PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 106, Number 2, June 1989

ON THE SINGULAR RANK OF A REPRESENTATION

JIAN-SHU LI

(Communicated by Jonathan M. Rosenberg)

ABSTRACT. Consider the reductive dual pair $(\mathrm{Sp}_{2n}, \mathrm{O}_{p,q})$. We prove that if π is a representation of Sp_{2n} coming from duality correspondence with some representation of $\mathrm{O}_{p,q}$ then the wave front set of π has rank $\leq p+q$. For $p+q<n$ this implies a result stated (but not proved) by Howe.

1. INTRODUCTION

Let G be a semi-simple Lie group. Let π be an admissible representation of G. The wave front set of π, denoted $\mathrm{WF}(\pi)$, is an important invariant attached to π. It measures the "singularities" of the representation π, and plays a basic role in the study of unipotent representations [1, 10].

Here we shall look at the case of the symplectic group $\mathrm{SP}_{2n}(\mathbf{R}) = \mathrm{Sp}_{2n}$. This is the subgroup of GL_{2n} preserving the standard symplectic form on $\mathbf{R}^{2n}$. Actually, it shall be convenient to work with $G = \widetilde{\mathrm{Sp}}_{2n}$, the metaplectic two-fold cover of Sp_{2n}. Let $\mathbf{g}$ be the Lie algebra of G and $\mathbf{g}^*$ its linear dual. Recall that $\mathrm{WF}(\pi)$ is a subset of $\mathbf{g}^*$. By means of the Killing form we may identify $\mathbf{g}^*$ with $\mathbf{g}$. In particular, an element of $\mathbf{g}^*$ is now a linear transformation on $\mathbf{R}^{2n}$, and as such it has a rank. Following [6] we define the *singular rank* of π to be the maximum rank of elements of $\mathrm{WF}(\pi)$. This is a (crude) measure of how singular the representation π is. The purpose of this note is to prove

Proposition 1. *Consider the reductive dual pair* $(\mathrm{Sp}_{2n}, \mathrm{O}_{p,q})$. *Let* π *be a representation of* $\widetilde{\mathrm{Sp}}_{2n}$ *coming from duality correspondence with* $\mathrm{O}_{p,q}$. *We have the inequality*

$$\text{singular rank}(\pi) \leq p+q \,. \tag{1}$$

Remarks. (a) Suppose $p+q<2n$—this being the range where Proposition 1 is non-empty. Then "generically" (1) should be an equality. (b). Analogous results are valid for other classical groups.

Received by the editors June 9, 1988 and, in revised form, September 28, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 22E46, 22E47; Secondary 22E45.
Key words and phrases. Wave front sets, reductive dual pairs.

Let X be a maximal isotropic subspace of $\mathbf{R}^{2n}$, and let N be the subgroup of G which leaves X pointwise fixed. For π unitary Howe [4] has also defined the N-rank of π. This is an integer between 0 and n, and is yet another invariant to tell how singular the representation π is. The following is stated as Proposition 2.7 in [6]:

Proposition 2. *Suppose π is irreducible unitary with N-rank less than n. One has the inequality*

$$\text{singular rank}(\pi) \leq N - \text{rank}(\pi). \tag{2}$$

However, it seems that the proof given in [6] does not imply the above statement, but rather the following: *If the singular rank of π is less than n then* (2) *holds.*

Now let π be an irreducible unitary representation of N-rank less than n. It was proved in [5, 7] (see also [8]) that π must come from duality correspondence with some $\mathrm{O}_{p,q}$, such that

$$p + q = N - \text{rank}(\pi).$$

Together with Proposition 1 this gives a proof of Proposition 2.

2. The proof

Let π be as in Proposition 1. Let $V(I_\pi)$ be the associated cone of the annihilator of π (see definition below). We shall actually show that $V(I_\pi)$ consists of elements of rank $\leq p+q$. Since it is well known that $\mathrm{WF}(\pi) \subseteq V(I_\pi) \cap \mathbf{g}^*$ (cf. [2, p. 159]), Proposition 1 will follow.

Let V be a vector space over $\mathbf{R}$ endowed with a nondegenerate symmetric bilinear form $(\ ,\ )$ of signature (p, q). Let $G' = \mathrm{O}_{p,q}(\mathbf{R})$ be the isometry group of $(\ ,\ )$. Set $W = \mathbf{R}^{2n} \otimes V \simeq V^{2n}$. In the usual fashion, W is endowed with a symplectic form (coming from the standard symplectic structure on $\mathbf{R}^{2n}$ and the form $(\ ,\ )$ on V). Let Sp be the group of symplectic transformations on W, and let sp be its Lie algebra. Consider the oscillator representation ω of $\widetilde{\mathrm{Sp}}$. Let K be a maximal compact subgroup of $\widetilde{\mathrm{Sp}}$. Using the so-called Fock model one sees that the K-finite vectors in ω form a space isomorphic to $\mathscr{P}(V_\mathbf{c}^n) = \mathscr{P}$, the space of polynomial functions on $V_\mathbf{c}^n$. (Here and later on, we use the subscript $\mathbf{c}$ to indicate various complexifications.) Acting on $\mathscr{P}$, the space $\omega(\mathrm{sp}_\mathbf{c})$ is seen to be spanned by operators of the form

$$z_i z_j, \quad \frac{\partial^2}{\partial z_i \partial z_j} \quad \text{and} \quad \frac{1}{2}\left(z_i \frac{\partial}{\partial z_j} + \frac{\partial}{\partial z_j} z_i\right).$$

Here z_i $(1 \leq i \leq n(p+q))$ are the coordinates with respect to some basis of $V_\mathbf{c}^n$.

Let $U(\mathbf{g}_\mathbf{c})$ be the universal enveloping algebra of $\mathbf{g}_c$. Let

$$\mathbf{C} = U(\mathbf{g}_\mathbf{c})_0 \subseteq U(\mathbf{g}_\mathbf{c})_1 \subseteq U(\mathbf{g}_\mathbf{c})_2 \subseteq \cdots$$

be the standard filtration on $U(\mathbf{g}_c)$. Put

$$\omega(U(\mathbf{g}_c))_i = \omega(U(\mathbf{g}_c)_i) \qquad (i \geq 0). \tag{3}$$

Finally let $\operatorname{gr}(\omega(U(\mathbf{g}_c)))$ be the graded algebra associated to the above filtration on $\omega(U(\mathbf{g}_c))$. One version of the First Fundamental Theorem of classical invariant theory asserts the isomorphism

$$\operatorname{gr}(\omega(U(\mathbf{g}_c))) \xrightarrow{\sim} S(V_c^{2n})^{G'_c}. \tag{4}$$

Here on the right-hand side $S(V_c^{2n})$ denotes the symmetric algebra associated to $V_c^{2n} = W_c$, $G'_c \approx \mathrm{O}_{p+q}(\mathbf{C})$ is the complexification of G', and $S(V_c^{2n})^{G'_c}$ denotes the algebra of invariants of G'_c in $S(V_c^{2n})$. (See [3], Theorems 6 and 7.)

The symmetric bilinear form $(\ ,\ )$ on V complexifies to one on V_c, which we continue to denote by $(\ ,\ )$. Write a typical vector in V_c^{2n} as

$$x = (x_1, \dots, x_{2n})$$

with $x_i \in V_c$. For $1 \leq i, j \leq 2n$ let (i, j) denote the map which takes x to (x_i, x_j). Then (i, j) defines a symmetric bilinear form on V_c^{2n}. Since V_c can be identified with its linear dual V_c^* by means of the form $(\ ,\ )$, we may view (i, j) as an element of $S^2(V_c^{2n})^{G'_c}$. In its original form [10] the First Fundamental Theorem of classical invariant theory asserts that the (i, j)'s generate the whole algebra $S(V_c^{2n})^{G'_c}$. Set $m = p + q$. Let $y_1, \dots, y_{m+1}$ be vectors in V_c. It is elementary that

$$\det \begin{pmatrix} (y_1, y_1) & \cdots & (y_1, y_{m+1}) \\ \vdots & \ddots & \vdots \\ (y_{m+1}, y_1) & \cdots & (y_{m+1}, y_{m+1}) \end{pmatrix} = 0. \tag{5}$$

Consider the matrix

$$\begin{pmatrix} (1,1) & (1,2) & \cdots & (1,2n) \\ (2,1) & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ (2n,1) & \cdots & \cdots & (2n,2n) \end{pmatrix}. \tag{6}$$

The relation (5) says that the determinant of any minor of (6) with size $(m+1) \times (m+1)$ must be zero. Indeed, the Second Fundamental Theorem of classical invariant theory asserts that these generate all the relations among the generators $(i, j) \in S(V_c^{2n})^{G'_c}$. (We shall not need this last fact.)

Let now π be a representation of G coming from duality correspondence with G'. This means π is a finitely generated, admissible quotient of $\omega|_{G'_c}$. Let $\mathscr{H}_\pi$ be the space of K-finite vectors for π (K being a maximal compact subgroup of G). Then $U(\mathbf{g}_c)$ acts on $\mathscr{H}_\pi$. We have a surjective homomorphism

$$\omega(U(\mathbf{g}_c)) \to \pi(U(\mathbf{g}_c)).$$

Define a filtration on $\pi(U(\mathbf{g}_c))$ by

$$\pi(U(\mathbf{g}_c))_i = \pi(U\mathbf{g}_c)_i) \qquad (i \geq 0)$$

(compare (3)). Since all the relevant filtrations come from the standard one on $U(\mathbf{g}_c)$, we have the following commutative diagram

$$\begin{array}{ccc} \omega(U(\mathbf{g}_c)) & \longrightarrow & \mathrm{gr}(\omega(U(\mathbf{g}_c)) \\ \downarrow & & \downarrow \\ \pi(U(\mathbf{g}_c)) & \longrightarrow & \mathrm{gr}(\pi(U(\mathbf{g}_c))) . \end{array} \tag{7}$$

Let $I_\pi \subseteq U(\mathbf{g}_c)$ be the annihilator of $\mathscr{H}_\pi$. Let $\mathrm{gr}(I_\pi)$ be the associated graded ideal in $S(\mathbf{g}_c) = \mathrm{gr}(U(\mathbf{g}_c))$. The associated cone $V(I_\pi)$ is defined to be the zero variety of $\mathrm{gr}(I_\pi)$ inside $\mathbf{g}_c^*$. As remarked earlier, we need only prove that $V(I_\pi)$ consists of elements of rank $\leq p+q$. Through the isomorphism (4), the relations (5) and what we said after (6) immediately translate into relations inside $\mathrm{gr}(\omega(U(\mathbf{g}_c)))$. In turn these translate into relations in $\mathrm{gr}(\pi(U(\mathbf{g}_c)))$ through the diagram (7). In a little more detail, let us choose elements e_{ij} in $U(\mathbf{g}_c)$ such that $e_{ij} = e_{ji}$, and e_{ij} goes to $(i, j) \in S(V_c^{2n})^{G_c'}$ under the sequence of maps

$$U(\mathbf{g}_c) \to \mathrm{gr}(\omega(U(\mathbf{g}_c))) \xrightarrow{\sim} S(V_c^2)^{G_c'} .$$

Obviously the e_{ij}'s may be chosen inside $\mathbf{g}_c$, and then $\{e_{ij} | 1 \leq i \leq j \leq 2n\}$ form a basis of $\mathbf{g}_c$. Let $\bar{e}_{ij}$ denote e_{ij} but viewed as an element in $S(\mathbf{g}_c)$. Consider the matrix

$$\begin{pmatrix} \bar{e}_{1\,1} & \cdots & \bar{e}_{1\,2n} \\ \vdots & & \vdots \\ \bar{e}_{2n\,1} & \cdots & \bar{e}_{2n\,2n} \end{pmatrix} . \tag{8}$$

Then the above discussion can be summarized as

Lemma 3. *For each minor*

$$M = \begin{pmatrix} \bar{e}_{i_1\, j_1} & \cdots & \bar{e}_{i_1\, j_{m+1}} \\ \vdots & \ddots & \vdots \\ \bar{e}_{i_{m+1}\, j_1} & \cdots & \bar{e}_{i_{m+1}\, j_{m+1}} \end{pmatrix} \tag{9}$$

of the matrix (8) *of size* $(m+1)\times(m+1)(m = p+q)$, *we have* $\det(M) \in \mathrm{gr}(I_\pi)$.

Note that we have a well-known isomorphism

$$\mathbf{g}_c^* \simeq S^2(C^{2n}) .$$

The latter can be identified with $2n$ by $2n$ symmetric matrices. The rank of an element in $\mathbf{g}_c^*$ is the same as the rank of its image in $S^2(C^{2n})$. It is now an elementary exercise to see that the common zeros in $\mathbf{g}_c^*$ of the polynomials $\det M$ (M as in (9)) is precisely the set of elements of $\mathbf{g}_c^*$ with rank $\leq p+q$. By the above lemma we see $V(I_\pi)$ indeed contains only elements of rank $\leq p+q$. This concludes the proof of Proposition 1.

Observe that the above argument really proves the following proposition, from which Proposition 1 follows immediately:

Proposition 4. *The rank of the associated cone of the annihilator of* $\omega|_G$ *is less than or equal to* $p+q$.

ACKNOWLEDGMENT

The author would like to thank the referee for some very helpful comments which improved the exposition of this paper.

REFERENCES

1. D. Barbasch and D. Vogan Jr., *Unipotent representations of complex semisimple Lie groups*, Ann. of Math. (2) **121** (1985), 41–110.
2. D. Barbasch and D. Vogan Jr., *Primitive ideals and orbital integrals in complex classical groups*, Math. Ann. **259** (1982), 153–199.
3. R. Howe, *Remarks on classical invariant theory* (preprint).
4. ——, *A notion of rank for unitary representations of classical groups*, C.I.M.E. Summer School on Harmonic Analysis, Cortona, 1980.
5. ——, *Small unitary representations of classical groups*, Group Representations, Ergodic Theory, Operator Algebras and Math. Physics, Proc. Conf. in honor of C. W. Mackey (C. C. Moore, ed.), M.S.R.I., Publ. No. 6, Springer-Verlag, New York, 1986, pp. 121–150.
6. ——, *Wave front sets of representations of Lie groups*, Automorphic Forms, Representations Theory, and Arithmetic, papers presented at the Bombay Colloquium, 1979, Tata Institute of Fundamental Research, Bombay, 1981, pp. 117–140.
7. J. Li, *Theta series and distinguished representations of classical groups*, Thesis, Yale University, 1987.
8. ——, *Singular unitary representations of classical groups* (preprint).
9. D. Vogan Jr., *Unitary representations of reductive Lie groups*, Ann. of Math. Stud. no. 118, 1987.
10. H. Weyl, *The classical groups*, Princeton University Press, Princeton, N.J., 1946.

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASSACHUSETTS 02139