PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 106, Number 2, June 1989

NONCOMPACT COMMUTATORS IN THE COMMUTANT OF A CYCLIC OPERATOR

JAMES D. STAFNEY

(Communicated by John B. Conway)

ABSTRACT. We show that the commutant of the operator $S \oplus (I + S^*)$, where S is the shift operator, contains two operators A and B such that $AB - BA$ is not compact operator.

Let S be the shift operator and let $T = S \oplus (I + S^*)$. Herrero [1] has shown that T and T^* are cyclic and the commutant of T is not commutative. In this paper we show that the commutant of T fails to be commutative in the following stronger sense.

Theorem. *The commutant of the operator $S \oplus (I + S^*)$ contains two operators M, M_1 such that $MM_1 - M_1M$ is not a compact operator.*

Proof. For convenience, let $W = S \oplus (I + S^*)$. So, W is an operator (= bounded linear operator) on the Hilbert space $H = H^2 \oplus H^2$. Each operator A on H can be expressed as a matrix

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where each $A_{ij} \in \mathscr{L}(H^2)$, the bounded linear operators on H^2. In particular,

$$W = \begin{pmatrix} S & 0 \\ 0 & I + S^* \end{pmatrix}.$$

An easy computation shows that the operators

$$E_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}$$

are in the commutant of W provided that $X = C$ satisfies the operator equation

$$(I + S^*)X = XS. \tag{1}$$

Received by the editors July 1, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 47B47.
Key words and phrases. Commutant, cyclic operator, commutator.

Furthermore, a simple computation shows that

$$E_1E_2 - E_2E_1 = \begin{pmatrix} 0 & 0 \\ -C & 0 \end{pmatrix}.$$

These observations show that the theorem follows from Lemma 1.

Lemma 1. *The operator equation* (1) *has a solution* X *in* $\mathscr{L}(H^2)$ *which is not a compact operator.*

To complete the proof of the theorem we now prove Lemma 1 based on the following lemmas and propositions. Let T and φ be as in Lemma 4. Let A be the operator on H^2 with matrix $(a_{k,n})$ corresponding to φ as in Proposition 3. Then Proposition 3 and Lemma 4 show that $X = A$ is a solution (in $\mathscr{L}(H^2)$) of (1) which is not compact.

Proposition 2. *The operator* $X = A$ *is a solution in* $\mathscr{L}(H^2)$ *of* (1) *if and only if the matrix* $(a_{k,n}$, k, $n = 0, 1, 2, \ldots$, *of* A *relative to the orthonormal basis* $u_n = e^{in\theta}$, $n = 0, 1, \ldots$, *satisfies*:

(i) $a_{k,n} + a_{k+1,n} = a_{k,n+1}$;

(ii) *the matrix* $(a_{k,n})$ *determines a bounded operator on* l^2.

Proof. Suppose A is a solution of (1) and $(a_{k,n})$ is its matrix. Then (ii) is clearly satisfied. To prove (i) we note

$$\sum_k a_{k,n+1}u_k = Au_{n+1} = ASu_n = (I + S^*)Au_n = (I + S^*)\sum_k a_{k,n}u_k$$
$$= \sum_k a_{k,n}u_k + \sum_k a_{k+1,n}u_k.$$

The converse follows from similar reasoning.

Let L^2 denote $L^2(\mu)$ where μ is normalized Lebesgue measure on the unit circle. We regard H^2 as a subspace of L^2 whenever appropriate. For each $\varphi \in L^2$, we define the transformation T_φ from L^2 into sequences by

$$T_\varphi f(k) = \int (e^{i\theta} - 1)^k \varphi(e^{i\theta}) f(e^{i\theta})\, d\theta/2\pi,$$

where $f \in L^2$, $k = 0, 1, \ldots$.

Proposition 3. *For* $\varphi \in L^2$, *the matrix*

$$a_{k,n} \equiv \int (e^{i\theta} - 1)^k e^{in\theta} \varphi(e^{i\theta})\, d\theta/2\pi$$

is the matrix, relative to the orthonormal basis in Proposition 2, of a solution (compact solution) $X = A$ *of (1) if and only if the restriction of* T_φ *to* H^2 *determines a bounded (compact) linear transformation of* H^2 *into* l^2.

Proof. If $(a_{k,n})$ is the matrix of a bounded operator, then T_φ clearly has the stated boundedness property. If T_φ has the stated boundedness property, then

$(a_{k,n})$ is the matrix of an operator $A \in \mathscr{L}(H^2)$. An easy computation shows that $(a_{k,n})$ satisfies (i) of Proposition 2. Thus, $X = A$ is a solution of (1).

Lemma 4. *Let $T = T_\varphi$ where φ is the characteristic function of the arc $\{e^{i\theta} : 0 < \theta < \pi/3\}$. Then*

(i) *T is a bounded linear transformation of L^2 into l^2;*
(ii) *the restriction of T to H^2 is not a compact operator.*

Proof. For $f \in L^2$,

$$Tf(k) = \int_0^{\pi/3} (e^{i\theta} - 1)^k f(e^{i\theta})\, d\theta/2\pi .$$

If $b_k = \int_0^1 x^k g(x)\, dx$, then since the Hilbert matrix is bounded,

$$\sum_k |b_k|^2 \le C \int_0^1 |g(x)|^2\, dx$$

for some constant C. From this and a routine change of variable argument with $x = |e^{i\theta} - 1|$, we conclude that T is bounded.

We now prove (ii). For convenience, let $v(\theta) = e^{i\theta} - 1$. A routine computation shows that

$$T^* T f(e^{i\theta_1}) = \int_0^{\pi/3} [1/(1 - \overline{v(\theta_1)} v(\theta))] f(\theta)\, d\theta/2\pi$$

for $f \in L^2$ and $0 \le \theta_1 \le \pi/3$; and, $T^* T f(e^{i\theta_1}) = 0$, otherwise. We will show that the restriction of $T^* T$ to H^2 is not compact. For convenience, let M denote the restriction of $T^* T$ to H^2. In particular, M is a bounded linear transformation from H^2 into L^2. We will use Lemma 5 to show that M is not compact.

Lemma 5. *Suppose that H, H' are Hilbert spaces, that A is a bounded linear transformation from H into H' and that $x_n \in H$ for $n = 1, 2, \ldots$ such that*

(i) $\|x_n\| = 1$;
(ii) $(x_m, x_n) \to 0$ *as* $n \to \infty$ *for each fixed* m;
(iii) $\|Ax_n\| \ge \delta > 0$ *for all* n.

Then A is not compact.

This lemma and Lemma 6 below show that M is not compact, which completes the proof of (ii) in Lemma 4.

For $0 < a < 1$, let $z_a = (1 + a)e^{i\pi/3}$ and $f_a(e^{i\theta}) = \sqrt{a}/(e^{i\theta} - z_a)$. Clearly, each $f_a \in H^2$.

Lemma 6. *There exist positive constants C_0, C_1, C_2 such that for $0 < a$, $d < 1$,*

(i) $C_0 \le \|f_a\| \le C_1$,

(ii) $(f_a, f_d) \to 0$ *as* $d \to 0$ *for each fixed* a, *and*

(iii) $\|Mf_a\| \geq C_2\|f_a\|$ *where* $\| \ \|$, $(\ , \ )$ *denote the norm and inner product on* L^2.

Proof. Let $b = 1 + a$ $(0 < a < 1)$ and let $I = \int_{-\pi}^{\pi} 1/|e^{i\theta} - b|^2 \, d\theta$. By noting that

$$|e^{i\theta} - b|^2 = a^2 + b\theta^2 - 2bR(\theta),$$

where $R(\theta) = \cos\theta - (1 - \theta^2/2)$ one can show that

$$\left(\frac{6}{11}\right)\left(\frac{1}{a}\right) \leq I \leq (3\sqrt{2}\pi)\left(\frac{1}{a}\right);$$

and, (i) easily follows.

To prove (ii) we note that (f_a, f_d) is an integral over the interval $-\pi < \theta < \pi$ which is the sum of integrals over the subsets $J_1 = \{\theta : |\theta - \pi/3| < \delta\}$ and its complement J_2 in $[-\pi, \pi]$. By applying the Schwarz inequality to the integrals over each interval J_1 and J_2, by noting that $\int_{J_1} |f_a|^2 \, d\theta \to 0$ as $\delta \to 0$ for a fixed and by noting that $f_d \to 0$ as $d \to 0$ uniformly in J_2, we obtain (ii).

We now prove (iii). From the definition of M,

$$Mf_a(\theta_1) = \int_0^{\pi/3} [1/(1 - \overline{v(\theta_1)}v(\theta))] f_a(\theta) \, d\theta/2\pi$$

for $0 < \theta_1 < \pi/3$. For convenience, let $F = F(a, \theta_1, \theta)$ denote the integrand in the latter integral. It is easily seen that

1. the argument of $f_a(\theta)$ varies over an interval of length $\leq \pi/2$ for $0 < a < 1$ and $0 \leq \theta \leq \pi/3$.

A geometric argument shows that $\overline{v(\theta_1)}v(\theta)$ lies in the intersection of two circles both of radius one and with centers at $e^{i\pi/3}$ and $e^{-i\pi/3}$ for $0 \leq \theta$, $\theta_1 \leq \pi/3$. From this we conclude that

2. the arugment of $1 - \overline{v(\theta_1)}v(\theta)$ varies over an interval of length $\leq \pi/3$ for $0 \leq \theta$, $\theta_1 \leq \pi/3$; and, for some $C > 0$,

$$|1 - \overline{v(\theta_1)}v(\theta)| \leq C(1 - |\overline{v(\theta_1)}v(\theta)|), \qquad 0 \leq \theta, \theta_1 \leq \pi/3. \tag{2}$$

Thus, in particular, the argument of F varies over an interval of length $\leq 5\pi/6$ for $0 < a < 1$ and $0 \leq \theta$, $\theta_1 \leq \pi/3$. So, for some complex number λ with $|\lambda| = 1$, the argument of λF lies in the interval $[-5\pi/12, 5\pi/12]$. Therefore,

$$|Mf_a(\theta_1)| \geq C \int_0^{\pi/3} |F(a, \theta_1, \theta)| \, d\theta, \qquad 0 < \theta_1 < \pi/3$$

for some $C > 0$.

Let $t = |v(\theta)|$ and $s = |v(\theta_1)|$. From (2) and the definition of f_a it follows that

$$|F| \geq C(1/(1 - st))(1/\sqrt{a}), \qquad 0 < s < 1, 1 - a < t < 1,$$

for some $C > 0$. By a change of variable we conclude that for some $C > 0$,

$$|Mf_a(\theta_1)| \geq (C/\sqrt{a}) \int_{1-a}^{1} 1/(1-st)\,dt \geq (C/\sqrt{a}) \log(a/2(1-s)) \geq 0$$

for $s > 1 - a/2$. Thus, for some $C > 0$, C independent of a, we have

$$\int_0^{\pi/3} |Mf_a(\theta_1)|^2\,d\theta_1 \geq (C/a) \int_{1-a/2}^{1} [\log(a/2(1-s))]^2\,ds = (C/2) \int_0^1 (\log t)^2\,dt,$$

which, together with (i), proves (ii).

Acknowledgment

The referee has pointed out that the operator $A = S^* \otimes I$ is cyclic and the commutant of A contains operators B and C such that $BC - CB$ is not compact. However, A^* is not cyclic. So, the significance of Herrero's example is that both T and T^* are cyclic. The referee has also noted that if A denotes the infinite direct sum of the operators $(1/n)I + 10^{-n}T$, $n = 1, 2, \ldots$, then A and A^* are cyclic and the commutant of A contains a pair of operators with noncompact commutator. The point of our theorem is that it is not so clear that the operator T has this property. Because of the referees comments, we changed our original abstract and first paragraph.

References

1. D. A. Herrero, *On multicyclic operators*, Integral Equations Operator Theory **1** (1978), pp. 57, 102.

University of California, Department of Mathematics & Computer Science, Riverside, California 92521