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A THEOREM ON POLYNOMIAL-STAR APPROXIMATION

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ABSTRACT. We prove that the unit ball of a Banach space is polynomial-star dense in the unit ball of its bidual. This strengthens Goldstine's theorem on weak-star density.

1. SUMMARY

Let $\mathscr{X}$ be a Banach space, with bidual $\mathscr{X}^{**}$. Goldstine's theorem [5] asserts that the open unit ball B of $\mathscr{X}$ is weak-star dense in the unit ball B^{**} of $\mathscr{X}^{**}$. In other words, for any $z \in \mathscr{X}^{**}$ with $\|z\| \leq 1$, there is a net $x_\alpha \in B$ with $L(x_\alpha) \to z(L)$ for all $L \in \mathscr{X}^*$. Our aim is to show that the net $\{x_\alpha\}$ can be chosen so that $P(x_\alpha)$ converges to a certain value $\widehat{P}(z)$ for any analytic polynomial P on $\mathscr{X}$. It will follow then that every $f \in H^\infty(B)$ extends in a canonical fashion to a function $\hat{f} \in H^\infty(B^{**})$, to give an isometric algebra embedding of $H^\infty(B)$ as a closed subalgebra of $H^\infty(B^{**})$. This settles a problem left open by Aron and Berner [1], who had shown that the canonically associated extension $\hat{f}$ of $f \in H^\infty(B)$ is analytic in the ball in $\mathscr{X}^{**}$ centered at 0 of radius $1/e$.

2. STATEMENT OF THE MAIN THEOREM

Recall that an m-homogeneous analytic polynomial P on $\mathscr{X}$ is the restriction to the diagonal of a continuous (complex-valued) m-linear form F on $\mathscr{X}$:

$$P(x) = F(x, \dots, x), \qquad x \in \mathscr{X}.$$

Each such P is the restriction of a unique symmetric m-linear form on $\mathscr{X}$, and the symmetric extension F can be recovered from P by the usual polarization formula [4, 6]. We denote

$$\|P\| = \sup\{|P(x)|: x \in B\}$$

and

$$\|F\| = \sup\{|F(x_1, \dots, x_m)|: x_1, \dots, x_m \in B\}.$$

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The polarization formula yields immediately the estimate

$$\|F\| \leq \frac{m^m}{m!}\|P\|,$$

and this estimate is sharp [6].

Each continuous m-linear form F on $\mathscr{X}$ can be extended to an m-linear form on the bidual $\mathscr{X}^{**}$, generally in many ways. One such extension, which we denote by $\widehat{F}$, is obtained by specifying that for each fixed j, $1 \leq j \leq m$, and for each fixed $x_1, \dots, x_{j-1} \in \mathscr{X}$ and $z_{j+1}, \dots, z_m \in \mathscr{X}^{**}$, the linear functional

$$z \to \widehat{F}(x_1, \dots, x_{j-1}, z, z_{j+1}, \dots, z_n), \qquad z \in \mathscr{X}^{**},$$

is weak-star continuous. In other words, one extends by weak-star continuity, beginning with the last variable and working backwards to the first. If one takes the variables in a different order for purposes of extending, one generally arrives at a different extension. However, the restriction

$$\widehat{P}(z) = \widehat{F}(z, \dots, z), \qquad z \in \mathscr{X}^{**},$$

of $\widehat{F}$ to the diagonal is independent of the order in which the variables are treated, and evidently

$$\|\widehat{P}\| \leq \|\widehat{F}\| = \|F\| \leq \frac{m^m}{m!}\|P\|. \tag{2.1}$$

The correspondence $P \to \widehat{P}$ is multiplicative. It extends to an algebra isomorphism of the analytic polynomials on $\mathscr{X}$ and an algebra of analytic polynomials on X^{**}. For details, see [1].

Our main theorem is the following.

Theorem 1. *Let S be a bounded subset of $\mathscr{X}$, and suppose $z \in X^{**}$ is weak-star adherent to S. Then there is a net $\{x_\alpha\}$ in $\mathscr{X}$ such that each x_α is an arithmetic mean of distinct elements of S, and $P(x_\alpha) \to \widehat{P}(z)$ for all analytic polynomials P on $\mathscr{X}$.*

For purposes of stating more succinctly the result, we define the *polynomial-star topology* of $\mathscr{X}^{**}$ to be the smallest topology for which a net $\{z_\alpha\}$ converges to z if and only if $\widehat{P}(z_\alpha)$ converges to $\widehat{P}(z)$ for all analytic polynomials P of $\mathscr{X}$. Evidently norm convergence implies polynomial-star convergence, and since continuous linear functionals are in particular analytic polynomials, polynomial-star convergence implies weak-star convergence. As an immediate consequence of Theorem 1 we obtain the following.

Theorem 2. *Let S be a bounded convex subset of $\mathscr{X}$. Then the weak-star closure of S in $\mathscr{X}^{**}$ coincides with the polynomial-star closure of S in $\mathscr{X}^{**}$.*

In particular, the unit ball B of $\mathscr{X}$ is polynomial-star dense in the unit ball B^{**} of $\mathscr{X}^{**}$. This is our extension of Goldstine's theorem.

Note that Theorem 2 is trivial if $\mathscr{X}$ is reflexive, since then the weak-star (=weak) closure of S coincides with the norm closure, even without the hypothesis of boundedness of S.

As another immediate application of Theorem 1, applied to the unit ball of $\mathscr{X}$, we obtain the following sharpened form of the estimate (2.1).

Theorem 3. *If P is an analytic polynomial on $\mathscr{X}$, and $\widehat{P}$ is its canonical extension to $\mathscr{X}^{**}$, then*

$$\|P\|_B = \|\widehat{P}\|_{B^{**}}.$$

Before giving the proof of Theorem 1, we make two observations.

It is essential that one be allowed to pass to arithmetic means in Theorem 1. To see this, let S be an orthogonal basis $\{e_j\}_{j=1}^{\infty}$ for l^2. For $t = \sum t_j e_j \in l^2$, define $P(t) = \sum t_j^2$. Then P is a 2-homogeneous analytic function on l^2. Since $P(e_j) = 1$ does not converge to $P(0) = 0$, 0 does not lie in the polynomial-star closure of S. However, since $e_j \to 0$ weakly, 0 lies in the weak (=weak-star) closure of S.

For some Banach spaces (such as l^p, $1 \le p < \infty$), it can be shown that a *sequence* converges in the polynomial-star topology if and only if it converges in norm. For this and related results, see [3]. Thus Theorem 1 is analogous to the theorem, valid in *some* Banach spaces, that every weakly convergent sequence has a subsequence whose arithmetic means converge in norm.

3. Proof of Theorem 1

The proof will depend on the following lemma.

Lemma. *Let $z \in X^{**}$, and let S be a subset of $\mathscr{X}$ which contains z in its weak-star closure. Let $\mathscr{F}$ be a finite family of continuous symmetric multilinear forms on $\mathscr{X}$. Let $\varepsilon > 0$ and $N \ge 1$. Then there exist $x_1, \dots, x_N \in S$ such that*

$$|F(x_{i_1}, \dots, x_{i_m}) - \widehat{F}(z, \dots, z)| < \varepsilon,$$

whenever $F \in \mathscr{F}$ is an m-form and $i_1, \dots, i_m$ are distinct indices between 1 and N.

Proof. Since each $F \in \mathscr{F}$ is symmetric, it suffices to obtain the estimate when $i_1 < \cdots < i_m$. Let $\varepsilon' > 0$ be small. The x_j's are selected inductively by the following procedure, which depends on the weak-star continuity property of the $\widehat{F}$'s described earlier.

First we choose $x_1 \in S$ such that

$$|\widehat{F}(x_1, z, \dots, z) - \widehat{F}(z, z, \dots, z)| < \varepsilon', \qquad F \in \mathscr{F}.$$

Then we chose $x_2 \in S$ such that

$$|\widehat{F}(x_2, z, \dots, z) - F(z, \cdots, z)| < \varepsilon', \qquad F \in \mathscr{F},$$

and moreover,

$$|\widehat{F}(x_1, x_2, z, \dots, z) - \widehat{F}(x_1, z, z, \dots, z)| < \varepsilon', \qquad F \in \mathscr{F}.$$

Proceeding in this fashion, we obtain x_j 's so that

$$|\widehat{F}(x_{i_1}, \dots, x_{i_{r-1}}, x_{i_r}, z, \dots, z) - \widehat{F}(x_{i_1}, \dots, x_{i_{r-1}}, z, z, \dots, z)| < \varepsilon', F \in \mathscr{F},$$

whenever $i_1 < \dots < i_r$. Then

$$|F(x_{i_1}, \dots, x_{i_m}) - \widehat{F}(z, \dots, z)|$$

is estimated by a sum of m terms

$$|F(x_{i_1}, \dots, x_{i_m}) - \widehat{F}(x_{i_1}, \dots, x_{i_{m-1}}, z)| + \dots + |\widehat{F}(x_{i_1}, z, \dots, z) - \widehat{F}(z, \dots, z)|,$$

each smaller than ε'. This is less than ε, for all $F \in \mathscr{F}$, providing ε' is sufficiently small. □

Now to prove Theorem 1, let $\varepsilon > 0$, and let $\mathscr{F}$ be a finite family of continuous symmetric multilinear forms on $\mathscr{X}$. It suffices to find an arithmetic mean x of elements of S such that

$$|\widehat{F}(z, \dots, z) - F(x, \dots, x)| \leq 2\varepsilon$$

for all $F \in \mathscr{F}$. For this, fix N large, choose $x_1, \dots, x_N \in S$ as in the lemma, and set

$$x = (x_1 + \dots + x_N)/N.$$

If $F \in \mathscr{F}$ is m-linear, we express

$$\begin{aligned} F(x, \dots, x) - \widehat{F}(z, \dots, z) &= \frac{1}{N^m} \sum_{i_1, \dots, i_m = 1}^{N} [F(x_{i_1}, \dots, x_{i_m}) - \widehat{F}(z, \dots, z)] \\ &= \Sigma_1 + \Sigma_2, \end{aligned}$$

where Σ_1 is the sum over m-tuples of distinct indices, and Σ_2 is the sum over the remaining indices. From the lemma we obtain

$$|\Sigma_1| < \varepsilon.$$

Since there are $N^m - N(N-1)\cdots(N-m+1)$ summands in Σ_2, each bounded by a constant C depending only on $\mathscr{F}$ and S, we obtain

$$|\Sigma_2| \leq \left[1 - \left(1 - \frac{1}{N}\right) \cdots \left(1 - \frac{m-1}{N}\right)\right] C.$$

For N sufficiently large this is also less than ε, so that we obtain the required estimates above. □

4. Embedding $H^\infty(B)$ in $H^\infty(B^{**})$

Consider a Taylor series

$$\sum_{m=0}^{\infty} P_m(x), \tag{4.1}$$

where P_m is an m-homogeneous analytic polynomial on $\mathscr{X}$, $m \geq 0$. Recall that the *radius of boundedness* of the series is the largest R, $0 \leq R \leq +\infty$, such that the series converges and represents a bounded analytic function on any ball $B_r = rB$ of radius $r < R$. In this case, the series (4.1) converges uniformly on each ball B_r, $r < R$. It is easy to check [4, 6] that the radius of boundedness is given by the expression

$$R = 1/\limsup \|P_m\|^{1/m}. \tag{4.2}$$

We may also form the series

$$\sum_{m=0}^{\infty} \widehat{P}_m(z) \tag{4.3}$$

on $\mathscr{X}^{**}$, for which the radius of boundedness is obtained by replacing P_m by $\widehat{P}_m$ in (4.2). From Theorem 3 we then obtain immediately the following.

Theorem 4. *The radius of boundedness of the Taylor series* (4.1) *in* $\mathscr{X}$ *coincides with that of the series* (4.3) *in* $\mathscr{X}^{**}$.

If now f is given by the Taylor series (4.1), we define $\hat{f}$ to be the sum of the Taylor series (4.3). Note that $\hat{f}$ is bounded on each ball $B_r^{**} = rB^{**}$, when r is strictly less than the radius of boundedness of (4.1).

Lemma. *Suppose f has Taylor series given by* (4.1) *with radius of boundedness $R > 0$, and that $0 < r < R$. Let $\{x_\alpha\}$ be a net in B_r converging to $z \in \mathscr{X}^{**}$ in the polynomial-star topology. Then $\{f(x_\alpha)\}$ converges to $\hat{f}(z)$.*

Proof. Fix $\varepsilon > 0$. Let $N \geq 1$, and let Q be the Nth partial sum of the series (4.1). Then $\widehat{Q}$ is the Nth partial sum of the series (4.3) for $\hat{f}$. Since the radius of boundedness of (4.3) is $R > r$, we can choose N so large that

$$\|\hat{f} - \widehat{Q}\|_{B_r^{**}} < \varepsilon.$$

Since $\widehat{Q} = Q$ on $\mathscr{X}$, we obtain

$$|f(x_\alpha) - \hat{f}(z)| \leq |f(x_\alpha) - Q(x_\alpha)| + |Q(x_\alpha) - \widehat{Q}(z)| + |\widehat{Q}(z) - \hat{f}(z)|.$$

Since $Q(x_\alpha) \to \widehat{Q}(z)$, we obtain

$$\limsup |f(x_\alpha) - \hat{f}(z)| \leq 2\varepsilon.$$

This establishes the lemma. □

Theorem 5. *If $f \in H^\infty(B)$ has Taylor series* (4.1), *then the series* (4.3) *converges on B^{**} to a function $\hat{f} \in H^\infty(B)$, satisfying*

$$\|\hat{f}\|_{B^{**}} = \|f\|_B.$$

The correspondence $f \to \hat{f}$ is an isometric isomorphism of $H^\infty(B)$ and a closed subalgebra of $H^\infty(B^{**})$.

Proof. The preceding lemma shows that

$$|\hat{f}(z)| \leq \|f\|_B$$

for all $z \in B^{**}$, so that the correspondence $f \to \hat{f}$ is an isometry. Since the operator $P \to \widehat{P}$ is linear and multiplicative on analytic polynomials, and these are dense in $H^\infty(B)$ in the topology of uniform convergence on the B_r's, $0 < r < 1$, the correspondence is linear and multiplicative on $H^\infty(B)$. $\square$

It would be of interest to characterize the value $\hat{f}(z)$ of the canonical extension of $f \in H^\infty(B)$, without resorting to the Taylor series of f. From the point of view of the spectrum of $H^\infty(B)$, the problem is to determine what distinguishes the complex-valued homomorphisms which arise from points of B^{**} from the remaining ones. For more on the spectrum of $H^\infty(B)$, see [2].

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