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A REMARK ON PSEUDO-JUMP OPERATORS

A. H. LACHLAN AND X. YI

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ABSTRACT. Let $\{W_n\}_{n\in\omega}$ be an enumeration of the recursively enumerable sets. In answer to a question of Jockusch and Shore we show that there exist i and j such that for all e either W_e is recursive or at least one of $\deg(W_e \oplus W_i^{W_e})$ and $\deg(W_e \oplus W_j^{W_e})$ is not recursively enumerable.

One of the questions asked by Jockusch and Shore in their paper [1] was: Given i, j does there always exist e such that W_e is nonrecursive, and $W_e \oplus W_i(W_e)$ and $W_e \oplus W_j(W_e)$ are both of r.e. degree? We use the notation $W_e(X)$ instead of the standard notation W_e^X (see [2] for example) to avoid excessively complicated superscripts later in the paper. Following [1] we will write $J_i(X)$ for $X \oplus W_i(X)$.

Soare and Stob [3] almost gave a negative answer by proving

Theorem. *From e one can effectively compute i and j such that, if W_e is nonrecursive, then at least one of $J_i(W_e)$ and $J_j(W_e)$ is not of r.e. degree.*

In this note we wish to prove the following

Metatheorem. *Let $P(e, i_1, \dots, i_n)$ be an $(n+1)$-ary relation on ω such that for all $e, i_1, \dots, i_n, e', i_1', \dots, i_n'$,*

$$[W_e = W_{e'} \;\&\; J_{i_k}(W_e) \equiv_T J_{i_k'}(W_{e'}) \qquad (1 \le k \le n)]$$
$$\Rightarrow [P(e, i_1, \dots, i_n) \leftrightarrow P(e', i_1', \dots, i_n')] .$$

If $\forall e\, \exists i_1 \cdots \exists i_n P(e, i_1, \dots, i_n)$ is true effectively in the sense that there are recursive functions f_k $(1 \le k \le n)$ such that $\forall e P(e, f_1(e), \dots, f_n(e))$ holds, then in fact

$$\exists i_1 \cdots \exists i_n \forall e P(e, i_1, \dots, i_n)$$

is true.

A negative answer to the question of Jockusch and Shore follows from the Soare-Stob Theorem by applying the Metatheorem with

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W_e recursive $\vee$ one of $J_i(W_e)$ and $J_j(W_e)$ is not of r.e. degree as the relation $P(e, i, j)$.

The Metatheorem follows immediately from the following

Lemma. *Let f be a recursive function. There exists $e < \omega$ such that for all j,*

$$(\forall k < j)[W_k \neq W_j] \Rightarrow W_e(W_j) \equiv_T J_{f(j)}(W_j) .$$

Proof. The simultaneous enumeration of the r.e. sets $\{W_{i,s}\}_{i,s\in\omega}$ is chosen as in [2, p. 18]. There is a recursive $j(s)$ taking every value infinitely often such that for each s, if $W_{i,s+1} - W_{i,s}$ is nonempty, then $i = j(s)$ and $|W_{i,s+1} - W_{i,s}| = 1$. Let w_s denote the unique member of $W_{j(s),s+1} - W_{j(s),s}$ when this set is nonempty. The notation for a subset of ω also serves to denote its characteristic function. We use $\preceq$ for the relation of extension on ${}^{\leq\omega}2$.

In stages $0, 1, \ldots$ we will effectively enumerate $E \subseteq ({}^{<\omega}2) \times \omega$. The set of pairs enumerated in E by the end of stage s is denoted E_s. For any $F \subseteq ({}^{<\omega}2) \times \omega$ and $\sigma \in^{\leq\omega} 2$, let $F(\sigma)$ denote $\{n : (\tau, n) \in F, \tau \subseteq \sigma\}$. At stage 0 we do nothing.

STAGE $s+1$. Let j denote $j(s)$. Let v_s denote w_s if $w_s \downarrow$ and s otherwise. Let x be the least number if any such that $x < v_s$, $x \in W_{f(j),s}(W_{j,s} \upharpoonright v_s)$, $\langle x, j\rangle \notin E_s(W_{j,s} \upharpoonright v_s)$, and

$$(\forall k < j)[(\exists y < v_s)[y \in W_{k,s} - W_{j,s}] \vee (\exists y < x)[y \in W_{j,s} - W_{k,s}]] .$$

Enumerate $(W_{j,s} \upharpoonright v_s, \langle x, j\rangle)$ in E. If there is no such x, pass immediately to the next stage.

Fix $j < \omega$ such that $W_j \neq W_k$ for all $k < j$. Let x_0 denote the least number such that

$$(\forall k < j)[W_k \subseteq W_j \Rightarrow (\exists y \leq x_0)[y \in W_j - W_k]] .$$

Call s true for j if $j(s) = j$ and $W_{j,s} \upharpoonright v_s \prec W_j$. Note that there are infinitely many s true for j. By inspection of the construction

$$\langle x, j\rangle \in E(W_j) \Rightarrow x \in W_{f(j)}(W_j) .$$

Consider the least $x \in W_{f(j)}(W_j)$ if any such that $x > x_0$ and $\langle x, j\rangle \notin E(W_j)$. Let s be true for j and large enough so that $x < v_s$,

$$\forall x'[x' < x \,\&\, x' \in W_{f(j)}(W_j)] \Rightarrow \langle x', j\rangle \in E_s(W_{j,s} \upharpoonright v_s)]$$

and

$$(\forall k < j)[[W_k \not\subseteq W_j \Rightarrow (\exists y < v_s)[y \in W_{k,s} - W_{j,s}]] \,\&$$
$$[W_k \subseteq W_j \Rightarrow (\exists y \leq x_0)[y \in W_{j,s} - W_{k,s}]] .$$

If $\langle x, j\rangle \notin E_s(W_{j,s} \upharpoonright v_s)$, then $(W_{j,s} \upharpoonright v_s, \langle x, j\rangle)$ is enumerated in E at stage $s + 1$. We conclude that

$$[x \in W_{f(j)}(W_j) \& x_0 < x] \Rightarrow \langle x, j\rangle \in E(W_j) .$$

Consider a particular number of the form $\langle x, k\rangle \in E(W_j)$ with $k \neq j$. Since $\langle x, k\rangle \in E(W_j)$, there exists t such that $x < v_t$, $(W_{k,t} \restriction v_t, \langle x, k\rangle)$ is enumerated in E at stage $t+1$, and $W_{k,t} \restriction v_t \preceq W_j$. If $k < j$, there are only a finite number of possibilities for t since $v_t \to \infty$ as $t \to \infty$ and $W_j \neq W_k$. If $j < k$, then from the condition satisfied at stage $t+1$,

$$(\exists y < x)[y \in W_{k,t} - W_{j,t}].$$

Clearly, some $y < x$ enters W_j at a stage $> t$. Thus a pair which causes $\langle x, k\rangle \in E(W_j)$, with $j < k$, cannot be enumerated in E once all members of W_j which are $< x$ have been enumerated.

It is now easy to deduce that $W_j \oplus E(W_j) \equiv_T W_j \oplus W_{f(j)}(W_j)$. By choosing e such that $W_e(X) = X \oplus E(X)$ for all $X \subseteq \omega$, we complete the proof of the lemma.

References

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Department of Mathematics and Statistics, Simon Fraser University, Burnaby, British Columbia, Canada V5A 1S6