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PROBABILITY MEASURE FUNCTORS PRESERVING THE ANR-PROPERTY OF METRIC SPACES

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ABSTRACT. Let $P_k(X)$ denote the set of all probability measures on a metric space X whose supports consist of no more than k points, equipped with the Fedorchuk topology. We prove that if $X \in \mathrm{ANR}$ then $P_k(X) \in \mathrm{ANR}$ for every $k \in \mathbf{N}$. This implies that for each $k \in \mathbf{N}$ the functor P_k preserves the topology of separable Hilbert space.

1. INTRODUCTION

For a metric space X let $F(X)$ denote the linear space of all functions of finite support on X equipped with the $C_b(X)$-topology, that is, the weak topology induced by the set $C_b(X)$ of all bounded continuous functions on X. Observe that the space $F(X)$ with the $C_b(X)$-topology is a locally convex space. Every function $\mu \in F(X)$ can be written in the form $\mu = \sum_{i=1}^{k} m_i \delta x_i$, where $m_i \in \mathbf{R}$ for $i = 1, \ldots, k$ and δ_x denotes the Dirac function with support at x, that is

$$\delta_x(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases}$$

We denote

$$\|\mu\| = \sum_{i=1}^{k} |m_i| \quad \text{and} \quad \operatorname{supp} \mu = \{x_1, \ldots, x_k\}.$$

Following Fedorchuk [Fe] let us say that a function $\mu \in F(X)$ is a probability measure iff $\mu(x) \geq 0$ for each $x \in X$ and $\|\mu\| = 1$. $\mu(x)$ is called the mass of μ at x.

For each $k \in \mathbf{N}$ let $P_k(X)$ denote the set of all probability measures on X whose supports consist of no more than k points and let $P_\infty(X) = \bigcup_{k=1}^{\infty} P_k(X)$.

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We shall see in §2 that the $C_b(X)$-topology and the topology introduced by Fedorchuk [Fe] are equivalent on $P_\infty(X)$.

Let us observe that $P_\infty(X)$ is a convex set in the locally convex space $F(X)$. Therefore $P_\infty(X)$ is an absolute extensor for any metric space X, hence $P_\infty(X) \in \mathrm{AR}$ iff it is metrizable.

In [Fe] Fedorchuk proved that if X is a compact ANR-space then $P_k(X) \in$ ANR for every $k \in N$. The aim of this note is to prove Fedorchuks' theorem without the compactness assumption on X. The compactness of X is essential in the proof of Fedorchuk [Fe]. In our case we use a different approach which involves a characterization of ANR-spaces established by the first author in [$N1$]. Our proof uses an idea of [$N1$], see also [$N1$, $N2$], however in our case masses of probability measures lead to more complicated situations.

The proof of the main result of this note is given in §3. In §2 we describe the Fedorchuk topology for the space $P_\infty(X)$ and show that $P_k(X)$ is metrizable for any $k \in \mathbf{N}$. This is the first step toward our result.

2. Fedorchuk's topology on $P_\infty(X)$ and the metrizability of $P_k(X)$

In this section we describe Fedorchuk's topology on $P_k(X)$ and prove that $P_k(X)$ is metrizable for any $k \in \mathbf{N}$. For each $\mu_0 = \sum_{i=1}^k m_i^0 \delta x_i^0 \in P_\infty(X)$ we define a neighborhood basis of μ_0 of the form $\mathscr{O}\langle \mu_0, U_1, \dots, U_k, \varepsilon\rangle$ where $\varepsilon > 0$ and $U_1, \dots, U_k$ are disjoint neighborhoods of $x_1^0, \dots, x_k^0$, respectively.

$$\begin{aligned}\mathscr{O}\langle \mu_0, U_1, \dots, U_k, \varepsilon\rangle &= \Big\{\mu \in P_\infty(X) : \mu = \sum_{i=1}^{k+1} \mu_i,\ \operatorname{supp}\mu_i \subset U_i \text{ and} \\ &\qquad |m_i^0 - \|\mu_i\|| < \varepsilon, i = 1, \dots, k \text{ and } \|\mu_{k+1}\| < \varepsilon\Big\}.\end{aligned}$$

Observe that U_i, $i = 1, \dots, k$ can be taken from a fixed basis for X. The following fact shows that $P_\infty(X)$ is an absolute extensor for any metric space X.

(2.1) **Proposition.** *The Fedorchuk topology and the $C_b(X)$-topology are equivalent on $P_\infty(X)$.*

Proof. For $\mu = \sum_{i=1}^k m_i \delta x_i \in P_\infty(X)$ and $f \in C_b(X)$ we denote $\sum_{i=1}^k m_i f(x_i)$ by $\int f\,d\mu$.

At first we assume that $\mu_n \to \mu = \sum_{i=1}^k m_i \delta x_i$ in the Fedorchuk topology. Let $\varepsilon > 0$ and $f \in C_b(X)$ be a bounded continuous function. For each $i = 1, \dots, k$ we take a neighborhood U_i of x_i such that $|f(x) - f(x_i)| < (1/3k)\varepsilon$ for every $x \in U_i$, $i = 1, \dots, k$. Since $\mu_n \to \mu$ in Fedorchuk's topology there is an $n_0 \in \mathbf{N}$ such that for every $n \geq n_0$ we have $|m_i - \|\mu_n^i\|| < \varepsilon/3Mk$ for $i = 1, \dots, k$ and $\|\mu_n^{k+1}\| < \frac{1}{3}\varepsilon$, where $M = \sup\{|f(x)| : x \in X\}$, $U_{k+1} = X \setminus \bigcup_{i=1}^k U_i$ and $\mu_n^i = \mu_n | U_i$, $i = 1, \dots, k+1$. Then for every $n \geq n_0$ we obtain $|\int f\,d\mu_n - \int f\,d\mu| \leq \varepsilon$. Therefore $\mu_n \to \mu$ in the $C_b(X)$-topology.

Conversely, assume that $\mu_n \to \mu = \sum_{i=1}^k m_i \delta x_i$ in the $C_b(X)$-topology. Let $\mathscr{O}\langle \mu, U_1, \dots, U_k, \varepsilon\rangle$ be a neighborhood of μ in the Fedorchuk topology. Take a Urysohn function $f_0 \colon X \to [0,1]$ such that $f_0|X \setminus \bigcup_{i=1}^k U_i = 1$ and $f_0(x_i) = 0$ for $i = 1, \dots, k$. Since $\int f_0\, d\mu_n \to \int f_0\, d\mu$ we infer that there is an $n_0 \in N$ such that setting $\mu_n^i = \mu_n | U_i$, $i = 1, \dots, k$ and $\mu_n^{k+1} = \mu_n | X \setminus \bigcup_{i=1}^k U_i$ we obtain

$$(1) \qquad \|\mu_n^{k+1}\| < \frac{1}{2}\varepsilon \text{ for every } n \geq n_0.$$

For each $i = 1, \dots, k$ let $f_i \colon X \to [0,1]$ be a Urysohn function such that $f_i \mid U_i = 1$ and $f_i \mid U_j = 0$ for $j \neq i$. Since $\int f_i d\mu_n \to \int f_i\, d\mu$ we infer that there is an $n_i \in \mathbf{N}$ such that for every $n \geq n_i$ we have

$$\left| \|\mu_n^i\| + \int_{X \setminus \cup_{i=1}^k U_i} f_i\, d\mu - m_i \right| < \frac{1}{2}\varepsilon \qquad \text{for } i = 1, \dots, k.$$

Since

$$\int_{X \setminus \cup_{i=1}^k U_i} f_i\, d\mu \leq \|\mu_n^{k+1}\| \qquad \text{for } i = 1, \dots, k$$

from (1) we get $|\|\mu_n^i\| - m_i| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$ for $i = 1, \dots, k$. Consequently, taking $\overline{n} = \max\{m_i,\ i = 0, \dots, k\}$ we obtain $\mu_n \in \mathscr{O}\langle \mu, U_1, \dots, U_k, \varepsilon\rangle$ for every $n \geq \overline{n}$.

The proposition is proved.

Now we shall prove the main result of this section.

(2.2) **Theorem.** *$P_k(X)$ is metrizable for any $k \in \mathbf{N}$.*

The proof of Theorem 2.2 is based on the following fact due to Frink [Fr], see also [M].

(2.3) **Theorem** [Fr]. *A T_1-space X is metrizable if and only if the following condition holds:*

(Fr) For each $x \in X$ there exists a neighborhood basis $\{U_n(x)\}_{n=1}^\infty$ such that if $U_n(x)$ is given there exists an $m = m(x, n)$ such that $U_m(y) \cap U_m(x) \neq \varnothing$ implies $U_m(y) \subset U_n(X)$.

Proof of Theorem 2.2. Obviously $P_k(X)$ is a T_1-space. Thus by Theorem 2.3 it suffices to verify the condition (Fr).

For each $\mu = \sum_{i=1}^q m_i \delta x_i \in P_k(X)$, $q \leq k$ we define a neighborhood basis $\{\mathscr{O}_n(\mu)\}_{n=1}^\infty$ satisfying the condition (Fr). For each $i = 1, \dots, q$ we take $\{U^n(x_i)\}_{n=1}^\infty$ such that

$$(2) \qquad \operatorname{diam} U^n(x_i) < \frac{1}{4}\min\{2^{-n},\ \operatorname{dist}(U^n(x_i),\ U^n(x_j))\ i \neq j\}$$

$$(3) \qquad \{U^n(x_i)\} \text{ satisfies the condition (Fr).}$$

We put

$$\mathscr{O}_n(\mu) = \mathscr{O}\langle \mu, U_1^n, \dots, U_k^n, \varepsilon_n(\mu)\rangle$$

where $U_i^n = U^n(x_i)$, $i = 1, \ldots, q$ and $\varepsilon_n(\mu) < \min\{2^{-n}, m_i, i = 1, \ldots, q\}$. Let us show that $\{\mathscr{O}_n(\mu)\}_{n=1}^{\infty}$ satisfies (Fr).

Given $\mathscr{O}_n(\mu)$. Since $\varepsilon_n(\gamma) < 2^{-n}$ for every $\gamma \in P_k(X)$ there exists an $m \in \mathbf{N}$ such that

$$(4) \qquad \varepsilon_m(\gamma) < \frac{1}{4k} \min\{\varepsilon_n(\mu),\ m_i\ i = 1, \ldots, q\} \text{ for every } \gamma \in P_k(X).$$

We shall prove that $m(\mu, n) = \max\{m, m(x_i, n),\ i = 1, \ldots, q\}$ satisfies the desired property of (Fr).

Assume that $\mathscr{O}_n(\gamma) = \mathscr{O}_m\langle\gamma, V_1^m, \ldots, V_r^m, \varepsilon_m(\gamma)\rangle$ with $\mathscr{O}_m(\gamma) \cap \mathscr{O}_m(\mu) \neq \varnothing$. Take $\theta \in \mathscr{O}_m(\gamma) \cap \mathscr{O}_m(\mu)$ and write $\theta_i = \theta | U_i^m$, $i = 1, \ldots, q$ and $\theta_{q+1} = \theta|_{X \setminus \cup_{i=1}^q U_i^m}$, $A_i = \operatorname{supp} \theta_i$, $i = 1, \ldots, q+1$. Since $\|\theta_i\| \geq m_1 - \varepsilon_m(\mu) > m_i - \frac{1}{4}m_i = \frac{3}{4}m_i > \varepsilon_m(\gamma)$, $i = 1, \ldots, q$ we infer that for every $i \leq q$ there exists at least $j \in \{1, \ldots, r\}$ such that $A_i \cap V_j \neq \varnothing$. Let

$$G_i = \cup\{V_j \colon V_j \cap A_i \neq \varnothing\},\ i = 1, \ldots, q;\ G_{q+1} = \cup\left\{V_j \colon V_j \subset X \setminus \bigcup_{i=1}^q A_i\right\}.$$

Since $A_i \subset U_i^m$ from (3) it follows that

$$(5) \qquad G_i \subset U_i^n \text{ for every } i = 1, \ldots, q.$$

We shall show that $\mathscr{O}_m(\gamma) \subset \mathscr{O}_n(\mu)$. For every $\omega \in \mathscr{O}_m(\gamma)$ we denote $\omega_i = \omega \mid G_i$ for $i = 1, \ldots, q+1$; $\omega_{i,j} = \omega_i \mid V_j$ for $V_j \subset G_i$; $\theta_{i,j} = \theta_i \mid V_j$ for $V_j \subset G_i$. Since ω, $\theta \in \mathscr{O}_m(\gamma)$ it follows that $|\|\omega_{i,j}\| - \|\theta_{i,j}\|| < 2\varepsilon_m(\gamma)$. Note that $\operatorname{Card}\{j \colon V_j \subset G_i\} \leq r \leq k$. From (4) we obtain

$$(6) \qquad |\|\omega_i\| - \|\theta_i\|| \leq \sum_{V_j \subset G_i} |\|\omega_{i,j}\| - \|\theta_{i,j}\|| < 2k\varepsilon_m(\gamma) < \frac{1}{2}\varepsilon_n(\mu)$$

for every $i = 1, \ldots, q+1$. Hence

$$|\|\omega_i\| - m_i| \leq |\|\omega_i\| - \|\theta_i\|| + \|\theta_i - m_i\| < \frac{1}{2}\varepsilon_n(\mu) + \varepsilon_m(\mu) < \varepsilon_n(\mu)$$

for every $i = 1, \ldots, q$ and by (6) we have

$$\|\omega_{q+1}\| \leq \theta_{q+1} + \frac{1}{2}\varepsilon_n(\mu) \leq \varepsilon_m(\mu) + \frac{1}{2}\varepsilon_n(\mu) < \varepsilon_n(\mu).$$

Consequently from (5) we infer that $\omega \in \mathscr{O}_n(\mu)$.

This completes the proof of Theorem 2.2.

3. The results

Our result in this note is the following

(3.1) **Theorem.** *If* $X \in \mathrm{ANR}$ *then* $P_k(X) \in \mathrm{ANR}$ *for each* $k \in \mathbf{N}$.

As a consequence of Theorem 2.1 we get

(3.2) **Corollary.** *$P_k(\ell_2) \cong \ell_2$ for each $k \in \mathbf{N}$.*

Here ℓ_2 denotes separable Hilbert space, and $X \cong Y$ means X is homeomorphic to Y.

Proof. Since $P_k(\ell_2) \in \mathrm{AR}$ the assertion follows from a result of [DT].

Let $\{\mathscr{U}_n\}$ be a sequence of open covers of a metric space X and let $\mathscr{U} = \bigcup_{n \in \mathbf{N}} \mathscr{U}_n$. By $\mathscr{N}(\mathscr{U})$ we denote the nerve of $\mathscr{U}$. We write $K \prec \{\mathscr{U}_n\}$ iff K is a subcomplex of $\mathscr{N}(\mathscr{U})$ and for each simplex $\sigma \in K$ we have $\sigma \subset \mathscr{U}_n \cup \mathscr{U}_{n+1}$ for some $n \in \mathbf{N}$. We write

$$N(\sigma) = \max\{n \in N : \sigma \subset \mathscr{U}_n \cup \mathscr{U}_{n+1}\}.$$

The proof of Theorem 3.1 uses the following fact which is a slight modification of a charcterization of ANR-spaces given in [N1].

(3.3) **Theorem** [N1]. *A metric space $X \in \mathrm{ANR}$ if and only if there exists a sequence of open covers $\{\mathscr{U}_n\}$ of X such that for any $K \prec \{\mathscr{U}_n\}$ and for any selection $f: K^0 \to X$ (i.e., $f(U) \in U$) there is a map $g: K \to X$ such that if $\{\sigma_n\}$ is a sequence of simplices of K for which $f(\sigma_n^0) \to x_0 \in X$ as $N(\sigma_n) \to \infty$ then we have $g(\sigma_n) \to x_0$.*

Here we say that a sequence $\{A_n\}$ of subsets of a metric space X tends to a point $x_0 \in X$ iff $\operatorname{diam}(A_n \cup \{x_0\}) \to 0$.

Note that the proof of the characterization theorem of ANR-spaces given in [N1] proves also Theorem 3.3.

Proof of Theorem 3.1. The remaining part of this section is devoted to the proof of our main result. We shall verify the conditions of Theorem 3.3 for $P_k(X)$. Assume that X is an ANR. Since every metric space can be embedded isometrically as a closed subset of a normed space, see [BP], by the ANR-property of X without loss of generality we may assume that X is an open subset of a normed space.

For each $n \in \mathbf{N}$ we take an open cover $\mathscr{W}_n$ of X such that $\mathscr{W}_{n+1} \prec \mathscr{W}_n$ and $\operatorname{diam} W < 2^{-n}$ for each $W \in \mathscr{W}_n$. Put $\mathscr{W} = \bigcup_{n=1}^{\infty} \mathscr{W}_n$. We shall assume that the Fedorchuk topology of $P_k(X)$ is induced by $\mathscr{W}$.

For each $n \in \mathbf{N}$ we take a cover $\mathscr{V}_n$ of X consisting of open convex sets such that

(7) $$\operatorname{conv} V \prec \mathscr{W}_n \quad \text{for each } V \in \operatorname{St} \mathscr{V}_n;$$

(8) $$\mathscr{V}_{n+1} \prec \mathscr{V}_n \quad \text{for each } n \in \mathbf{N}.$$

Set

(9) $\widetilde{\mathscr{U}}_n = \{\mathscr{O}\langle\mu, U_1, \dots, U_q, 2^{-n}\rangle \mid U_i \in \mathscr{V}_n, \operatorname{dist}(U_i, U_j) \geq 3.2^{-n}$

for $U_i \neq U_j$ and $\min\{m_i, i = 1, \dots, q\} \geq (k+1)^k 2^{-n}\}$,

$$\mathscr{U}_n = \bigcup_{i \geq n} \widetilde{\mathscr{U}}_i \text{ and } \mathscr{U} = \bigcup_{n=1}^{\infty} \mathscr{U}_n.$$

Observe that $\mathscr{U}_n$ is an open cover of $P_k(X)$ for every $n \in \mathbf{N}$.

Note that for each simplex $\sigma \in \mathscr{N}(\mathscr{U})$ we have $\sigma = \langle V_1, \dots, V_p\rangle$, where $V_i = \mathscr{O}\langle\mu_i, U_1^i, \dots, U_{q(i)}^i, \varepsilon\rangle \in \mathscr{U}$, $\mu_i = \sum_{j=1}^{q(i)} m_j^i \delta x_j^i$, $x_j^i \in U_j^i$, $j = 1, \dots, q(i)$; $i = 1, \dots, p$ and $\bigcap_{i=1}^p V_i \neq \varnothing$. Obviously we may assume that $V_i \in \widetilde{\mathscr{U}}_{n(i)}$ with $n(1) \leq n(2) \leq \cdots \leq n(p)$. Let us put $F_i = \{U_1^i, \dots, U_{q(i)}^i\}$ for $i = 1, \dots, p$. Now define

(10)

$$A(\sigma) = \left\{ L = \{U^i\} \mid U^i \in F_i, \bigcap_{U^i \in L} U^i \neq \varnothing, \bigcap_{U^i \in L} U^i \cap U = \varnothing \text{ for } U \notin L \right\}.$$

(3.4) **Lemma.** $\operatorname{Card} A(\sigma) \leq k$.

Proof. In fact we shall prove that $\operatorname{Card} A(\sigma) = q(p) \leq k$. From (9) it follows that every $U \in F_p$ belongs to at most one member $L \in A(\sigma)$. Therefore it suffices to establish that for every $L \in A(\sigma)$ there is a $U \in F_p$ such that $U \in L$. This follows from the following fact which is a crucial step in the proof of our result.

(3.5) *Fact*. If $m_{j(\bar{i})}^{\bar{i}} \geq (k+1)^k \varepsilon$, where $\varepsilon = \max\{\varepsilon_i \mid i = 1, \dots, p\}$, then for each $i \neq \bar{i}$ there exists $U_{j(i)}^i$, $j(i) \leq q(i)$, such that $\bigcap_{i=1}^p U_{j(i)}^i \neq \varnothing$.

Proof. Let $\mu \in \bigcap_{i=1}^p V_i$ with $\mu = \sum_{i=1}^r m_i \delta_{x_i}$, $r \leq k$. For simplicity we assume that $\bar{i} = 1$. Since $\mu \in V_1$ it can be written in the form $\mu = \sum_{i=1}^{q(1)+1} \mu_i^1$, where $\mu_i^1 = \mu \mid U_i^1$, $i = 1, \dots, q(1)$, and $\mu_{q(1)+1}^1 = \mu \mid X \setminus \bigcup_{i=1}^{q(1)} U_i^1$. Observe that

$$\|\mu_{q(1)+1}^1\| < 2^{-n(1)} \leq \varepsilon \quad \text{and} \quad \|\mu\| = \sum_{i=1}^{q(1)+1} \|\mu_i^1\| = \sum_{i=1}^r m_i = 1.$$

Write

$$\mu_1 = \sum_{i=1}^{q(1)} m_i^1 \delta_{x_i^1}, \qquad x_i^1 \in U_i^1, \qquad i = 1, \dots, q(1).$$

Since $\mu \in V_1$ we infer that $|\|\mu_{j(1)}^1\| - m_{j(1)}^1| < 2^{-n(1)} \leq \varepsilon$. Therefore by (9)

(11) $$\|\mu_{j(1)}^1\| \geq m_{j(1)}^1 - 2^{-n(1)} \geq (k+1)^k \varepsilon - \varepsilon = ((k+1)^k - 1)\varepsilon.$$

We write

$$\mu_{j(1)}^1 = \sum_{i=1}^s m_i^1 \delta_{x_i^1}, \qquad x_i^1 \in U_{j(1)}^1 \text{ for } i = 1, \dots, s.$$

Then we have $\|\mu^1_{j(1)}\| = \sum_{i=1}^{s} m^1_i$. Put $A_1 = \operatorname{supp}\mu^1_{j(1)}$.

(3.6) *Claim.* Let $V_2 = \mathscr{O}\langle\mu_2, U^2_1, \dots, U^2_{q(2)}, \varepsilon_2\rangle$. Then there exists $j(2) \le q(2)$ such that $A_1 \cap U^2_{j(2)} \ne \varnothing$.

Proof. Write $\mu = \sum_{i=1}^{q(2)+1}$, where $\mu^2_i = \mu \mid U^2_i$, $i = 1, \dots, q(2)$, and $\mu^2_{q(2)+1} = \mu \mid_{X\setminus\bigcup_{i=1}^{q(2)}} U^2_i$. Assume to the contrary that $A_1 \cap U^2_i = \varnothing$ for every $i = 1, \dots, q(2)$. Then we have $A_1 \subset \operatorname{supp}\mu^2_{q(2)+1}$.

Since $\mu \in V_2$, it follows that $\|\mu^1_{j(1)}\| = \|\mu \mid A_1\| \le \|\mu|\operatorname{supp}\mu^2_{q(2)+1}\| < \varepsilon_2 \le \varepsilon$ which contradicts (11), so the claim is proved.

Denote $I(2) = \{i \in \{1, \dots, q(2)\} \mid U^2_i \cap A_1 \ne \varnothing\}$, $B(2) = \bigcup_{i\in I(2)} U^2_i$, and $\widetilde{A}_2 = A_1 \cap B(2)$. Since $A_1\setminus\widetilde{A}_2 \subset X\setminus\bigcup_{i=1}^{q(2)} U^2_i$ it follows that $\|\mu \mid A_1\setminus\widetilde{A}_2\| \le \varepsilon_2 \le \varepsilon$. Therefore from (11) we get

$$\|\mu \mid \widetilde{A}_2\| \ge \|\mu \mid A_1\| - \|\mu \mid A_1\setminus\widetilde{A}_2\| \ge ((k+1)^k - 1)\varepsilon - \varepsilon \ge k(k+1)^{k-1}\varepsilon.$$

Since $\operatorname{Card} I(2) \le k$ there exists a $j(2) \in I(2)$ such that $\|\mu \mid \widetilde{A} \cap U^2_{j(2)}\| \ge (k+1)^{k-1}\varepsilon$. Let us put $A_2 = \widetilde{A}_2 \cap U^2_{j(2)}$. Continuing this process we get a finite sequence $A_1 \supset A_2 \supset \cdots \supset A_p$. Since $\operatorname{Card} A_1 \le k$ we infer that for any $p \in N$, the family $\{A_1, \dots, A_p\}$ consists of at most k different sets. Therefore we have $\|\mu \mid A_p\| \ge (k+1)^{k-(k-1)}\varepsilon = (k+1)\varepsilon > 0$. In particular we get $\bigcap_{i=1}^{p} U^i_{j(1)} \supset A_p \ne \varnothing$.

This proves 3.5.

Now we are already in a position to complete the proof of Theroem 3.1.

Assume that $K \prec \{\mathscr{U}_n\}$. Let $f\colon K^0 \to P_k(X)$ be a selection. For each $V = \mathscr{O}\langle\mu, U_1, \dots, U_q, \varepsilon\rangle \in K^0$ we put $g_0(V) = \mu$. Assume that $\sigma = \langle V_1, \dots, V_p\rangle \in K$ with

$$V_i = \mathscr{O}\langle u_i, U^i_1, \dots, U^i_{q(1)}, \varepsilon_i\rangle \in \widetilde{\mathscr{U}}_{n(i)}, \qquad \text{(see (9))},$$

and $n(1) \le n(2) \le \cdots \le n(p)$. Then $\bigcap_{i=1}^{p} V_i \ne \varnothing$. Observe that μ_i can be written in the form

$$\mu_i = \sum_{U\in F_i} m(U)\delta_{x(U)}$$

where $F_i = \{U^i_1, \dots, U^i_{q(i)}\}_{i=1}^{p}$. As we have seen, for every $U^p \in F_p$ there exists a unique $L = L(U^p) \in A(\sigma)$ such that $U^p \in L$. For every $U \in F_i$ we put $J(U) = \{j \mid (U^p_j, U) \subset L(U^p_j)\}$. We shall define for each $i = 1, \dots, p$ a sequence $\{\hat{m}^i_j, \hat{x}^i_j\}_{j=1}^{q(p)}$ as follows: put

$$\hat{x}^i_j = \begin{cases} x^p_j, & \text{if } L(U^p_j) \cap F_i = \varnothing, \\ x(U), & \text{if } U \in L(U^p_j) \cap F_i. \end{cases}$$

Note that $\hat{x}^i_j$ is well defined, since $L(U^p) \cap F_i$ consists of at most one element for every $U^p \in F_p$ and for every $i = 1, \dots, p$.

Now we put $\hat{m}_j^i = 0$ if $L(U_j^p) \cap F_i = \emptyset$, and if $L(U_j^p) \cap F_i \neq \emptyset$ we define $\hat{m}_j^i$ by the following equalities:

$$\sum_{j \in J(U)} m_j^i = \hat{m}(U), \qquad \frac{\hat{m}_j^i}{m_j^p} = \frac{\hat{m}_{j'}^i}{m_{j'}^p} \text{ for all } j, \ j' \in J(U).$$

Then μ_i can be written in the form

$$\mu_i = \sum_{j=1}^{q(p)} \hat{m}_j^i \delta_{\hat{x}_j^i} \qquad \text{for } i = 1, \dots, p.$$

We now define $g_\sigma : \sigma \to P_k(X)$ as follows: For each $x \in \sigma$ we have $x = \sum_{i=1}^p \lambda_i V_i$, $\lambda_i \geq 0$ and $\sum_{i=1}^p \lambda_i = 1$. We put

$$g_\sigma(x) = \sum_{j=1}^{q(P)} m_j \delta_{x_j}$$

where

$$x_j = \sum_{i=1}^p \lambda_i \hat{x}_j^i \quad \text{and} \quad m_j = \sum_{i=1}^p \lambda_i \hat{m}_j^i$$

for every $j = 1, \dots, q(P)$. Observe that by (7), x_j is well defined. Now

$$\sum_{j=1}^{q(p)} m_j = \sum_{j=1}^{q(p)} \sum_{i=1}^p \lambda_i \hat{m}_j^i = \sum_{i=1}^p \lambda_i \sum_{j=1}^{q(p)} \hat{m}_j^i$$
$$= \sum_{i=1}^p \lambda_i \sum_{U \in F_i} m(U) = \sum_{i=1}^p \lambda_i = 1.$$

Therefore $g_\sigma(x) \in P_k(X)$.

It is easy to see that for every σ, $\sigma' \in K$ we have $g_\sigma \mid \sigma \cap \sigma' = g_{\sigma'} \mid \sigma \cap \sigma'$ and $g_\sigma \mid \sigma^0 = g_0$. Therefore the family $\{g_\sigma\}_{\sigma \in K}$ induces a map $g : K \to P_k(X)$ such that $g \mid K^0 = g_0$.

We show that $\{\sigma_n\}$ satisfies Theorem 3.3. Assume that $\{\sigma_n\}$ is a sequence of simplices of K such that $f(\sigma_n^0) \to \mu_0 \in P_k(X)$ as $N(\sigma_n) \to \infty$.

Let $V = \mathscr{O}\langle \mu_0, W_1, \dots, W_q, \varepsilon \rangle$ be a neighborhood of $\mu_0 = \sum_{i=1}^q m_i^0 \delta_{x_i^0}$, where $W_i \in \mathscr{W}$ for $i = 1, \dots, q$ are disjoint neighborhoods of x_i^0, $i = 1, \dots, q$, respectively. Since $f(\sigma_n^0) \to \mu_0$ and $N(\sigma_n) \to \infty$, we infer that $g(\sigma_n^0) \to \mu_0$.

Note that by (7), (8), and the definition of g, there is an $n_0 \in \mathbf{N}$ such that if $N(\sigma_n) \geq n_0$ and $x \in \sigma_n$, then we have

$$g(x) = \sum_{i=1}^{q+1} \mu_i(x),$$

where $\mu_i(x) = g(x) \mid W_i$, $i = 1, \dots, q$, $\mu_{q+1} = g(x) \mid X \setminus \bigcup_{i=1}^q W_i$, $|m_i^0 - \|\mu_i(x)\|| < \varepsilon$ for every $i = 1, \dots, q$, and $\|\mu(x)_{q+1}\| < \varepsilon$.

Therefore $g(\sigma_n) \to \mu_0$ as $N(\sigma_n) \to \infty$. Consequently by Theorem 3.3 we conclude that $P_k(X)$ is an ANR. This completes the proof of the theorem.

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