CONTINUITY PROPERTIES OF THE SPECTRUM OF OPERATORS ON LEBESGUE SPACES

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Abstract. Fix $1 < p < s < \infty$. Let $T_x, x \in [p, s]$, be the collection of bounded linear operators on the Lebesgue spaces $L^x$ determined by some fixed operator $T$. This paper concerns continuity properties of the map $x \rightarrow \sigma(T_x)$.

1. Introduction

Let $\Omega$ be a nonempty set, and let $\mu$ be a $\sigma$-finite measure on $\Omega$. If $1 \leq p \leq \infty$, then let $L^p$ denote the usual Lebesgue space $L^p(\Omega, \mu)$ with the usual $p$-norm. Now fix $p$ and $s$ with $1 \leq p < s \leq \infty$. Let $B_{p,s}$ be the algebra of all linear operators $T: L^p \cap L^s \rightarrow L^p \cap L^s$ with the property that $T$ is continuous on $L^p \cap L^s$ with respect to both the $p$-norm and the $s$-norm. The algebra $B_{p,s}$ is a Banach algebra which is studied in [2]. Also, some applications involving $B_{p,s}$ can be found in [3]. If $T \in B_{p,s}$, then by the Riesz Convexity Theorem [6, Theorem 11, p. 525] for each $x$ in the interval $[p, s]$ the linear operator $T$ has a unique extension to a bounded linear operator $T_x$ on $L^x$ (in the case where $s = \infty$, $T_\infty$ is a bounded linear operator on the closure of $L^p \cap L^\infty$ in $L^\infty$). Let $\sigma(T_x)$ and $r(T_x)$ denote the spectrum and spectral radius of this extension. It is well known that $\sigma(T_x)$ can be different for different $x$ in $[p, s]$. An example of this phenomenon is given in [4]: Let $\Omega = (0, \infty)$ and $\mu$ be the Lebesgue measure. Take $T$ to be the Cesàro operator, so

$$T_y(f)(t) = t^{-1} \int_0^t f(x)dx \quad (f \in L^y),$$

$1 < y \leq \infty$. Then $\sigma(T_y)$ is the circle with center and radius $(2(1 - y^{-1}))^{-1}$. Note that $\sigma(T_y)$ varies continuously in $y$ in this case.

In this paper we study continuity properties of the map $y \rightarrow \sigma(T_y)$. A key theorem here is that for $T \in B_{p,s}$, the function $y \rightarrow r(T_y)$ is continuous on the open interval $(p, s)$ (it need not be continuous on $[p, s]$). A number of other continuity results are derived using this basic theorem as a tool.

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There are only a few examples in the literature where $\sigma(T_x)$ is specifically computed and this set is different for different values of $y$. Specific examples where this occurs can be found in [4], [5], and [7]. Next we give a number of additional examples.

**Example 1.** Let $\Omega$ be the positive integers, and let $\mu$ be the measure with

$$\mu(k) = \frac{1}{k!} \quad (k \geq 1).$$

For $1 \leq p < \infty$, let $T_p$ be the linear operator on $l^p(\mu)$ defined by $T_p(\{a_k\}) = \{b_k\}$ where $b_k = (k + 1)^{-1} a_{k+1}$, $k \geq 1$. Also, let $T_\infty$ act on $c_0$ by the same rule.

**Note.** For $1 < p \leq \infty$, $T_p$ is compact and $\sigma(T_p) = \{0\}$.

**Proof.** Fix $p$, $1 < p < \infty$. Define $M_n$ on $l^p(\mu)$ by $M_n(\{a_k\}) = \{b_k\}$ where $b_k = a_k$ for $1 \leq k \leq n$, and $b_k = 0$ for $k > n$. For $\{a_k\} \in l^p(\mu)$,

$$\| (T_p - M_n T_p) \{a_k\} \|^p_p = \sum_{k=n+1}^{\infty} \frac{1}{k!} [(k + 1)^{-p} a_{k+1}]^p \leq (n + 2)^{1-p} \| \{a_k\} \|^p_p.
$$

Therefore

$$\| T_p - M_n T_p \|_p \leq (n + 2)^{p-1} \rightarrow 0$$
as $n \rightarrow \infty$. This proves $T_p$ is compact. A similar computation shows that $T_\infty$ is compact.

Now it is easy to see that the operator $M_n T_p$ is nilpotent, so $\sigma(M_n T_p) = \{0\}$ for $n \geq 1$. Since $T_p$ is compact, this implies $\sigma(T_p) = \{0\}$ [8, Theorem 3].

Now we consider the case where $p = 1$. Let $l^1$ denote the usual sequence space (the $L^1$-space of the positive integers with respect to counting measure). Define an isometry $W_1 : l^1 \rightarrow l^1(\mu)$ by $W_1(\{a_k\}) = \{c_k\}$ where $c_k = k! a_k$, $k \geq 1$. The isometry $W_1$ maps $l^1$ onto $l^1(\mu)$. Let $B$ be the unilateral backward shift on $l^1$; $B(\{a_k\}) = \{b_k\}$ where $b_k = a_{k+1}$, $k \geq 1$. A straightforward computation shows

$$W_1^{-1} T_1 W_1 = B \quad \text{on } l^1.$$

As is well known, $\sigma(B) = D$, the closed unit disk, so $\sigma(T_1) = D$.

To summarize: in this case $T \in \mathcal{B}_{1,\infty}$, $T_x$ is compact with $\sigma(T_x) = \{0\}$ for $1 < x \leq \infty$, while $T_1$ is not compact with $\sigma(T_1) = D$. The spectral radius function is given by

$$r(T_x) = 0 \quad \text{for } 1 < x \leq \infty, \quad r(T_1) = 1.$$

**Example II.** Let $\Omega$ be the positive integers, and let $\mu$ be the measure with

$$\mu(k) = 2^k \quad (k \geq 1).$$
Let $S$ and $B$ be defined on any sequence space by

$$S\{\{a_k\}\} = \{b_k\} \quad \text{where } b_1 = 0, b_k = a_{k-1} \text{ for } k > 1;$$

$$B\{\{a_k\}\} = \{c_k\} \quad \text{where } c_k = a_{k+1} \text{ for } k \geq 1.$$ 

Let $S_p$ and $B_p$ denote the operators $S$ and $B$ acting on the sequence space $l^p(\mu), 1 \leq p < \infty$. Let $l^p$ be the usual $L^p$-space on the positive integers with counting measure. For $1 \leq p < \infty$ define $W_p: l^p \to l^p(\mu)$ by $W_p(\{a_k\}) = \{d_k\}$ where $d_k = 2^{-k/p}a_k, k \geq 1$. Then $W_p$ is a linear isometry of $l^p$ onto $l^p(\mu)$. Each of the following properties is established by a simple computation.

(II.1) $S^* = S$ and $B^* = B$, so $S + B$ is selfadjoint on $l^2(\mu)$;

(II.2) $W_p^{-1}S_pW_p = 2^{1/p}S$ on $l^p$;

(II.3) $W_p^{-1}B_pW_p = 2^{-1/p}B$ on $l^p$.

Now the spectrum of an operator of the form $a + bS + cB$, $a, b, c \in \mathbb{C}$, on the space $l^p$ has been specifically computed by I. Gohberg and M. Zambitsky. This result can be found in [9, Proposition 2]. Combining this information with (II.1)–(II.3), we can compute the spectrum of any operator of the form $a + bS_p + cB_p, 1 \leq p < \infty$. We state these results for a specific operator:

Set $T_p = S_p + 2B_p \in \mathcal{B}(l^p(\mu)), 1 \leq p < \infty$. Then

(II.4) $\sigma(T_2) = \{\lambda \in \mathbb{R} : -2\sqrt{2} \leq \lambda \leq 2\sqrt{2}\}$;

(II.5) For $p \neq 2$, $p^{-1} + q^{-1} = 1(q^{-1} = 0$ when $p = 1), \sigma(T_p)$ is the set of all $\lambda = x + iy, x, y \in \mathbb{R}$, which lie inside or on the ellipse $x^2/a^2 + y^2/b^2 = 1$ where $a = (2^{1/p} + 2^{1/q})$ and $b = (2^{1/p} - 2^{1/q})$.

(II.6) $r(T_p) = 2^{1/p} + 2^{1-1/p}, 1 \leq p < \infty$.

The operator $T_2$ in this example is selfadjoint. In this case it is always true that when $2 \leq x \leq y$ or $y \leq x \leq 2$, then $\sigma(T_x) \subseteq \sigma(T_y)$; see [2, Theorem 4.4]. Also, the fact that $\sigma(T_p) \neq \sigma(T_2)$ for $p \neq 2$ shows that the hypothesis (4.1) in §4 of [1] cannot be omitted and still obtain the results in [1, Theorem 4.8].

2. Continuity properties

The main result of this paper concerns the continuity of the spectral radius function $x \to r(T_x)$; this result is stated in Theorem 3. On the way to proving this theorem, we derive some additional interesting continuity properties of the spectrum.

**Proposition 1.** Assume $T \in \mathcal{B}_{p,s}$ and $r(T_y) = 0$ for some $y \in [p,s]$. Then $r(T_x) = 0$ for all $x \in (p,s)$.

**Proof.** Fix $x \in (p,s)$. We may assume $x < y$. Choose $z$ such that $z \in (p,s)$ and $z < x$. Then $\exists t \in (0,1)$ with $x^{-1} = tz^{-1} + (1-t)y^{-1}$. Applying the Riesz Convexity Theorem, we have for all $n \geq 1$

$$\|T^n_x\| \leq \|T^n_z\|t^n\|T^n_y\|^{1-t}.$$
Therefore
\[ r(T_x) \leq r(T_z)^ir(T_y)^{1-i} = 0. \]

**Corollary 2.** Assume \( T \in B_{p,s} \) and for some \( y \in [p,s] \) \( \sigma(T_y) \) is finite. Then \( \sigma(T_x) = \sigma(T_y) \) for all \( x \in (p,s) \).

**Proof.** Assume \( \sigma(T_y) = \{\lambda_1, \ldots, \lambda_n\} \). Let \( q \) be the polynomial \( q(z) = \prod_{k=1}^{n}(z - \lambda_k) \). Then \( \sigma(q(T_y)) = q(\sigma(T_y)) = \{0\} \). It follows from Proposition 1 that \( \sigma(q(T_x)) = \{0\} \) for all \( x \in (p,s) \). Thus, \( \sigma(T_x) \subseteq \{\lambda_1, \ldots, \lambda_n\} \) for all such \( x \). Fix any \( x \in (p,s) \), \( x \neq y \). We may assume \( x < y \). Now \( T \in B_{x,y} \), and both \( \sigma(T_x) \) and \( \sigma(T_y) \) are finite. Then by [2, Cor. 5.2], \( \sigma(T_x) = \sigma(T_y) \).

Now we prove the main result.

**Theorem 3.** Assume \( T \in B_{p,s} \). Then \( r(T_x) \) is a continuous function on \((p,s)\).

**Proof.** If \( r(T_y) = 0 \) for some \( y \in (p,s) \), then \( r(T_x) = 0 \) for all \( x \in (p,s) \) by Proposition 1. Thus, we may assume \( r(T_x) > 0 \) for all \( x \in (p,s) \). For each \( n \geq 1 \), define
\[ \phi_n(w) = n^{-1} \log(||(T^n)w||). \]

By the Riesz Convexity Theorem [6, Theorem 11, p. 525] \( \phi_n \) is a convex function on \((s^{-1},p^{-1})\). Now \( \phi_n(w) \) converges pointwise to \( \phi(w) = \log(r(T_{w-1})) \).

By hypothesis \( \phi(w) \) has only finite values, and \( \phi \) must be convex on \((s^{-1},p^{-1})\).

It follows from [10, Theorem 3.2] that \( \phi(w) \) is continuous on \((s^{-1},p^{-1})\), and thus, \( r(T_x) \) is continuous on \((p,s)\).

Concerning this result, note that Example 1 shows that \( x \to r(T_x) \) need not be continuous on the closed interval \([p,s]\).

Next we derive some consequences of the continuity of the spectral radius function. The first is a type of upper semicontinuity property for the function \( x \to \sigma(T_x) \). For \( K \) a compact subset of \( C \), let \( \hat{K} \) denote the polynomial convex hull of \( K \). E. Stout’s book [11] is a good source for information on polynomial convexity.

**Theorem 4.** Assume \( T \in B = B_{p,s} \) and fix \( x \in (p,s) \). Let \( U \) be an open set in \( C \) with \( \sigma(T_x) \subseteq U \). Then \( \exists \delta > 0 \) such that \( \sigma(T_y) \subseteq U \) whenever \( |y-x| < \delta \).

**Proof.** Suppose no such \( \delta \) exists. Then we can choose a sequence \( \{y_n\} \subseteq (p,s) \) with \( y_n \to x \) such that for each \( n \) there exists \( \lambda_n \in \sigma(T_{y_n}) \) with \( \lambda_n \not\in U \). The sequence \( \{\lambda_n\} \) is contained in the compact set \( \sigma(T_x) \), and so some subsequence converges to a number \( \lambda \not\in U \). We may assume that \( \lambda_n \to \lambda \).

Set \( K = \sigma(T_x) \), and if \( q \) is a polynomial, then let \( ||q||_K = \sup\{|q(\mu)|: \mu \in K\} \). Now \( \lambda \not\in K \), so by definition there is a polynomial \( q \) such that \( |q(\lambda)| > ||q||_K = 1 \). Choose a positive integer \( n \) sufficiently large that \( |q^n(\lambda)| > 2 \). Set \( p = q^n \). Thus,
\[ |q(\lambda)| > 2 \quad \text{and} \quad ||p||_K = 1. \]
Since \( r(p(T_x)) = 1 \), by the continuity of the spectral radius (Theorem 3) \( r(p(T_{y_k})) \to 1 \). Also, \( p(\lambda_k) \in \sigma(p(T_{y_k})) \) and \( p(\lambda_k) \to p(\lambda) \). This provides a contradiction since for \( k \) sufficiently large
\[
r(p(T_{y_k})) < 2 \quad \text{and} \quad |p(\lambda_k)| > 2.
\]

Next we prove a stronger upper semicontinuity property of the map \( x \to \sigma(T_x) \) which holds under the assumption that the spectrum of \( T \) in the algebra \( \mathcal{B}_p, s \) is “thin”. First we need a preliminary result that uses the continuity of the spectral radius function.

**Lemma 5.** Assume \( T \in \mathcal{B} = \mathcal{B}_p, s \) and \( \lambda \notin \sigma_\mathcal{B}(T) \). Then \( d(x) = \text{dist}(\lambda; \sigma(T_x)) \) is a continuous function on \( (p, s) \).

**Proof.** For all \( x \in (p, s) \), \( ((\lambda - T)^{-1})_x = (\lambda - T_x)^{-1} \). Thus since
\[
\sigma((\lambda - T_x)^{-1}) = \{((\lambda - \mu)^{-1} : \mu \in \sigma(T_x)\},
\]
it follows that for \( x \in (p, s) \)
\[
r((\lambda - T_x)^{-1})_x = \sup\{|\lambda - \mu|^{-1} : \mu \in \sigma(T_x)\} = (d(x))^{-1}.
\]
By Theorem 3 this function is continuous on \( (p, s) \) which proves the lemma.

**Theorem 6.** Assume \( T \in \mathcal{B} = \mathcal{B}_p, s \) and that \( \sigma_\mathcal{B}(T) \) has empty interior. If \( x \in (p, s) \) and \( U \) is an open set with \( \sigma(T_x) \subseteq U \), then \( \exists \delta > 0 \) such that \( \sigma(T_y) \subseteq U \) whenever \( |x - y| < \delta \).

**Proof.** Suppose no such \( \delta \) exists. Then there is a sequence \( \{y_n\} \subseteq (p, s) \) such that \( y_n \to x \) and for each \( n \) there exists \( \lambda_n \in \sigma(T_{y_n}) \) with \( \lambda_n \notin U \). We may assume (by taking a subsequence if necessary) that \( \{\lambda_n\} \) converges to a number \( \mu \notin U \). Let \( \varepsilon = \text{dist}(\mu; \sigma(T_x)) > 0 \). Set \( D = \{\alpha \in \mathbb{C} : |\alpha - \mu| < \varepsilon/4\} \). Since \( \sigma(T_x) \) has no interior, we can choose \( \lambda \in D \) with \( \lambda \notin \sigma_\mathcal{B}(T) \). By Lemma 5 \( d(y) = \text{dist}(\lambda; \sigma(T_y)) \) is a continuous function on \( (p, s) \). By the choices of \( \varepsilon \) and \( \lambda \), \( d(x) > \varepsilon/2 \). But since \( \lambda_n \to \mu \), we have \( d(y_n) < \varepsilon/2 \) for all \( n \) sufficiently large, a contradiction.

Next we turn to some results concerning the situation where \( \sigma(T_x) \) is disconnected or has an isolated point at some \( x \in (p, s) \). What we prove is that this property extends to all \( y \) in some neighborhood of \( x \).

We need the following technical lemma.

**Lemma 7.** Let \( K \) and \( J \) be compact subsets of \( \mathbb{C} \) with \( \hat{K} \) and \( \hat{J} \) disjoint. Then there exists \( U \) and \( V \), open sets with compact closure, such that

(i) \( \hat{K} \subseteq U \) and \( \hat{J} \subseteq V \); and
(ii) \( U^- \) and \( V^- \) are disjoint.

**Proof.** For each \( \mu \in \hat{J} \), choose a polynomial \( q_\mu \) such that \( \|q_\mu\|_K = 1 \) and \( |q_\mu(\mu)| > 3 \). Let
\[
V_\mu = \{\lambda \in \mathbb{C} : |q_\mu(\lambda)| > 3\}, \quad \text{and}
\]
\[
U_\mu = \{\lambda \in \mathbb{C} : |q_\mu(\lambda)| < 2\}.
\]
Let \( \{V_1, \ldots, V_n\} \) be a finite cover for \( \hat{J} \). For convenience, in the notation that follows the subscript \( \mu_k \) will be replaced by \( k \). Note that \( \hat{J} \subseteq \bigcup_{k=1}^n V_k \) and \( \hat{K} \subseteq \bigcap_{k=1}^n U_k \). Choose \( U \) with compact closure such that

\[
\hat{K} \subseteq U \subseteq \overline{U} \subseteq \bigcap_{k=1}^n U_k.
\]

If \( \lambda \in \overline{U} \), then \( \|q_k\|_{\overline{U}} < 2 \), so \( |q_k(\lambda)| < 2 \) for \( 1 \leq k \leq n \). Thus, \( \overline{U} \subseteq \bigcap_{k=1}^n U_k \). Since \( \bigcap_{k=1}^n U_k \) and \( \bigcup_{k=1}^n V_k \) are disjoint, it follows that \( \overline{U} \) and \( \hat{J} \) are disjoint.

Now repeat the argument above with \( \overline{U} \) in place of \( J \) and \( J \) in place of \( K \). The argument proves \( \exists V \) an open set with compact closure such that \( \hat{J} \subseteq V \) and \( \overline{V} \) and \( \overline{U} \) are disjoint.

**Theorem 8.** Let \( T \in \mathcal{B}_{p,s} \). Assume for some \( x \in (p,s) \), \( \sigma(T_x) \) is disconnected. Then \( \sigma(T_y) \) is disconnected for all \( y \) in some neighborhood of \( x \).

**Proof.** Assume \( K \) and \( J \) are disjoint nonempty compact sets with \( \sigma(T_x) = K \cup J \). Then \( \hat{K} = K \) and \( \hat{J} = J \), so by Lemma 7, there exist open sets \( U \) and \( V \) with compact closure such that \( K \subseteq U \), \( J \subseteq V \), and \( \overline{U} \) and \( \overline{V} \) are disjoint. By Theorem 4 \( \exists \delta > 0 \) such that \( \sigma(T_y) \subseteq U \cup V \) whenever \( |x-y| < \delta \). Fix \( y \) with this property, and assume (without loss of generality) that \( x < y \). Then \( T \in \mathcal{B} = \mathcal{B}_{x,y} \). By [2, Theorem 5.1(4)]

\[
\sigma(T_x) \subseteq \sigma(T_y) = (\sigma(T_x) \cup \sigma(T_y)) \subseteq (\overline{U} \cup \overline{V}) = \overline{U} \cup \overline{V},
\]

where the last equality follows from [11, Lemma 29.21(b)]. Now since \( \sigma(T_x) \) has nonempty intersection with both \( U \) and \( V \), the same is true for \( \sigma(T_y) \). Then using the fact that \( S \to S_y \) is a continuous algebra monomorphism of \( \mathcal{B} \) into \( \mathcal{B}(L^2) \), it follows from [2, Theorem 4.5] that \( \sigma(T_y) \cap U \) and \( \sigma(T_y) \cap V \) are nonempty. Thus, \( \sigma(T_y) \) is disconnected.

**Corollary 9.** Let \( T \in \mathcal{B}_{p,s} \). Assume for some \( x \in (p,s) \) that \( \lambda_0 \) is an isolated point of \( \sigma(T_x) \). Then \( \lambda_0 \) is an isolated point of \( \sigma(T_y) \) for all \( y \) in some neighborhood of \( x \).

**Proof.** Set \( K = \{\lambda_0\} \) and \( J = \sigma(T_x) \setminus \{\lambda_0\} \). Choose \( U, V, \delta > 0, y, \) and \( \mathcal{B} \) is in the proof of Theorem 8. Then \( \sigma(T_x) \subseteq \overline{U} \cup \overline{V} \). Let \( Q \in \mathcal{B} \) be the spectral projection corresponding to the set \( \sigma(T_y) \cap \overline{U} \) which is a nonempty open and closed subset of \( \sigma(T_y) \). Then by the Spectral Mapping Theorem

\[
\sigma((QT)_x) = \sigma(Q_x T_x) = \{\lambda_0\}.
\]

It follows from Corollary 2 that for all \( w \in (x,y) \), \( \sigma((QT)_w) = \{\lambda_0\} \). Therefore \( \lambda_0 \) is an isolated point of \( \sigma(T_w) \) when \( x < w < y \). This argument proves that \( \lambda_0 \) is an isolated point of \( \sigma(T_w) \) whenever \( |x-w| < \delta \).

**Added in proof.** It has been pointed out to me that a general upper semicontinuity property of the map \( y \to \sigma(T_y) \) has been proved by I. Ya. Sneiberg in
Both Theorems 4 and 6 are special cases of this result. On the positive side, this same result implies that Theorem 8 and Corollary 9 have more general forms. Specifically, $\sigma(T_x)^A$ can be replaced by $\sigma(T_x)$ in the hypotheses of these two results without altering the conclusions.

REFERENCES

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