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CONVOLUTION PROPERTIES OF A CLASS OF STARLIKE FUNCTIONS

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ABSTRACT. Let R denote the class of functions $f(z) = z + a_2 z^2 + \cdots$ that are analytic in the unit disc $E = \{z : |z| < 1\}$ and satisfy the condition $\mathrm{Re}(f'(z) + zf''(z)) > 0$, $z \in E$. It is known that R is a subclass of S_t, the class of univalent starlike functions in E. In the present paper, among other things, we prove (i) for every $n \geq 1$, the nth partial sum of $f \in R$, $s_n(z, f)$, is univalent in E, (ii) R is closed with respect to Hadamard convolution, and (iii) the Hadamard convolution of any two members of R is a convex function in E.

1. INTRODUCTION

Let A denote the class of functions f that are regular in the unit disc $E = \{z : |z| < 1\}$ and satisfy the conditions $f(0) = f'(0) - 1 = 0$. We denote by S the subclass of A consisting of univalent functions and by K, S_t, and C the usual subclasses of S whose members are convex, starlike (w.r.t. the origin) and close-to-convex, respectively. Finally, denote by R the family of functions $f \in A$ which satisfy the condition $\mathrm{Re}(f'(z) + zf''(z)) > 0$, $z \in E$. Chichra [1] proved that if $f \in R$, then $\mathrm{Re}\, f'(z) > 0$, $z \in E$, and hence f is univalent in E. R. Singh and S. Singh [8] showed that if $f \in R$ then f is also starlike in E.

In the present paper we improve Chichra's result and show that the assertion of Singh and Singh holds under a much weaker hypothesis. We also prove that for every integer $n \geq 1$, the nth partial sum of $f \in R$, $s_n(z, f)$, is close-to-convex in E. Finally, we prove that R is closed with respect to Hadamard convolution and that if f, $g \in R$, then their Hadamard convolution is convex in E. The significance of the last two results will be made clear later on at the appropriate place.

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2. Preliminaries

We shall need the following definitions and results. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are analytic in $|z| < \rho$, then their Hadamard product/convolution, $f * g$, is the function defined by the power series

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

The function $f * g$ is also analytic in $|z| < \rho$.

A sequence $\{c_n\}_0^{\infty}$ of non-negative numbers is said to be a convex null sequence if $c_n \to 0$ as $n \to \infty$ and

$$c_0 - c_1 \geq c_1 - c_2 \geq \cdots \geq c_n - c_{n+1} \geq \cdots \geq 0.$$

If f is analytic in $|z| < \rho$, g is analytic and univalent in $|z| < \rho$ and $f(0) = g(0)$, then we say that f is subordinate to g in $|z| < \rho$, in symbols, $f \prec g$ in $|z| < \rho$, if $f(|z| < \rho) \subset g(|z| < \rho)$.

Lemma 1. *Let $\{c_n\}_0^{\infty}$ be a convex null sequence. Then the function*

$$q(z) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n z^n$$

is analytic in E and $\operatorname{Re} q(z) > 0$, $z \in E$.

Lemma 2. *Suppose that w is a nonconstant function analytic in $|z| < \rho$ and $w(0) = 0$. Then, if $|w(z)|$ attains its maximum value on the circle $|z| = r < \rho$ at the point z_0, we can write $z_0 w'(z_0) = k w(z_0)$, where $k \geq 1$.*

Lemma 3. *For $0 \leq \theta \leq \pi$,*

$$\frac{1}{2} + \sum_{k=1}^{n} \frac{\cos k\theta}{k+1} \geq 0.$$

Lemma 4. *If $P(z)$ is analytic in E, $P(0) = 1$, and $\operatorname{Re} P(z) > \frac{1}{2}$, $z \in E$, then for any function F, analytic in E, the function $P * F$ takes values in the convex hull of the image of E under F.*

Lemmas 1, 2, and 3 are due to Fejěr [2], Jack [3], and Rogosinski and Szegö [6], respectively. The assertion of Lemma 4 readily follows by using the Herglotz' representation for $P(z)$.

3. Theorems and their proofs

Theorem 1. *Let $f \in R$. Then we have*

(a) $\operatorname{Re} f'(z) > -1 + 2\log 2 \doteqdot 0.\cdot 39\ldots$, $(z \in E)$. *The constant $-1 + 2\log 2$ cannot be replaced by any larger one.*
(b) $\operatorname{Re} \frac{f(z)}{z} > \frac{1}{2}$, $z \in E$.
(c) *For every $n \geq 1$, the nth partial sum of f, $s_n(z, f)$, satisfies $\operatorname{Re} s_n'(z, f) > 0$, $z \in E$, and hence $s_n(z, f)$ is univalent in E.*

(d) *For every* $n \geq 1$

$$\operatorname{Re} \frac{s_n(z,f)}{z} > \frac{1}{3} \qquad (z \in E).$$

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Since $\operatorname{Re}(f'(z) + zf''(z)) > 0$, $z \in E$, we have

$$\operatorname{Re}\left[1 + \sum_{n=2}^{\infty} n^2 a_n z^{n-1}\right] > 0 \qquad (z \in E), \tag{1}$$

and hence

$$\operatorname{Re}\left[1 + \frac{1}{2}\sum_{n=2}^{\infty} n^2 a_n z^{n-1}\right] > \frac{1}{2} \qquad (z \in E). \tag{2}$$

Consider the function

$$P(z) = 1 + 2\sum_{n=2}^{\infty} \frac{1}{n} z^{n-1}.$$

Clearly $P(z)$ is analytic in E, $P(0) = 1$ and

$$\begin{aligned} \operatorname{Re} P(z) &= \operatorname{Re}\left[-1 - \frac{2}{z}\log(1-z)\right] \\ &> -1 + 2\log 2 [5]. \end{aligned} \tag{3}$$

Now, since we can write

$$f'(z) = \left[1 + \frac{1}{2}\sum_{n=2}^{\infty} n^2 a_n z^{n-1}\right] * \left[1 + 2\sum_{n=2}^{\infty} \frac{1}{n} z^{n-1}\right],$$

it follows, in view of (2), (3) and Lemma 4, that $\operatorname{Re} f'(z) > -1 + 2\log 2$, $z \in E$. That the constant $-1 + 2\log 2$ cannot be replaced by any larger one follows from the fact that the function f_0 defined by $zf_0'(z) = -z - 2\log(1-z)$ is in the class R.

(b) We observe that since the sequence $\{c_n\}_0^{\infty}$ defined by $c_0 = 1$, $c_n = 2/(n+1)^2$, $n \geq 1$, is a convex null sequence, we have, in view of Lemma 1.

$$\operatorname{Re}\left[1 + 2\sum_{n=2}^{\infty} \frac{1}{n^2} z^{n-1}\right] > \frac{1}{2} \qquad (z \in E). \tag{4}$$

Writing $f(z)/z$ as

$$\frac{f(z)}{z} = \left[1 + \frac{1}{2}\sum_{n=2}^{\infty} n^2 a_n z^{n-1}\right] * \left[1 + 2\sum_{n=2}^{\infty} \frac{1}{n^2} z^{n-1}\right]$$

and making use of (2), (4) and Lemma 4, we conclude that $\operatorname{Re}(f(z)/z) > \frac{1}{2}$, $z \in E$.

(c) We can write

$$s_n'(z,f) = \left[1 + \sum_{k=2}^{\infty} k^2 a_k z^{k-1}\right] * \left[1 + \sum_{k=2}^{n} \frac{1}{k} z^{k-1}\right]. \tag{5}$$

Putting $z = re^{i\theta}$, $0 \le r < 1$, $0 \le |\theta| \le \pi$, and making use of the minimum principle for harmonic functions along with Lemma 3, we obtain

$$
\begin{aligned}
\operatorname{Re}\left[1 + \sum_{k=2}^{n} \frac{1}{k} z^{k-1}\right] &= \operatorname{Re}\left[1 + \sum_{k=1}^{n-1} \frac{z^k}{k+1}\right] \\
&= 1 + \sum_{k=1}^{n-1} \frac{r^k \cos k\theta}{(k+1)} \qquad (0 \le \theta \le \pi) \\
&> 1 + \sum_{k=1}^{n-1} \frac{\cos k\theta}{k+1} \\
&\ge \tfrac{1}{2}.
\end{aligned} \tag{6}
$$

In view of (1), (6), (5) and Lemma 4, we deduce that $\operatorname{Re} s_n'(z, f) > 0$, $z \in E$, and so $s_n(z, f)$ is close-to-convex in E for every $n \ge 1$.

(d) Let

$$q(z) = 1 + \sum_{k=2}^{n} \frac{1}{k} z^{k-1}.$$

Then by (6), we have $\operatorname{Re} q(z) > \frac{1}{2}$ in E. An application of Lemma 2 readily provides that

$$
\begin{aligned}
\operatorname{Re} S(z) &= \operatorname{Re}\left[\frac{1}{z}\int_0^z q(t)\,dt\right] \\
&= \operatorname{Re}\left[1 + \sum_{k=2}^{n} \frac{1}{k^2} z^{k-1}\right] > \frac{2}{3} \qquad (z \in E).
\end{aligned} \tag{7}
$$

Writing $s_n(z, f)/z$ as

$$
\begin{aligned}
\frac{s_n(z, f)}{z} &= \left[1 + \frac{1}{2}\sum_{k=2}^{\infty} k^2 a_k z^{k-1}\right] * \left[1 + 2\sum_{k=2}^{n} \frac{1}{k^2} k^{z-1}\right] \\
&= \left[1 + \frac{1}{2}\sum_{k=2}^{\infty} k^2 a_k z^{k-1}\right] * \left[-1 + 2\left(1 + \sum_{k=2}^{n} \frac{1}{k^2} z^{k-1}\right)\right],
\end{aligned}
$$

and making use of (2), (7) and Lemma 4, the conclusion (d) follows at once.

Remark 1. It is clear that $f \in R$ if and only if $\operatorname{Re} g'(z) > 0$, $z \in E$, where $g(z) = zf'(z)$. From this it follows that if $f \in R$, then f has the integral representation

$$f'(z) = -1 - \int_{|x|=1} \frac{2}{xz} \log(1 - xz)\, d\mu(x), \tag{8}$$

where μ is a probability measure on $|x| = 1$.

Since $\operatorname{Re}[(-2/xz)\log(1 - xz)] > 2\log 2$[5], part (a) also follows from (8).

If $f \in R$, then as seen in Theorem 1, part (c), each $s_n(z, f)$ is univalent in E. It is, therefore, natural to ask for the largest number λ_n, $0 < \lambda_n < 1$, such that $\lambda_n s_n(z, f) \prec s_{n+1}(z, f)$, $z \in E$. Using part (d) of the above theorem along with the fact that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in R$, then $|a_n| \leq 2/n^2$, $n \geq 2$, we readily obtain the following result which provides a lower bound for λ_n.

Corollary 1. *Let $f \in R$. Then*

(i) $\frac{1}{2}z = \frac{1}{2}s_1(z, f) \prec s_2(z, f)$, $\quad z \in E$,

(ii) $\frac{(n+1)^2-6}{(n+1)^2} s_n(z, f) \prec s_{n+1}(z, f)$, $\quad z \in E$, $n \geq 2$.

The constant $\frac{1}{2}$ in (i) *is the best possible one.*

Our next result shows that the assertion of R. Singh and S. Singh mentioned in the Introduction holds under a much weaker hypothesis.

Theorenm 2. *If $f \in A$ and*

$$\mathrm{Re}(f'(z) + zf''(z)) > -\tfrac{1}{4} \qquad (z \in E),$$

then $f \in S_t$.

Proof. Letting $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, it follows from the hypothesis of the theorem that

$$\mathrm{Re}\left[1 + \frac{2}{5}\sum_{k=2}^{\infty} k^2 a_k z^{k-1}\right] > \frac{1}{2}, \qquad z \in E. \tag{9}$$

Also one can easily see that the sequence $\{c_n\}_0^{\infty}$, where $c_0 = 1$ and $c_n = 5/2(n+1)^2$, $n \geq 1$, is a convex null sequence and as such

$$\mathrm{Re}\left[1 + \frac{5}{2}\sum_{k=2}^{\infty} \frac{1}{k^2} z^{k-1}\right] > \frac{1}{2} \qquad (z \in E). \tag{10}$$

From (9) and (10) and Lemma 4 we deduce that

$$\begin{aligned} \mathrm{Re}\frac{f(z)}{z} &= \mathrm{Re}\left[\left(1 + \frac{2}{5}\sum_{k=2}^{\infty} k^2 a_k z^{k-1}\right) * \left(1 + \frac{5}{2}\sum_{k=2}^{\infty} \frac{1}{k^2} z^{k-1}\right)\right] \\ &> \frac{1}{2}, \qquad z \in E. \end{aligned} \tag{11}$$

An application of Lemma 2 readily yields that if f satisfies the hypothesis of Theorem 2, then $\mathrm{Re} f'(z) > 0$, $z \in E$, and hence f is univalent in E. Define a function w in E by

$$\frac{zf'(z)}{f(z)} = \frac{1 + w(z)}{1 - w(z)}. \tag{12}$$

Clearly w so defined is meromorphic in E, $w(0) = 0$ and since f is univalent in E, we have $w(z) \neq 1$ in E. From (12) we obtain

$$f'(z) + zf''(z) = \left(\frac{f(z)}{z}\right)\left[\left(\frac{1 + w(z)}{1 - w(z)}\right)^2 + \frac{2zw'(z)}{(1 - w(z))^2}\right]. \tag{13}$$

We claim that $|w(z)| < 1$ in E. If possible, suppose that there exists a point $z_0 \in E$ such that $\max_{|z|\le|z_0|} |w(z)| = |w(z_0)| = 1$. Then from Lemma 2 it follows that $zw'(z_0) = kw(z_0)$, where $k \ge 1$ and $w(z_0) = e^{i\theta}$, $0 < \theta < 2\pi$. Putting $z = z_0$ in (13), we get

$$\begin{aligned} \operatorname{Re}[f'(z_0) + z_0 f''(z_0)] &= \operatorname{Re}\left[\left(\frac{f(z_0)}{z_0}\right)\left\{\left(\frac{1+e^{i\theta}}{(1-e^{i\theta}}\right)^2 + \frac{2ke^{i\theta}}{(1-e^{i\theta})^2}\right\}\right] \\ &\le -\frac{k}{2\sin^2(\theta/2)} \operatorname{Re}\left(\frac{f(z_0)}{z_0}\right) \qquad (14) \\ &\le -\frac{1}{4}, \end{aligned}$$

since $k \ge 1$ and, in view of (11), $\operatorname{Re}(f(z)/z) > \frac{1}{2}$, $z \in E$. As (14) contradicts our hypothesis, we conclude that $|w(z)| < 1$ in E. Equation (12) then implies that f must belong to S_t.

Corollary 2. *If $g \in A$ and*

$$\operatorname{Re}[g'(z) + 3zg''(z) + z^2 g'''(z)] > -\tfrac{1}{4} \qquad (z \in E),$$

then $g \in K$.

It is known [7] that if $f \in S_t$ and $g \in K$, then $f * g \in S_t$ and that if f, $g \in S_t$, then $f * g$ need not be in S_t. In the following theorem we prove that if f, $g \in R$, a subclass of S_t, then so does $f * g$, i.e. R is closed with respect to Hadamard product.

Theorem 3. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ belong to R, then so does their Hadamard product*

$$h(z) = (f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Proof. Since $h(z) = (f * g)(z)$, we have

$$zh'(z) = zf'(z) * g(z)$$

and hence

$$h'(z) + zh''(z) = (f'(z) + zf''(z)) * g(z)/z. \qquad (15)$$

Since $\operatorname{Re}(f'(z) + zf''(z)) > 0$, $z \in E$, and by Theorem 1, part (b), $\operatorname{Re}(g(z)/z) > \frac{1}{2}$, $z \in E$, the desired result follows at once from (15) and Lemma 4.

From the proof of Theorem 3 it is clear that in fact the following more general result holds:

Theorem 3$'$. *If $f \in R$, $g \in A$ and $\operatorname{Re}(g(z)/z) > \frac{1}{2}$, $z \in E$, then $f * g \in R$.*

We observe that $\operatorname{Re}(g(z)/z) > \frac{1}{2}$, $z \in E$, need not even imply the univalence of g in E.

Corollary 3. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in R$, *then so does*

$$f_k(z) = z + \sum_{n=1}^{\infty} a_{nk+1} z^{nk+1}, \qquad k = 1, 2, 3, \ldots.$$

In the next theorem we prove that if f, $g \in R$, then $f * g \in K$. Since the class K is closed with respect to Hadamard convolution [7], the significance of our result will be apparent only if we show that R (which has hitherto been shown to be a subset of S_t) is not contained in K. To prove that $R \not\subseteq K$, denote by P' the family of functions $f \in A$ which satisfy the condition $\operatorname{Re} f'(z) > 0$, $z \in E$. Krzyż [4] has demonstrated a function $f_0 \in P'$ such that $f_0 \notin S_t$ (space does not permit us to carry out the construction of f_0). Clearly the function f^*, defined by $f^*(z) = \int_0^z (f_0(\zeta)/\zeta)\, d\zeta$, is a member of R which is not in K, showing that $R \not\subseteq K$.

Theorem 4. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *and* $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ *are in* R, *then*

$$h(z) = (f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \in K.$$

Proof. In view of Corollary 2 it suffices to show that

$$\operatorname{Re}[h'(z) + 3zh''(z) + z^2 h'''(z)] > -\frac{1}{4}, \qquad z \in E,$$

or, equivalently,

$$\operatorname{Re}\left[1 + \sum_{n=2}^{\infty} n^3 a_n b_n z^{n-1}\right] > -\frac{1}{4}, \qquad z \in E. \tag{16}$$

Since f, $g \in R$, we have

$$\operatorname{Re}\left[1 + \frac{1}{2}\sum_{n=2}^{\infty} n^2 a_n z^{n-1}\right] > \frac{1}{2}, \quad z \in E,$$

and

$$\operatorname{Re}\left[1 + \frac{1}{2}\sum_{n=2}^{\infty} n^2 b_n z^{n-1}\right] > \frac{1}{2}, \qquad z \in E.$$

Therefore, in view of Lemma 4, it follows that

$$\operatorname{Re}\left[1 + \frac{1}{4}\sum_{n=2}^{\infty} n^4 a_n b_n z^{n-1}\right] > \frac{1}{2}, \qquad z \in E. \tag{17}$$

Now, we can write

$$\left[1 + \sum_{n=2}^{\infty} n^3 a_n b_n z^{n-1}\right] = \left[1 + \frac{1}{4}\sum_{n=2}^{\infty} n^4 a_n b_n z^{n-1}\right] * \left[1 + 4\sum_{n=2}^{\infty} \frac{1}{n} z^{n-1}\right]. \tag{18}$$

Since

$$\begin{aligned}\operatorname{Re}\left[1+4\sum_{n=2}^{\infty}\frac{1}{n}z^{n-1}\right] &= \operatorname{Re}\left[-3-\frac{4}{z}\log(1-z)\right] \\ &> -3+4\log 2[5] \\ &\doteqdot -0.231 > -\tfrac{1}{4} \qquad (z\in E),\end{aligned} \tag{19}$$

it follows from (17), (18), (19) and Lemma 4 that (16) holds for all $z \in E$. The proof of Theorem 4 is, therefore, complete.

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