

ON QUASIHHEREDITARY RINGS

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ABSTRACT. The notion of a quashereditary semiprimary ring was introduced by Cline, Parshall and Scott and was studied extensively by Dlab and Ringel. It is known that if R is semiprimary of global dimension ≤ 2 then it is quasihereditary and that there is a serial ring of global dimension 4 which is not. This paper establishes three principal results: If R is quasihereditary then the Cartan determinant conjecture is true for R (i.e., $\det C(R) = 1$); a serial ring is quasihereditary iff it has a heredity ideal; in particular, every serial ring of global dimension 3 is quasihereditary; and there is a ring (a 0-relations algebra in fact) of global dimension 3 which is not quasihereditary.

INTRODUCTION

Throughout all rings, which are denoted by R , are assumed to be unitary and semiprimary (i.e., the radical N is nilpotent and R/N is semisimple artinian). Following [DR1] we say that an ideal I of R is *heredity* if $I^2 = I$, ${}_R I$ is projective and $INI = 0$. Further, R is *quasihereditary* if there exists a chain of ideals $0 = I_0 \subset I_1 \subset \cdots \subset I_m = R$ such that I_t/I_{t-1} is a heredity ideal R/I_{t-1} , $t = 1, \dots, m$. (Such a chain is called a *heredity chain*.) Of course, if $m = 1$, R is semisimple. These rings were introduced by Cline, Parshall and Scott ([CPS]) in the context of representation theory. Their importance lies also in the fact every semiprimary ring is the endomorphism ring of a projective module over a quasihereditary ring, as was shown by Dlab and Ringel in [DR3]. They form a natural generalization of hereditary rings but as shown in [DR1] they can be of arbitrary finite global dimension.

All semiprimary rings of global dimension ≤ 2 are quasihereditary ([DR2]) but [DR2] also contains an example of an 11 dimensional serial algebra over a field which is of global dimension 4 but is not quasihereditary. We show below (2.3) that a serial ring is quasihereditary iff it contains a heredity ideal and, as a consequence (2.7), that if R is serial of global dimension 3 then it is quasihereditary. The example in section 3 shows that even a 0-relations algebra over a field of global dimension 3 need not be quasihereditary.

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Let R be a basic left artinian ring with a basic set of primitive idempotents e_1, \dots, e_n . Then the (left) Cartan matrix $C(R)$ of R is the $n \times n$ integral matrix whose (i, j) -entry is the number of copies of the simple $S_i = Re_i/Ne_i$ which occur in a composition series for Re_j . It has long been known that $\text{gl. dim } R < \infty$ implies $\det C(R) = \pm 1$ and it is conjectured that finite global dimension implies $\det C(R) = 1$. This is called the Cartan determinant conjecture (see [BFVZ] and [FZ] for recent information about it). It is shown below that the conjecture is true for quasihereditary rings.

We shall keep the above notation throughout: e_1, \dots, e_n is a basic set of primitive idempotents and S_1, \dots, S_n the corresponding simples.

1. PRELIMINARY RESULTS AND THE CARTAN DETERMINANT

We first show that if R has a heredity ideal then it has one of the form Re_jR .

Lemma 1.1. *Suppose that I is a heredity ideal of R . Then I can be written $I = ReR$ where (with suitable renumbering) $e = e_1 + \dots + e_k$. Then for each $j = 1, \dots, n$, Re_1Re_j is either 0 or a direct sum of copies of Re_1 . In particular, Re_1R is heredity.*

Proof. Since I is idempotent it has the form ReR for some $e = e^2$ (see e.g. [DR1]), but equivalent idempotents generate the same ideal, so e may be taken to have the form $e_1 + \dots + e_k$, for some k . For any j , $ReRe_j$ is a direct summand of I , so it is projective and is generated by Re . Hence $ReRe_j = Q_1 \oplus \dots \oplus Q_k$, where each Q_i is isomorphic to a direct sum of copies of Re_i . Since $e_1 = e_1e$, $Re_1Re_j = Re_1ReRe_j$ and then $Re_1Re_j = \sum_{i=1}^k \oplus Re_1Q_i$. Now $Re_1Re_j = 0$ if $j = 2, \dots, k$ since $INI = 0$, giving that $Re_1Re_j = Re_1Q_1 = Q_1$. Since $Re_1Re_1 = Re_1$, Re_1R is projective. Hence it is heredity. \diamond

Corollary 1.2. *If R is quasihereditary, the primitive idempotents can be numbered so that $0 \subset Re_1R \subset R(e_1 + e_2)R \subset \dots \subset R(e_1 + \dots + e_n)R = R$ is a heredity chain.*

Proof. Given a heredity chain $0 = I_0 \subset I_1 \subset \dots \subset I_t = R$, we have seen that we can write $I_1 = R(e_1 + \dots + e_k)R$ (with a suitable renumbering) and, by the lemma, Re_1R is heredity. Now $0 \subset I_1/Re_1R \subset I_2/Re_1R \subset \dots \subset R/Re_1R$ is a heredity chain and $e_2 + Re_1R, \dots, e_n + Re_1R$ is a complete orthogonal basic set of primitive idempotents for R/Re_1R . The process can be continued. \diamond

Note that if R is quasihereditary then $\text{gl. dim } R \leq 2(n - 1)$ and the Loewy length is $\leq 2^n - 1$ by [DR1, Statement 9]. Of course this particular heredity chain may be longer than necessary but it does give bounds as functions of the number of basic primitive idempotents.

Proposition 1.3. *Let R be a left artinian ring. Let $C(R)$ be its Cartan matrix. If R is quasihereditary then $\det C(R) = 1$.*

Proof. Without loss of generality, we may take R to be basic. Indeed the notion “quasihereditary” is a Morita invariant since the lattice isomorphism between the lattices of two Morita equivalent rings preserves multiplication of ideals. Hence a heredity chain is taken to a heredity chain. The Cartan matrix is also a Morita invariant.

Using the lemma, let $I = Re_1R$ be heredity. Since $e_1Ne_1 = 0$, $c_{11} = 1$ in $C(R)$. For $j > 1$, $Ie_j \cong (Re_1)^{m_j}$, again by the lemma. Hence elementary column operations transform $C(R)$ into the form

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ c_{21} & & & \\ \cdot & & C(R/I) & \\ \cdot & & & \\ c_{n1} & & & \end{bmatrix}.$$

The result follows by induction on n since when $n = 1$, R is simple. \diamond

Another result which may turn out to be true for all rings of finite global dimension is true for quasihereditary rings. It was established for global dimension ≤ 2 in [GGZ].

Proposition 1.4. *Let R be a quasihereditary semiprimary ring. Then for an simple module S , $\text{Ext}^1(S, S) = 0$.*

Proof. With the usual notation, we may assume that $I = Re_1R$ is heredity (by (1.1)). Then $e_1Ne_1 = 0$, giving $\text{Ext}^1(S_1, S_1) = 0$. Now $\text{Rad } R/I = (N + I)/I$ and if we work by induction on n (again the result is clear when $n = 1$), we get that

$$(e_i + I) \frac{N + I}{I} (e_i + I) \subseteq \frac{N^2 + I}{I}$$

for $i > 1$. That is, $e_iNe_i \subseteq N^2 + I$ for $i > 1$; then $e_iNe_i \subseteq e_iN^2e_i + e_iRe_1Re_i \subseteq e_iN^2e_i$. Hence, again, $\text{Ext}^1(S_i, S_i) = 0$. \diamond

Before going on to the serial case, let us note that l -hereditary rings (and, in particular, finite dimensional incidence algebras) are quasihereditary. These are rings such that for any $i \neq j$ any $0 \neq h: Re_i \rightarrow Re_j$ is monic (see [M-V]). First there is a lemma which will be used again.

Lemma 1.5. *Let Re_i be simple. Then $Re_iR = I$ is hereditary. Further, if e_iR is simple, then Re_iR is heredity.*

Proof. Since $I \subseteq \text{Soc}({}_R R)$, $NI = 0$. Further I is the S_i -homogeneous component of $\text{Soc}({}_R R)$. It is projective. The second part follows from [DR1, Statement 7] which shows that the notion of heredity ideal is left-right symmetric. \diamond

Proposition 1.6. *If R is a semiprimary ring which is l -hereditary then R is quasihereditary.*

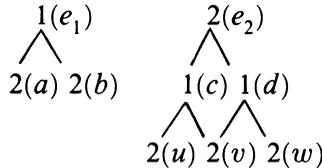
Proof. One sees readily that there are simple projectives. Suppose Re_1 is one. Then $I = Re_1R$ is heredity. Put $f = 1 - e_1$. The claim is that $R/I \cong fRf$. Indeed $r \rightarrow frf$ induces a homomorphism since $frsf = frfsf + fre_1sf$. The second term is zero since $NI = 0$. The kernel is I since if $frf = 0$ then $r = frf + e_1rf + fre_1 + e_1re_1 \in I$. Now fRf will also be l -hereditary so the process can continue. \diamond

In fact heredity ideals always give rise to primitive idempotents which behave like those of l -hereditary rings.

Lemma 1.7. *Let R be a semiprimary ring and $I = ReR$ a heredity ideal with $e = e_1 + \dots + e_k$. Then for each $i = 1, \dots, k$, any homomorphism $0 \neq h: Re_i \rightarrow R$ is monic.*

Proof. Since I is the trace of Re , $\text{Hom}_R(Re_i, R) = \text{Hom}_R(Re_i, I)$. Further the projective $I \cong (Re_1)^{m_1} \oplus \dots \oplus (Re_k)^{m_k} := P$, via, say, q . Now $eNe \subseteq INI = 0$, so if $\pi: P \rightarrow Re_j$ is a projection, $1 \leq j \leq k$, such that $\pi qh(e_i) \neq 0$ it follows that $\pi qh(e_i) = e_i \pi p h(e_i) e_j \notin Ne_j$. But then $\pi qh: Re_i \rightarrow Re_j$ is an isomorphism and $i = j$. Hence h is monic. \diamond

The converse is false however, even for diagram algebras ([F2]), as the following example shows.



Let R be the algebra over a field K given by the accompanying diagram, where the numbers represent the simples in composition series for the indecomposable projectives and the letters are the elements of a multiplicative basis (see [F2] for more on this kind of presentation). The parts of the multiplication table which are not obvious from the diagrams are: $ac = v$, $bc = u$, $ad = w$ and $bd = v$. Then it is readily seen that $\text{Hom}(Re_1, Re_2) = Kc + Kd$ and that any $0 \neq h: Re_1 \rightarrow Re_i$ is monic for $i = 1, 2$. Equally clear is that Re_1R is not projective.

The converse does hold for 0-relations algebras (see e.g. [GHZ] for the definition); i.e., if $0 \neq h: Re_i \rightarrow Re_j$ is monic for all j , then Re_iR is heredity.

2. QUASIHEDERITARY SERIAL RINGS

In this section the notions and notations of [F1] are used. Two cases need separate study, when R has a simple projective and when it does not.

Lemma 2.1. *Let R be an indecomposable serial ring. If R has no simple projectives and e_i is a primitive idempotent then R/Re_iR is indecomposable with a simple projective $Re_{i+1} + Re_iR$.*

Proof. We arrange the primitive idempotents to give a Kupisch series ([F1]) and since there are no simple projectives we may assume $i \neq n$ (for convenience

of notation). Now $Re_iRe_j \subseteq N^2e_j$ for $j = i + 2, \dots, n, 1, \dots, i - 1$. If $\bar{}$ indicates “modulo Re_iR ”, it follows that $\overline{Re}_{i+1}, \dots, \overline{Re}_n, \overline{Re}_1, \dots, \overline{Re}_{i-1}$ is a Kupisch series for \overline{R} and there is no non-zero map $\overline{Re}_{i-1} \rightarrow \overline{Re}_{i+1}$. Hence \overline{Re}_{i+1} is simple. \diamond

Lemma 2.2. *Let R be an indecomposable serial ring with a simple projective. Then R is quasihereditary.*

Proof. Let Re_1, \dots, Re_n be a Kupisch series where Re_1 is simple. Then by (1.5), $I = Re_1R$ is heredity. Further, $\overline{Re}_2, \dots, \overline{Re}_n$ is a Kupisch series for \overline{R} and it is clear that \overline{Re}_2 is simple (the notation of the previous lemma is used). The process continues. \diamond

PROPOSITION 2.3. *An indecomposable serial ring R is quasihereditary iff it has a heredity ideal.*

Proof. One direction is obvious. For the other, if R has a simple projective then (2.2) gives the result. If not, R has a heredity ideal which may be taken to be of the form Re_iR (1.1). Now (2.1) says that R/Re_iR has a simple projective. \diamond

This result may be translated into terms of the composition lengths of the indecomposable projectives as follows. (Here $[m]$ is the class of m modulo n from among $1, \dots, n$.)

Proposition 2.4. *Let R be an indecomposable serial ring. Then R is quasihereditary iff a Kupisch series for R may be written in such a way that there is a portion Re_i, \dots, Re_{i+k-1} with composition lengths $c_i < \dots < c_{i+k-1}$ and $c_{[i+k]} \leq k$.*

Proof. If there is such a portion, then $Re_iRe_{i+p} \cong Re_i$ for $p = 1, \dots, k - 1$ while $Re_iRe_j = 0$ if $j \neq i, i + 1, \dots, i + k - 1$. Hence Re_iR is projective. Furthermore, $e_iNe_i = 0$, since Ne_i is an epimorph of $Re_{[i-1]}$.

In the other direction if Re_iR is heredity, then $h: Re_i \rightarrow Re_j$ is either 0 or is monic. This readily translates into the condition. \diamond

We have already seen how a simple projective gives a heredity ideal. The next lemma shows that in the serial case a simple of projective dimension 2 also plays a crucial rôle.

Lemma 2.5. *If R is an indecomposable serial ring which has a simple of projective dimension 2 then it is quasihereditary.*

Proof. It suffices by (2.3) to find a heredity ideal. Suppose $\text{p. dim } S_i = 2$. Let Re_1, \dots, Re_n be Kupisch series chosen for convenience so that $i = 1$. Set $k = c_i$, the composition length of Re_i . We have an exact sequence $0 \rightarrow N^{k-1}e_{i-1} \rightarrow Re_{i-1} \rightarrow Re_i \rightarrow S_i \rightarrow 0$, and, by hypothesis, $N^{k-1}e_{i-1}$ is projective. By [F1, Lemma 2.1], $N^{k-1}e_{i-1} \cong Re_{[i-k]}$. It follows that $c_{i-1} = c_{[i-k]} + k - 1$. but there are always the inequalities $c_{i-1} \leq c_{[i-2]} + 1 \leq c_{[i-3]} + 2 \leq \dots \leq c_{[i-k]} + k - 1 (= c_{i-1})$. Hence, $c_{i-1} = c_{[i-2]} + 1 = \dots = c_{[i-k]} + k - 1$, and, in particular, $c_{i-1} > c_{[i-2]} > \dots > c_{[i-k]} > \dots > c_{[i-k]}$. Now since $c_i = k$, (2.4) gives the result. \diamond

In order to use this in the case of global dimension 3 we also need the following:

Lemma 2.6. *If R is any left artinian ring of left global dimension ≤ 3 then it has a simple projective dimension 0 or 2.*

Proof. We recall the following ([M, p. 204, Ex. 3]). If $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is exact then

- (i) $\text{p. dim } A > \text{p. dim } B \Rightarrow \text{p. dim } C = \text{p. dim } A + 1$, and
- (ii) $\text{p. dim } A < \text{p. dim } B \Rightarrow \text{p. dim } C = \text{p. dim } B$.

Now suppose that $\text{p. dim } S_i = 3$ and let $0 = K_0 \subset K_1 \subset \dots \subset K_s = Ne_i \subset Re_i$ be the composition series for Re_i and set $T_j = K_j/K_{j-1}$ and $L_j = Re_i/K_j$. Suppose further that no simple is of projective dimension 0 or 2. From the exact sequence $0 \rightarrow T_1 \rightarrow Re_i \rightarrow L_1 \rightarrow 0$ and i) we get, using the fact that $\text{gl. dim } R = 3$ and the hypothesis, $\text{p. dim } T_1 = 1$ and $\text{p. dim } L_1 = 2$. Now from the exact sequence $0 \rightarrow T_2 \rightarrow L_1 \rightarrow L_2 \rightarrow 0$ we must have $\text{p. dim } T_2 = 1$ and by ii) $\text{p. dim } L_2 = 2$. Continuing in this fashion we arrive at the exact sequence $0 \rightarrow T_s \rightarrow L_{s-1} \rightarrow L_s \rightarrow 0 (L_s = s_i)$ and, as above, $\text{p. dim } L_s = 2$. But this is a contradiction. \diamond

Proposition 2.7. *Let R be a serial ring of global dimension ≤ 3 . Then R is quasihereditary.*

Proof. Lemmas 2.2, 2.5 and 2.6 give the result. \diamond

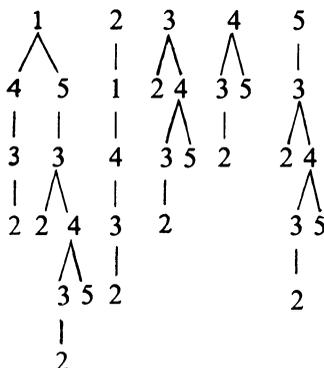
It is quite easy to reduce the case of a left serial ring to that of a serial ring. As pointed out in [BFVZ, p. 160], the (left) quiver of an indecomposable serial ring is either a rooted tree or like a rooted tree but with the root replaced by an oriented cycle. The tips of the branches represent simple projectives and, hence, give heredity ideals. More precisely, if there are no arrows into i in the quiver, say with $i \rightarrow j \dots$, then $Re_i R$ is heredity and dividing by it has the effect of pruning the twig $i \rightarrow$ from the quiver. This procedure may be continued until the quiver is that of a serial ring and (2.3) or (2.4) may be used.

3. GLOBAL DIMENSION 3 DOES NOT IMPLY QUASIHEREDITARY

The sole purpose of this section is to present an example of a 0-relations algebra R of global dimension 3 which is not quasihereditary*. The language of [F2] is used and the algebra is presented as a diagram algebra as in the example of Section 1. Here $n = 5$, the Loewy length is 6 and the algebra is of dimension 33 over any field K . In this case there is no ambiguity about the multiplication table for the multiplicative basis so the symbols for the basis

*The authors have been informed that K. Yamagata has also found an example of an algebra of global dimension 3 which is not quasihereditary.

elements have been omitted.



Projective resolutions for the simples can be read off from the diagrams (see [F2, Theorem 2.5]).

$$\begin{aligned}
 0 \rightarrow Re_3 \rightarrow Re_5 \rightarrow Re_4 \oplus Re_5 \rightarrow Re_1 \rightarrow S_1 \rightarrow 0 \\
 0 \rightarrow Re_5 \rightarrow Re_1 \rightarrow Re_2 \rightarrow S_2 \rightarrow 0 \\
 0 \rightarrow Re_5 \rightarrow Re_1 \rightarrow Re_2 \oplus Re_4 \rightarrow Re_3 \rightarrow S_3 \rightarrow 0 \\
 0 \rightarrow Re_3 \oplus Re_4 \rightarrow Re_3 \oplus Re_5 \rightarrow Re_4 \rightarrow S_4 \rightarrow 0 \\
 0 \rightarrow Re_3 \rightarrow Re_5 \rightarrow S_5 \rightarrow 0.
 \end{aligned}$$

Moreover, R is a 0-relations algebra by [F2, Theorems 3.3 and 4.2] and it is apparent that each $Re_i R$ contains a proper epimorph of Re_i as a direct summand, so R is not quasihereditary.

Added in proof: the authors have recently found an example of the type in §3 with $n = 2$ and $N^4 = 0$.

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