{d}(x_{k-1})
\]
\[
\leq x_{k-2} + \frac{1}{19} \tilde{d}(x_{k-2}) + \frac{1}{19} \frac{20}{19} \tilde{d}(x_{k-2}) - \frac{7}{38} \left( \frac{18}{19} \right)^2 \tilde{d}(x_{k-2})
\]
\[
\leq x_{k-2},
\]
and similarly
\[
x + \frac{\tilde{d}(x)}{4} \geq x_k + \frac{1}{4} \cdot \frac{18}{19} \tilde{d}(x_k) \geq x_k + \frac{1}{19} \left[ 1 + \frac{20}{19} + \left( \frac{20}{19} \right)^2 \right] \tilde{d}(x_k)
\]
\[
\geq x_k + \frac{1}{19} (\tilde{d}(x_k) + \tilde{d}(x_{k+1}) + \tilde{d}(x_{k+2}))
\]
\[
\geq x_{k+3}.
\]

By [1, Theorem 3] there exist two absolute constants \( C_1, c_1 \) such that for every \( n \) there exists a polynomial \( \chi^*_n = \chi \) of degree at most \( n/2 \) such that on \([-1, 1]\) the polynomial \( \chi \) is monotone increasing, satisfies \( 0 < \chi \leq 1 \) there and, with \( \gamma := (1 + \beta)/2 > 1 \),
\[
\left| \chi(x) - \frac{1 + \sign x}{2} \right| \leq C_1 \exp(-c_1 n |x|^\gamma), \quad x \in [-1, 1],
\]
holds. With this \( \chi_n^* = \chi \) (\( n \) is fixed) we define
\[
\chi_0(x) := 1 - \chi(x - x_1),
\]
\[
\chi_j(x) := \chi(x - x_j) - \chi(x - x_{j+1}), \quad j = 1, \ldots, m - 1,
\]
and
\[
\chi_m(x) := \chi(x - x_m).
\]

Clearly we have
\[
\sum_{j=0}^{m} \chi_j(x) = 1,
\]
and \( 0 \leq \chi_j(x) \leq 1 \) on \([0, 1]\). Furthermore, \((1 + 19/18)^{-\gamma} > 1/4\) gives
\[
\chi_j(x) \leq 2C_1 \exp(-c_1 n \min\{|x - x_j|^\gamma; |x - x_{j+1}|^\gamma\})
\]
\[
\leq 2C_1 \exp \left( -\frac{1}{4} c_1 n |x - x_j|^\gamma \right)
\]
provided \( x \in [0, 1]\setminus[x_{j-1}, x_{j+2}] \). This “partition of unity” will be used together with local best approximants to produce the required polynomial of degree \( n \).
In fact, let \( P_j(f) = P_j, \ j = 0, \ldots, m, \) be the best uniform polynomial approximant of \( f \) on \([x_{j-1}, x_{j+2}]\) of degree
\[
n_j := \left[ -\frac{c_1}{800} n(d(x_j)^\gamma) \right],
\]
and set
\[
P_n^*(x) = P(x) := \sum_{j=0}^{m} x_j(x) P_j(x).
\]
Then \( P \) is a polynomial of degree at most \( n \) and below we show that \( P_n^* = P \) satisfies the requirements set forth in Theorem 1.

Let \( x \in [x_k, x_{k+1}] \) with \( k \) arbitrary and let \( j \) be different from \( k - 1, k \) and \( k + 1 \). From (9) we get
\[
0 \leq x_j(x) \leq 2C_1 \exp \left[ -\frac{c_1}{4} n|x - x_j|^\gamma \right],
\]
and the estimate (cf. [4, 2.13.27])
\[
|\mathcal{Q}(y)| \leq (2|y|^\alpha) \|\mathcal{Q}\|_{[-1, 1]}, \quad y \in \mathbb{R} \setminus [-1, 1],
\]
transformed to the interval \([x_{j-1}, x_{j+2}]\) together with (5) and (6) easily implies
\[
|f(x) - P_j(x)| \leq \|f\|_{[0, 1]} + (100|x - x_j|/d(x_j))^n_j \|P_j\|_{[x_{j-1}, x_{j+2}]}
\leq 3\|f\|_{[0, 1]} (100|x - x_j|/d(x_j))^n_j
\leq 3\|f\|_{[0, 1]} \exp \left[ -\frac{c_1}{800} n d(x_j)^\gamma \log \left( 100|x - x_j|/d(x_j) \right) \right]
\leq 3\|f\|_{[0, 1]} \exp \left[ \frac{c_1}{8} n e^{-1} d(x_j)^{\gamma - 1}|x - x_j| \right]
\leq 3\|f\|_{[0, 1]} \exp \left( \frac{c_1}{8} n|x - x_j|^\gamma \right)
\]
because \( 100|x - x_j|/d(x_j) > 1 \) and \( \log u \leq e^{-1} u \) for \( u \geq 1 \). Thus
\[
\chi_j(x)|f(x) - P_j(x)| \leq 6C_1 \|f\|_{[0, 1]} \exp \left( -\frac{c_1}{8} n|x - x_j|^\gamma \right)
\leq 6C_1 \|f\|_{[0, 1]} \exp \left( -\frac{c_1}{16} n|x - x_j|^\gamma \right) \exp(-c_2 n^{-\gamma/\beta})
\leq 6C_1 \|f\|_{[0, 1]} \exp(-c_2 n d(x)^\beta) \exp(-c_2 n^{-\gamma/\beta}),
\]
where \( c_2 := (c_1/16) \cdot (1/150) > 0 \) and where we used that \( |x - x_j| \geq (1/40) n^{-1/\beta} \)
(note that \( x \notin [x_{j-1}, x_{j+2}] \) and \( |x_{t+1} - x_t| \geq (1/40) n^{-1/\beta} \) for any \( 0 \leq t \leq m \)
and \( |x - x_j| \geq (1/50) d(x) \). This immediately implies that for \( x \in [x_k, x_{k+1}] \),
\[
(10) \sum_{j \neq k-1, k, k+1}^m \chi_j(x)|f(x) - P_j(x)| \leq C_2 \|f\|_{[0, 1]} \exp(-c_2 n[d(x)]^\beta)
\]
for some constant \( C_2 \) depending only on \( \beta \).
Since (cf. (8))

\[ f(x) - P(x) = \sum_{j=0}^{m} \chi_j(x)(f(x) - P_j(x)), \]

we have to estimate \( f(x) - P_j(x) \) for \( j = k - 1, k, k + 1 \), as well.

Again, let \( x \in [x_k, x_{k+1}] \). If \( d(x) \leq n^{-1/\beta} \), then we just write

\[ |f(x) - P_j(x)| \leq \|f\|_{[0,1]} \leq e\|f\|_{[0,1]} \exp(-n[d(x)]^\beta). \]

If, however, \( d(x) > n^{-1/\beta} \), then \( d(x) < 2d(x) \), which, together with (7) yields

\[ |f(x) - P_j(x)| \leq \frac{E_{n_j}(f)}{[x_k, x_{k+1}]} \]
\[ \leq \frac{E_{n_j}(f)}{[x-d(x)/4, x+d(x)/4]} \]
\[ \leq \frac{E_{n_j}(f)}{[x-d(x)/2, x+d(x)/2]}, \]

where \( E_{n_j}(f)_{[a,b]} \) denotes the error in best uniform approximation to \( f \) on \([a, b]\) out of \( \Pi_m \). To estimate the right-hand member of (13) consider the Taylor expansion of \( f \) about \( x \). For the absolute value of the \( \nu \)-th Taylor coefficient, Cauchy’s inequality gives the bound

\[ \sup_{|z-x| \leq 3d(x)/4} |f(z)| \left( \frac{3}{4} d(x) \right)^{-\nu} =: C(x) \left( \frac{3}{4} d(x) \right)^{-\nu}, \]

and so the \( n_j \)-th partial sum approximates \( f \) on \([x - d(x)/2, x + d(x)/2]\) with error at most

\[ C(x) \sum_{\nu=n_j+1}^{\infty} \left( \frac{2}{3} \right)^\nu = 2C(x) \exp \left( \left( \log \frac{2}{3} \right) n_j \right) \leq 2C(x) \exp(-c_3 n[d(x)]^\beta), \]

with \( c_3 := (\log \frac{3}{2})c_1/1000 \) and where we used the fact that \( n_j > (c_1/1000)n[d(x)]^\beta \) whenever \( n_j \geq N_0 \) and \( N_0 \) satisfies \( (c_1/800)N_0^{1-\gamma/\beta} \geq 4 \).

Thus

\[ E_{n_j}(f)_{[x-d(x)/2, x+d(x)/2]} \leq 2C(x) \exp(-c_3 n[d(x)]^\beta). \]

The relations (10)-(14) yield

\[ |f(x) - P_n^*(x)| \equiv |f(x) - P(x)| \]
\[ \leq C_2\|f\| \exp(-c_2 n[d(x)]^\beta) + 3\|f\| \leq C_3\|f\| \]

and

\[ |f(x) - P_n^*(x)| \equiv |f(x) - P(x)| \leq C_4(x) \exp(-c_3 n[d(x)]^\beta), \]

which proves (3). Property (2) also can easily be obtained from (15). In fact, notice that every \( n_j \) is at least as large as

\[ \frac{c_1}{1000} n(d(x))^\gamma \geq \left[ \frac{c_1}{1000} n^{1-\gamma/\beta} \right] =: m_n, \]
and \( m_n \to \infty \) as \( n \to \infty \). If \( Q_{m_n} \) denotes the best polynomial approximation of \( f \) on \([0, 1]\) by polynomials of degree at most \( m_n \), then we get from \( n_j \geq m_n \)

\[
P_j(f; \cdot)_{[x_j-1, x_j+1]} - Q_{m_n}(\cdot) \equiv P_j(f - Q_{m_n}; \cdot)_{[x_j-1, x_j+1]},
\]

and so (cf. (8))

\[
f(\cdot) - \sum_{j=0}^{m} \chi_j(\cdot)P_j(f; \cdot) \equiv (f - Q_{m_n})(\cdot) - \sum_{j=0}^{m} \chi_j(\cdot)P_j(f - Q_{m_n}; \cdot).
\]

Hence we obtain from (15), with \( f \) replaced by \( f - Q_{m_n} \) on the right, that

\[
|f(x) - P^n_n(x)| \leq C_3\|f - Q_{m_n}\|_{[0,1]}.
\]

Since \( \|f - Q_{m_n}\|_{[0,1]} \to 0 \) as \( n \to \infty \), we obtain (2). \( \Box \)

3. Proof of Theorem 2

Suppose to the contrary that for any \( a \in (0, 1/2] \) the estimate

\[
|P_n(x) - |x|| \leq C_a \exp(-2cn|x|), \quad x \in [-1, 1] \setminus [-a, a],
\]

is possible with some positive constants \( C_a \) and \( c \) and polynomials \( P_n \in \Pi_n \). Then (16) holds for \( (P_n(x) + P_n(-x))/2 \), as well, so we may assume each \( P_n \) to be even.

Consider the derivative \( P'_n \) of \( P_n \), which is odd. Let \( x \in [a, 1] \) and \( b := \min\{x, 1/2\} \). Since

\[
|P_n(u) - u| \leq C_a \exp(-2cnb)
\]

if \( u \in [b, 1] \), we get from Markoff's inequality, applied to the interval \([b, 1]\), that

\[
|P'_n(x) - 1| \leq 4C_a n^2 \exp(-2cnb)
\]

\[
\leq 4C_a n^2 \exp(-cnx), \quad a < x < 1.
\]

In a similar way

\[
|P'_n(x) + 1| \leq 4C_a n^2 \exp(-cn|x|), \quad -1 \leq x \leq -a,
\]

and so for \( a \leq |x| \leq 1 \) we have

\[
|P'_n(x) - \text{sign} x| \leq 4C_a n^2 \exp(-cn|x|),
\]

where \( C_a \) is independent of \( n \).

This implies that the polynomials

\[
Q_n(x) := 1 - (P'_n(\sqrt{x}))^2
\]

of degree at most \( n \) satisfy \( Q_n(0) = 1 \) and, for large \( n \),

\[
|Q_n(x)| \leq 8C_a n^2 \exp(-cn\sqrt{x}), \quad a \leq x \leq 1.
\]
However, this is impossible, since in [1, Section 4.1] it was proved that if \( Q_n \in \Pi_n \) and \( Q_n(0) = 1 \), then for \( 0 < a \leq 1 \) the inequality \( |Q_n(x)| \leq 1 \) on \([a, 1]\) implies
\[
\int_a^1 \frac{-\log|Q_n(x)|}{x^{3/2}} \, dx \leq 4n.
\]
This contradiction proves the theorem. □

4. Proof Theorem 3

Let \( a_k := 2^{-k}(1 + i) \), where \( i = \sqrt{-1} \), and set
\[
f(z) := \sum_{k=1}^{\infty} n_k^{-1/8} [2^k(z - a_k)]^{-n_k} =: \sum_{k=1}^{\infty} g_k(z),
\]
where \( \{n_k\} \) will be completely specified later in the proof. We set \( n_1 = 1 \), and require \( n_{k+1} > 2^k n_k \) for all \( k \) so that for every \( \ell > 0 \) we have
\[
n_k/2^k \to \infty.
\]

Since
\[
||[2^k(z - a_k)]^{-1}|| \leq 1 \quad \text{if } |z - a_k| \geq 2^{-k},
\]
it is easy to see that the series defining \( f \) uniformly converges on the real line and also on every compact subset of the complex plane not containing the points \( 0, \{a_k\}_{k=1}^{\infty} \); thus these are the singular points of \( f \). In particular,
\[
d(x) \geq \frac{1}{\sqrt{2}} x \quad \text{for } x \in [0, 1].
\]

Now we need some easy estimates:
\[
|g_k(2^{-k})| = n_k^{-1/8},
\]
\[
|g_k(2^{-k} + n_k^{-1/4})| = n_k^{-1/8} (1 + 4^k n_k^{-1/2})^{-n_k/2} < \frac{1}{2} n_k^{-1/8}
\]
and so
\[
n_k^{1/4} |g_k(2^{-k}) - g_k(2^{-k} + n_k^{-1/4})| > \frac{1}{2} n_k^{-1/8}.
\]
Using this, by standard gliding-hump arguments we can select \( n_1, n_2, \ldots \) one after the other in such a way that
\[
n_k^{1/4} |f(2^{-k}) - f(2^{-k} + n_k^{-1/4})| > \frac{1}{4} n_k^{-1/8}
\]
is satisfied for all \( k \).

Now if (4) is true for some \( C, c > 0 \) then
\[
|f(x) - P_m(x)| \leq 2^k \exp(-2^{-k} m[d(x)]^\gamma), \quad m = 1, 2, \ldots, \quad x \in [0, 1],
\]
is also true for any large \( k, k \geq \max\{\ln C; -\ln c\}/\ln 2 \), and below we show that this is impossible.
Fix a \( k \) so large that (20) is satisfied and also \((1/16)n_k^{-1/8} < 2^k \exp \{-2^{-\beta/2} n_k^{1/16}\}\), which is always possible because of (17).

Set \( m = [2^{k(1+\beta)}n_k^{1/16}] + 1 \) into (20). Then for \( x \in [2^{-k}, 1] \) we get from (18) and (20)

\[
|f(x) - P_m(x)| \leq 2^k \exp(-2^{-k} 2^{k(1+\beta)} n_k^{1/16} (2^{-k}/\sqrt{2})^\beta) < \frac{1}{16} n_k^{-1/8};
\]

therefore (cf. (19)),

\[
(21) \quad n_k^{1/4} |P_m(2^{-k}) - P_m(2^{-k} + n_k^{-1/4})| > \frac{1}{8} n_k^{1/8}.
\]

On the other hand, \( \{P_m\} \) are uniformly bounded on \([0, 1]\), say \(|P_m| \leq K\), \( m = 1, 2, \ldots \), (cf. (4)); hence Bernstein’s inequality shows that for \( x \in [2^{-k}, 2^{-k} + n_k^{-1/4}] \),

\[
|P'_m(x)| \leq Km2^k,
\]

which implies

\[
(22) \quad n_k^{1/4} |P_m(2^{-k}) - P_m(2^{-k} + n_k^{-1/4})| \leq Km2^k
\]

\[
\leq 2K2^{k(2+\beta)} n_k^{1/16}.
\]

However, (17), (21) and (22) are not compatible which proves our theorem. \( \Box \)

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