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MULTI-STATES ON C^*-ALGEBRAS

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ABSTRACT. This paper is concerned with the study of the dual of a C^*-algebra as a matrix ordered space. It is shown that an $n \times n$ matrix of linear functionals of a C^*-algebra, satisfying the generalized positivity condition, induces a representation of the algebra that generalizes the classical Gelfand-Naimark-Segal representation. This allows analysis of the relationship between the comparability of cyclic representations of the algebra and the matricial order structure of the dual. We consider the problem of unitary diagonalization of linear functionals and show that positive normal functionals on a matrix algebra over a semifinite von Neumann algebra can always be diagonalized.

1. INTRODUCTION AND PRELIMINARIES

The dual of a C^*-algebra has the natural matrix order. An $n \times n$ matrix $[\rho_{jk}]$ of linear functionals on a C^*-algebra $\mathscr{U}$ is said to be an *n-positive linear functional* on $\mathscr{U}$ if $[\rho_{jk}(A_{jk})]$ is a positive $n \times n$ matrix whenever $[A_{jk}]$ is a positive element of the C^*-algebra of $n \times n$ matrices over $\mathscr{U}$. If $[\rho_{jk}]$ satisfies the normalization condition $\rho_{jj}(I) = 1$ for each j in $\{1, \dots, n\}$, we say that $[\rho_{jk}]$ is an *n-state* (*multi-state*, when n is not specified).

In this paper we develop basic theory of multi-states, and our approach is motivated by the case of a single state. Thus with each n-positive linear functional of $\mathscr{U}$ we associate a representation of $\mathscr{U}$ that generalizes the classical GNS representation (Theorem 2.1). This enables to clarify the relationship between the comparability of cyclic representations of $\mathscr{U}$ and the matricial order structure of $\mathscr{U}^*$. We characterize irreducible and factor representations engendered by multi-states in terms of component functionals, and use this characterization to locate a certain class of extreme points in the set of identity preserving completely positive maps of $\mathscr{U}$ into $M_n(\mathbf{C})$. Finally, we consider the problem of unitary diagonalization of linear functionals on a matrix algebra over a C^*-algebra (see Section 3). We give a necessary and sufficient condition for the diagonalization. While the answer is negative in general, it is shown that

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positive normal functionals on the von Neumann algebra of $n \times n$ matrices over a semifinite countably decomposable von Neumann algebra can always be diagonalized.

Throughout the paper all C^*-algebras are assumed to be unital and I will always denote the identity. If L is a linear space and n is a positive integer, $\mathrm{M}_n(L)$ will denote the linear space of $n \times n$ matrices over L. If $\mathscr{H}$ is a Hilbert space and x, y are two vectors in $\mathscr{H}$, we shall denote by $\omega_{x,y}$ the linear functional on $\mathscr{B}(\mathscr{H})$, the algebra of all bounded operators on $\mathscr{H}$, given by $\omega_{x,y}(A) = \langle Ax, y\rangle$ $(A \in \mathscr{B}(\mathscr{H}))$. ω_x is the abbreviation of $\omega_{x,x}$.

The n-positive linear functionals of a C^*-algebra $\mathscr{U}$ form a positive cone that induces the natural order on $\mathrm{M}_n(\mathscr{U}^*)$. This order is also induced by the positive linear functionals on the C^*-algebra $\mathrm{M}_n(\mathscr{U})$, as seen from the following well-known fact.

Proposition 1.1. *The map* $T: \mathrm{M}_n(\mathscr{U}^*) \to (\mathrm{M}_n(\mathscr{U}))^*$ *given by*

$$(T([\rho_{jk}]))([A_{jk}]) = \sum_{j,k=1}^{n} \rho_{jk}(A_{jk}) \qquad ([\rho_{jk}] \in \mathrm{M}_n(\mathscr{U}^*),\ [A_{jk}] \in \mathrm{M}_n(\mathscr{U}))$$

is an order isomorphism between $\mathrm{M}_n(\mathscr{U}^*)$ *and* $(\mathrm{M}_n(\mathscr{U}))^*$.

Proof. Each ρ in $(\mathrm{M}_n(\mathscr{U}))^*$ is uniquely determined by the linear functionals ρ_{jk} on $\mathscr{U}$ given by $\rho_{jk}(A) = \rho(A \otimes E_{jk})$, where $A \otimes E_{jk}$ denotes the matrix in $\mathrm{M}_n(\mathscr{U})$ whose (j,k) entry is A and all the other entries are 0; and we have:

$$(T([\rho_{jk}]))([A_{jk}]) = \sum_{j,k=1}^{n} \rho_{jk}(A_{jk}) = \sum_{j,k=1}^{n} \rho(A_{jk} \otimes E_{jk}) = \rho([A_{jk}])$$

If $[\rho_{jk}]$ is an n-positive linear functional on $\mathscr{U}$ and $[A_{jk}] \in \mathrm{M}_n(\mathscr{U})^+$, then

$$(T([\rho_{jk}]))([A_{jk}]) = \sum_{j,k=1}^{n} \rho_{jk}(A_{jk}) = \langle [\rho_{jk}(A_{jk})]e,\ e\rangle \geq 0,$$

where e is the vector $(1, 1, \ldots, 1)$ in $\mathbf{C}^n$. Hence $T([\rho_{jk}])$ is positive. Conversely, if $\rho \in ((\mathrm{M}_n(\mathscr{U}))^*)^+$, then $[\rho_{jk}] = T^{-1}(\rho)$ is an n-positive linear functional. Indeed, given $[A_{jk}]$ in $\mathrm{M}_n(\mathscr{U})^+$ and $a = (a_1, \ldots, a_n) \in \mathbf{C}^n$, the matrix $[\overline{a_j} a_k A_{jk}]$ $(= (\mathrm{diag}(a_j))^*[A_{jk}](\mathrm{diag}(a_j)))$ belongs to $\mathrm{M}_n(\mathscr{U})^+$; so that $0 \leq \rho([\overline{a_j} a_k A_{jk}]) = \sum_{j,k=1}^n \overline{a_j} a_k \rho_{jk}(A_{jk}) = \langle [\rho_{jk}(A_{jk})]a, a\rangle$. Therefore T is an order isomorphism.

Remark. Suppose $\mathscr{U}$ is a C^*-algebra acting on a Hilbert space $\mathscr{H}$. Then $\mathrm{M}_n(\mathscr{U})$ is represented faithfully on $\oplus_{i=1}^n \mathscr{H}$, the direct sum of n copies of $\mathscr{H}$, through the usual matrix action on "column vectors"; and we see that for each vector $x = (x_1, \ldots, x_n)$ in $\oplus_{k=1}^n \mathscr{H}$ the functional $\omega_x | \mathrm{M}_n(\mathscr{U})$ corresponds through the isomorphism T to the n-positive linear functional $[\omega_{jk}]$, where $\omega_{jk} = \omega_{x_k, x_j} | \mathscr{U}$ $(j, k \in \{1, \ldots, n\})$.

2. The associated representation

The following theorem establishes the analogue of the GNS representation for n-positive linear functionals. The construction involved is similar to the Stinespring construction for completely positive maps.

Theorem 2.1. *If $\mathscr{U}$ is a C^*-algebra and $[\rho_{jk}]$ is an n-positive linear functional of $\mathscr{U}$, then there is a representation π of $\mathscr{U}$ on a Hilbert space $\mathscr{H}$, and n vectors $x_1, \dots, x_n$ in $\mathscr{H}$ such that $\{x_1, \dots, x_n\}$ is a generating set for $\pi(\mathscr{U})$ on $\mathscr{H}$ and*

$$\rho_{jk} = \omega_{x_k, x_j} \circ \pi \qquad (j, k \in \{1, \dots, n\}).$$

Proof. Let $\mathscr{L}$ be the linear space of all n-tuples of elements of $\mathscr{U}$. Define the conjugate-bilinear form $\langle , \rangle$ on $\mathscr{L}$ by:

$$\langle \overline{A}, \overline{B} \rangle = \sum_{j,k=1}^{n} \rho_{jk}(B_j^* A_k)(\overline{A} = (A_i), \overline{B} = (B_i)).$$

We shall show that $\langle , \rangle$ is an inner product on $\mathscr{L}$. For this note first that $\rho_{kj}(C^*) = \overline{\rho_{jk}(C)}$ for each C in $\mathscr{U}$ and $j, k \in \{1, \dots, n\}$. Indeed, this is apparent when $j = k$. If, say, $j < k$, consider the $n \times n$ matrix M whose (j, j) and (j, k) entries are C^* and I respectively, and all the other entries are 0. The nonzero entries of $M^* M \in \mathrm{M}_n(\mathscr{U})^+$ form the matrix

$$\begin{bmatrix} CC^* & C \\ C^* & I \end{bmatrix}.$$

Consequently the matrix

$$\begin{bmatrix} \rho_{jj}(CC^*) & \rho_{jk}(C) \\ \rho_{kj}(C^*) & \rho_{kk}(I) \end{bmatrix}$$

is positive, and $\rho_{kj}(C^*) = \overline{\rho_{jk}(C)}$ for all C in $\mathscr{U}$. Thus,

$$\langle \overline{A}, \overline{B} \rangle = \sum_{j,k=1}^{n} \rho_{jk}(B_j^* A_k) = \sum_{j,k=1}^{n} \overline{\rho_{kj}(A_k^* B_j)} = \overline{\langle \overline{B}, \overline{A} \rangle}.$$

Given $\overline{A} = (A_i)_{i=1}^n$ in $\mathscr{L}$, the matrix $[A_j^* A_k]$ belongs to $\mathrm{M}_n(\mathscr{U})^+$, since it is the product

$$\begin{bmatrix} A_1 \cdots A_n \\ \mathbf{0} \end{bmatrix}^* \begin{bmatrix} A_1 \cdots A_n \\ \mathbf{0} \end{bmatrix}.$$

Hence the matrix $[\rho_{jk}(A_j^* A_k)]$ is positive, and with $e = (1, \dots, 1) \in \mathbb{C}^n$ we have: $0 \leq \langle [\rho_{jk}(A_j^* A_k)] e, e \rangle = \sum_{j,k=1}^n \rho_{jk}(A_j^* A_k) = \langle \overline{A}, \overline{A} \rangle$. Therefore $\langle , \rangle$ is an inner product on $\mathscr{L}$. Let $\mathscr{N} = \{\overline{A} \in \mathscr{L} | \langle \overline{A}, \overline{A} \rangle = 0\}$. By standard arguments the equation $\langle \overline{A} + \mathscr{N}, \overline{B} + \mathscr{N} \rangle = \langle \overline{A}, \overline{B} \rangle$ defines a definite inner product on the quotient space $\mathscr{L}/\mathscr{N}$ whose completion is a Hilbert space $\mathscr{H}$.

Each element A of $\mathscr{U}$ defines the linear transformation $\pi(A)$ on $\mathscr{L}/\mathscr{N}$ by $\pi(A)((A_i) + \mathscr{N}) = (AA_i) + \mathscr{N}$. To see that $\pi(A)$ is bounded note that

$$R = \begin{bmatrix} A^* A \; 0 \cdots 0 \\ \mathbf{0} \end{bmatrix} \leq \begin{bmatrix} \|A\|^2 I \; 0 \cdots 0 \\ \mathbf{0} \end{bmatrix} = S$$

so that with

$$T = \begin{bmatrix} A_1 \cdots A_n \\ \mathbf{0} \end{bmatrix}$$

we have: $[A_j^* A^* A A_k] = T^* R T \le T^* S T = [\|A\|^2 A_j^* A_k]$, and $[\rho_{jk}(A_j^* A^* A A_k)] \le \|A\|^2 [\rho_{jk}(A_j^* A_k)]$, by positivity of $[\rho_{jk}]$. Therefore

$$\begin{aligned} \langle (AA_i), (AA_i) \rangle &= \sum_{j,k=1}^{n} \rho_{jk}(A_j^* A^* A A_k) = \langle [\rho_{jk}(A_j^* A^* A A_k)] e,\ e \rangle \\ &\le \|A\|^2 \langle [\rho_{jk}(A_j^* A_k)] e,\ e \rangle = \|A\|^2 \langle (A_i),\ (A_i) \rangle . \end{aligned}$$

Consequently, $\pi(A)$ is bounded on $\mathscr{L}/\mathscr{N}$ and extends by continuity to a bounded linear operator on $\mathscr{H}$ (also denoted by $\pi(A)$). It is easy to see that π is an algebra homomorphism of $\mathscr{U}$. Furthermore, for any $\overline{A} = (A_i)_{i=1}^n$ and $\overline{B} = (B_i)_{i=1}^n$ in $\mathscr{L}$,

$$\begin{aligned} \langle \pi(A)(\overline{A} + \mathscr{N}), \overline{B} + \mathscr{N} \rangle &= \sum_{j,k=1}^{n} \rho_{jk}(B_j^* A A_k) = \sum_{j,k=1}^{n} \rho_{jk}((A^* B_j)^* A_k) \\ &= \langle \overline{A} + \mathscr{N}, \pi(A^*)(\overline{B} + \mathscr{N}) \rangle , \end{aligned}$$

which implies that $\pi(A)^* = \pi(A^*)$. Therefore π is a representation of $\mathscr{U}$ on $\mathscr{H}$.

Finally, for each i in $\{1, \dots, n\}$ let x_i denote the vector $\overline{I}_i + \mathscr{N}$, where $\overline{I}_i$ is the element of $\mathscr{L}$ whose i-th component is I and all the other components are 0. Since $\overline{A} + \mathscr{N} = \sum_{i=1}^n \pi(A_i) x_i$ when $\overline{A} = (A_i)_{i=1}^n$, $\{x_1, \dots, x_n\}$ is a generating set for $\pi(\mathscr{U})$ on $\mathscr{H}$. Also $\rho_{jk}(A) = \langle \pi(A)(\overline{I}_k + \mathscr{N}), (\overline{I}_j + \mathscr{N}) \rangle = \langle \pi(A) x_k, x_j \rangle = \omega_{x_k, x_j}(\pi(A))$ for all $A \in \mathscr{U}$ and all $j, k \in \{1, \dots, n\}$. This completes the proof.

In the sequel the representation π of Theorem 2.1 will be called *the representation engendered by* $[\rho_{jk}]$.

Remark. In the case when $[\rho_{jk}]$ is diagonal n-positive linear functional (that is, $\rho_{jk} = 0$ for $j \ne k$) the representation engendered by $[\rho_{jk}]$ is (equivalent to) the n-fold direct sum of the GNS representations engendered by ρ_{jj} $(j = 1, \dots, n)$.

The following proposition provides a link between comparability of cyclic representations of $\mathscr{U}$ and the order structure of $M_n(\mathscr{U}^*)$. Recall that two positive linear functionals of $\mathscr{U}$ are called disjoint (quasi-equivalent) if the corresponding GNS representations are disjoint (quasi-equivalent). ([3, 10.31]).

Proposition 2.2. *If* ρ_{11} *and* ρ_{22} *are positive linear functionals of a* C^**-algebra* $\mathscr{U}$, *then* ρ_{11} *and* ρ_{22} *are disjoint if and only if there are no nonzero linear functionals* $\rho_{12}, \rho_{21} (= \rho_{12}^*)$ *on* $\mathscr{U}$ *such that the matrix* $[\rho_{jk}]$ $(j, k = 1, 2)$ *is a* 2*-positive linear functional.*

Proof. Suppose ρ_{11} and ρ_{22} are disjoint and $[\rho_{jk}]$ is a 2-positive linear functional of $\mathscr{U}$. From Theorem 2.1 $\rho_{jk}(A) = \langle \pi(A) x_k, x_j \rangle$ $(A \in \mathscr{U}, j, k = 1, 2)$, where π is the representation engendered by $[\rho_{jk}]$ and x_1, x_2 are vectors in the Hilbert space of π. Let P_1' and P_2' be the projections in $\pi(\mathcal{U})'$ whose ranges are $[\pi(\mathcal{U})x_1]$ and $[\pi(\mathcal{U})x_2]$ respectively. Then the GNS representation engendered by ρ_{jj} $(j = 1, 2)$ is equivalent to the subrepresentation $A \to \pi(A)P_j'$ ([3, 4.5.3]). Since ρ_{11} and ρ_{22} are disjoint, the central carriers of projections P_1' and P_2' are orthogonal. In particular $P_1'P_2' = 0$, so that $0 = \langle \pi(A)P_2'x_2, P_1'x_1 \rangle = \rho_{12}(A) = \rho_{21}(A^*)$ for each A in $\mathcal{U}$.

Conversely, suppose there are no nonzero functionals ρ_{12} and ρ_{21} such that $[\rho_{jk}]$ is 2-positive. Letting π_1 and π_2 denote the representations of $\mathcal{U}$ engendered by ρ_{11} and ρ_{22}, consider the direct sum representation $\Phi = \pi_1 \oplus \pi_2$ on a Hilbert space $\mathcal{H}$. Assuming, as we may, that ρ_{11}, $\rho_{22} \neq 0$, we have $\pi_1(\mathcal{U}) = \Phi(\mathcal{U})E_1'$, $\pi_2(\mathcal{U}) = \Phi(\mathcal{U})E_2'$ for some nonzero projections E_1' and E_2' in $\Phi(\mathcal{U})'$; and $\rho_{11} = \omega_{z_1} \circ \Phi$, $\rho_{22} = \omega_{z_2} \circ \Phi$ for some vectors z_1 and z_2 such that $[\Phi(\mathcal{U})z_j] = E_j'(\mathcal{H})$ $(j = 1, 2)$. If π_1 and π_2 are not disjoint, then there are nonzero projections $F_1' \leq E_1'$, $F_2' \leq E_2'$ and a partial isometry V' in $\Phi(\mathcal{U})'$ such that $V'^*V' = F_1'$ and $V'V'^* = F_2'$. Since the projections F_1' and F_2' have ranges $[\Phi(\mathcal{U})F_1'z_1]$ and $[\Phi(\mathcal{U})F_2'z_2]$ respectively, the vectors $y_1 = F_1'z_1$ and $y_2 = V'^*z_2 = V'^*F_2'z_2$ are nonzero. Furthermore, as y_1 is a generating vector for the C^*-algebra $\Phi(\mathcal{U})F_1'$ acting on $F_1'(\mathcal{H})$, the functional $\omega_{y_2, y_1} \circ \Phi$ is nonzero. The map $T\colon \mathrm{M}_2(\mathcal{U}) \to \mathrm{M}_2(\Phi(\mathcal{U})F_1')$, given by $T([A_{jk}]) = [\Phi(A_{jk})F_1']$ is a $*$ homomorphism. Therefore the matrix $[\omega_{jk}]$, where $\omega_{jk} = \omega_{y_k, y_j} \circ \Phi$ is a 2-positive linear functional on $\mathcal{U}$. Letting $\rho_{12} = \omega_{12}$ and $\rho_{21} = \omega_{21}$, we have

$$\begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} - \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{11} & \omega_{12} \end{bmatrix} = \begin{bmatrix} \omega_{z_1} \circ \Phi - \omega_{y_1} \circ \Phi & \mathbf{0} \\ \mathbf{0} & \omega_{z_2} \circ \Phi - \omega_{y_2} \circ \Phi \end{bmatrix}.$$

But

$$\omega_{z_1} \circ \Phi - \omega_{y_1} \circ \Phi = \omega_{F_1'z_1} \circ \Phi + \omega_{(E_1'-F_1')z_1} \circ \Phi - \omega_{F_1'z_1} \circ \Phi = \omega_{(E_1'-F_1')z_1} \circ \Phi$$

$$\omega_{z_2} \circ \Phi - \omega_{y_2} \circ \Phi = \omega_{F_2'z_2} \circ \Phi + \omega_{(E_2'-F_2')z_2} \circ \Phi - \omega_{V'^*F_2'z_2} \circ \Phi = \omega_{(E_2'-F_2')z_2} \circ \Phi$$

Thus $[\rho_{jk}] \geq [\omega_{jk}] \geq 0$, while $\rho_{12} = \omega_{y_2, y_1} \circ \Phi \neq 0$ contradicting the assumption. Consequently ρ_{11} and ρ_{22} are disjoint.

In the following proposition we characterize irreducible and factor representations engendered by n-positive linear functionals in terms of component functionals. For simplicity of exposition we consider the case of n-states. The general statement can be derived along the same lines.

Proposition 2.3. *Let* $[\rho_{jk}]$ *be an* n*-state of a* C^**-algebra* $\mathcal{U}$ *and* π *be the representation of* $\mathcal{U}$ *engendered by* $[\rho_{jk}]$.

(i) π *is irreducible if and only if* ρ_{jj} $(j = 1, \ldots, n)$ *are (equivalent) pure states and for each* j *and* k *there is a unitary* $U_{jk} \in \mathcal{U}$ *such that* $\rho_{jk}(U_{jk}) = 1$.

(ii) *π is a factor representation if and only if ρ_{jj} $(j = 1, \dots, n)$ are quasi-equivalent factor states.*

Proof. By Theorem 2.1 $\rho_{jk} = \omega_{x_k, x_j} \circ \pi$ for some generating set of unit vectors $x_1, \dots, x_n$. As noted before, the GNS representation engendered by ρ_{jj} $(j = 1, \dots, n)$ is equivalent to the subrepresentation $A \to \pi(A)P_j'$, where P_j' is the projection in $\pi(\mathscr{U})'$ whose range is $[\pi(\mathscr{U})x_j]$. If π is irreducible, then $P_j' = I$ and so $\pi(\mathscr{U}) = \pi(\mathscr{U})P_j'$ for each j. Thus the representations $A \to \pi(A)P_j'$ are irreducible and ρ_{jj} are equivalent pure states. By transitivity of $\pi(\mathscr{U})$, for each j and k there is a unitary U_{jk} in $\mathscr{U}$ such that $\pi(U_{jk})x_k = x_j$ ([3,5.4.5]); so that $\rho_{jk}(U_{jk}) = \langle \pi(U_{jk})x_k, x_j \rangle = 1$.

Assuming the converse we have $\langle \pi(U_{jk})x_k, x_j \rangle = \rho_{jk}(U_{jk}) = 1$. As $\|\pi(U_{jk})x_k\| = \|x_j\| = 1$, it follows that $\pi(U_{jk})x_k = x_j$. Therefore $[\pi(\mathscr{U})x_j] = [\pi(\mathscr{U})x_k]$, and $P_j' = P_k'$ for each j and k. Since $x_1, \dots, x_n$ form a generating set for $\pi(\mathscr{U})$, this implies that $P_j' = I$ for each j. Consequently $\pi(\mathscr{U}) = \pi(\mathscr{U})P_j'$ acts irreducibly.

If $\pi(\mathscr{U})''$ is a factor, then the central carrier, $C_{P_j'}$, of P_j' is the identity for each j; so that $\pi(\mathscr{U})$ is isomorphic to $\pi(\mathscr{U})P_j'$, and ρ_{jj} are quasi-equivalent factor states ([3, 10.3.3]).

Conversely, if ρ_{jj} are quasi-equivalent factor states, then $\pi(\mathscr{U})''P_j'$ is a factor and $C_{P_j'} = I$ for each j. If C is a nonzero projection in the center of $\pi(\mathscr{U})''$, then $CP_i' \neq 0$ for some i. Since CP_i' belongs to the center of $\pi(\mathscr{U})''P_i'$, $CP_i' = P_i'$; so that $C \geq C_{P_i'} = I$. Hence $C = I$, and $\pi(\mathscr{U})''$ is a factor.

Proposition 2.4. *Let $[\rho_{jk}]$ and $[\sigma_{jk}]$ be n-positive linear functionals on a C^*-algebra $\mathscr{U}$, and π be the representation engendred by $[\rho_{jk}]$ on a Hilbert space $\mathscr{H}$, so that $\rho_{jk} = \omega_{x_k, x_j} \circ \pi$ for some generating set of vectors $x_1, \dots, x_n$ for $\pi(\mathscr{U})$. If $[\sigma_{jk}] \leq [\rho_{jk}]$, then there is a positive operator H' in the unit ball of $\pi(\mathscr{U})'$ such that $\sigma_{jk}(A) = \omega_{x_k, x_j}(H'\pi(A))$ $(A \in \mathscr{U};\ j, k = 1, \dots, n)$.*

Proof. Let $\mathscr{S}$ denote the linear span of the set $\{\pi(\mathscr{U})x_i | i = 1, \dots, n\}$. Define a conjugate-bilinear form ψ on $\mathscr{S}$ by :

$$\psi(x, y) = \sum_{j,k=1}^{n} \sigma_{jk}(B_j^* A_k),$$

where $x = \sum_{i=1}^n \pi(A_i)x_i$ and $y = \sum_{i=1}^n \pi(B_i)x_i$. Applying the Cauchy-Schwarz inequality,

$$|\psi(x,y)|^2 = \left|\sum_{j,k=1}^n \sigma_{jk}(B_j^* A_k)\right|^2 \le \sum_{j,k=1}^n \sigma_{jk}(A_j^* A_k) \sum_{j,k=1}^n \sigma_{jk}(B_j^* B_k)$$
$$\le \sum_{j,k=1}^n \rho_{jk}(A_j^* A_k) \sum_{j,k=1}^n \rho_{jk}(B_j^* B_k) = \|x\|^2\|y\|^2,$$

since $[\sigma_{jk}] \le [\rho_{jk}]$. Thus ψ is bounded by 1 on $\mathscr{S}$ and extends by continuity to a conjugate-bilinear form on $\mathscr{H}$ (also denoted by ψ). Therefore there is a positive operator H' of norm 1 on $\mathscr{H}$ such that $\psi(x,y) = \langle H'x, y\rangle$ for all x and y in $\mathscr{H}$. Since

$$\begin{aligned}\langle H'\pi(A)\pi(B)x_k,\ \pi(C)x_j\rangle &= \psi(\pi(AB)x_k,\ \pi(C)x_j) = \sigma_{jk}(C^*AB)\\ &= \sigma_{jk}((A^*C)^*B) = \psi(\pi(B)x_k,\ \pi(A^*C)x_j)\\ &= \langle \pi(A)H'\pi(B)x_k,\ \pi(C)x_j\rangle\end{aligned}$$

for all A, B, C in $\mathscr{U}$ and $x_1, \dots, x_n$ is a generating set, it follows that $H' \in \pi(\mathscr{U})'$. Also we have $\sigma_{jk}(A) = \psi(\pi(A)x_k, x_j) = \langle H'\pi(A)x_k, x_j\rangle$.

The set of all n-states of a C^*-algebra $\mathscr{U}$ is convex and compact in the topology of weak* convergence on $\mathrm{M}_n(\mathscr{U}^*)$. From the Krein-Milman theorem it is the weak* closure of the convex hull of extremal n-states.

Proposition 2.5. *Let $[\rho_{jk}]$ be an n-state of a C^*-algebra $\mathscr{U}$. Suppose ρ_{jj} $(j = 1, \dots, n)$ are pure states of $\mathscr{U}$ and whenever ρ_{ss} is equivalent to ρ_{tt} for some s and t in $\{1, \dots, n\}$, there is a unitary U_{st} in $\mathscr{U}$ such that $\rho_{st}(U_{st}) = 1$. Then $[\rho_{jk}]$ is an extremal n-state.*

Proof. Let m be the number of equivalence classes of the pure states ρ_{jj}. By Proposition 2.2 the matrix $[\rho_{jk}]$ is a block-diagonal sum of the matrices $[\rho_{j(i),k(i)}]$ $(1 \le i \le m)$ corresponding to each equivalence class; and for each i $[\rho_{j(i),k(i)}]$ is an n_i-state for some n_i. By Proposition 2.3(i) the representation π_i engendered by $[\rho_{j(i),k(i)}]$ is irreducible for each i, and is equivalent to the GNS representation induced by any one of the states $\rho_{j(i),j(i)}$. Since π_i is (equvialent to) the subrepresentation of the representation π engenered by $[\rho_{jk}]$, π is the sum of disjoint representations π_i. Let C_i denote the central support of π_i (so that $\pi_i(\mathscr{U}) = \pi(\mathscr{U})C_i$).

Suppose $a \cdot [\sigma_{jk}] \le [\rho_{jk}]$ for some n-state $[\sigma_{jk}]$ and $0 < a \le 1$. Then $a \cdot \sigma_{jk}(A) = \omega_{x_k, x_j}(H'\pi(A))$ $(A \in \mathscr{U})$ for some H' in $\pi(\mathscr{U})'$, from Proposition 2.4. But, since $H'C_i$ belongs to $\pi_i(\mathscr{U}')$ and $H'C_i \ne 0$, $H'C_i = a \cdot C_i$ for each i. Consequently $H'C_i = aI$, $[\sigma_{jk}] = [\rho_{jk}]$, and $[\rho_{jk}]$ is extremal.

Proposition 2.6. *There is a one-to-one correspondence between the set of all n-positive linear functionals $\rho = [\rho_{jk}]$ of a C^*-algebra $\mathscr{U}$ and the set of all completely positive maps $\Psi: \mathscr{U} \to M_n(\mathbf{C})$, given by $\Psi_\rho(A) = [\rho_{jk}(A)]$ $(\rho = [\rho_{jk}], A \in \mathscr{U})$.*

Proof. First we show that Ψ_ρ is completely positive. Let $[A_{im}] = \sum_{i,m=1}^q A_{im} \otimes F_{im}$ be a positive element of $M_q(\mathscr{U})$ for some integer q. Since $[A_{im}]$ is a sum of positive elements of $M_q(\mathscr{U})$ of the form $[B_i^* B_m]$, we may assume that $[A_{im}] = \sum_{i,m=1}^q A_i^* A_m \otimes F_{im}$. Moreover, by Theorem 2.1 we may assume that $\mathscr{U}$ acts on a Hilbert space $\mathscr{H}$ and $\rho_{jk} = \omega_{x_k, x_j} | \mathscr{U}$ for some vectors $x_1, \dots, x_n$ in $\mathscr{H}$. Then $[\Psi_\rho(A_{im})]$ is the matrix

$$\sum_{i,m=1}^{q} \left(\sum_{j,k=1}^{n} \langle A_m x_k, A_i x_j \rangle \otimes E_{jk} \right) \otimes F_{im} \qquad \text{in } M_q(M_n(\mathbf{C})).$$

This matrix can be identified with the representing matrix of the operator T^*T in some basis $\{u_s\}$ $(s = 1, \dots, qn)$ for a qn-dimensional subpsace $\mathscr{H}_0$ of $\mathscr{H}$ containing all the vectors $A_m x_k$ $(m = 1, \dots, q; k = 1, \dots, n)$, where T is the operator on $\mathscr{H}_0$ given by $Tu_s = A_m x_k$ when $s = (m-1)n + k$. Consequently $[\Psi_\rho(A_{im})] \in M_q(M_n(\mathbf{C}))^+$, and Ψ_ρ is completely positive.

On the other hand, given a completely positive map Ψ from $\mathscr{U}$ into $M_n(\mathbf{C})$, let ρ_{jk} be the linear functional on $\mathscr{U}$ given by $\rho_{jk}(A) = (\Psi(A))_{jk}$, where $(\Psi(A))_{jk}$ denotes the (j,k) entry of the matrix $\Psi(A)$. Given $[B_{jk}]$ in $M_n(\mathscr{U})^+$, the matrix $S = \sum_{j,k=1}^n \Psi(B_{jk}) \otimes F_{jk}$ belongs to $M_n(M_n(\mathbf{C}))^+$ by the complete positivity of Ψ. Thus, with $\{e_i\}$ the standard basis for $\mathbf{C}^n$ and $z = \oplus_{i=1}^n e_i$ in $\oplus_{i=1}^n \mathbf{C}^n$, we have $0 \le \langle Sz, z \rangle = \sum_{j,k=1}^n \rho_{jk}(B_{jk})$. From Proposition 1.1 it now follows that $[\rho_{jk}]$ $(= \rho)$ is an n-positive linear functional, and it is clear that $\Psi = \Psi_\rho$.

The following corollary is an immediate consequence of Propositions 2.5 and 2.6.

Corollary 2.7. *Let $[\rho_{jk}]$ be an n-state of a C^*-algebra $\mathscr{U}$. If $[\rho_{jk}]$ satisfies the assumptions of Proposition 2.5 and $\rho_{jk}(I) = 0$ for all $j \neq k$, then the map $\Psi: \mathscr{U} \to M_n(\mathbf{C})$ given by $\Psi_\rho(A) = [\rho_{jk}(A)]$ is an extreme point in the set of all identity preserving completely positive maps from $\mathscr{U}$ into $M_n(\mathbf{C})$.*

3. Unitary diagonalization

This section is concerned with the problem of diagonalization of linear functionals on a matrix algebra over a C^*-algebra. Given a (unital) C^*-algebra $\mathscr{U}$, we shall denote by E_i $(i = 1, \dots, n)$ the projection in $M_n(\mathscr{U})$ whose matrix has I on the ith diagonal entry and has all the other entries 0. A linear functional φ on $M_n(\mathscr{U})$ is called *diagonal* if $\varphi(E_j A E_k) = 0$ for all $j \neq k$ and $A \in M_n(\mathscr{U})$ In terms of the matrix $[\varphi_{jk}]$ in $M_n(\mathscr{U}^*)$ corresponding to

φ this is equivalent to the condition that $[\varphi_{jk}]$ is diagonal. The problem we consider is whether for a linear functional ω on $\mathrm{M}_n(\mathscr{U})$ there exist a unitary $U \in \mathrm{M}_n(\mathscr{U})$ such that the functional φ given by $\varphi(A) = \omega(UAU^*)$ is diagonal. In this case we shall say that ω is *unitarily diagonalizable*. Recall that the centralizer of a linear functional ω on a C^*-algebra $\mathscr{B}$ is the set $\mathscr{C}_\omega = \{H \in \mathscr{B} | \omega(AH) = \omega(HA)$ for al $A \in \mathscr{B}\}$. $\mathscr{C}_\omega$ is a C^*-subalgebra of $\mathscr{B}$ if ω is positive.

Proposition 3.1. *Let $\mathscr{U}$ and $E_i(i$ $(i = 1, \dots, n)$ be as above. A linear functional ω on $\mathrm{M}_n(\mathscr{U})$ is unitarily diagonalizable if and only if there are n orthogonal projections F_i $(i = 1, \dots, n)$ with sum I in the centralizer $\mathscr{C}_\omega$, such that E_i is (Murray-von Neumann) equivalent to F_i in $M_n(\mathscr{U})$.*

Proof. If such a family F_i $(i = 1, \dots, n)$ exists, let V_i be the partial isometry such that $V_iF_iV_i^* = E_i$. The element $U = \sum_{i=1}^n V_i$ is a unitary and $UF_iU^* = E_i$. Therefore for each $A \in \mathrm{M}_n(\mathscr{U})$ and $j \neq k$ we have:

$$\begin{aligned}\omega(UE_jAE_kU^*) &= \omega(UE_jU^*UAU^*UE_kU^*) = \omega(F_jUAU^*F_k) = \\ &= \omega(F_kF_jUAU^*) = 0;\end{aligned}$$

so that ω is unitarily diagonalizable.

Conversely, if there exists a unitary U in $\mathrm{M}_n(\mathscr{U})$ such that $\omega(UE_jAE_kU^*) = 0$ for all $j \neq k$ and $A \in \mathrm{M}_n(\mathscr{U})$, let $F_i = UE_iU^*$ $(i = 1, \dots, n)$. Since $\sum_{i=1}^n F_i = I$ and $F_jF_k = 0$ for all $j \neq k$, we have:

$$\begin{aligned}\omega(AF_i) &= \omega\left(\sum_{j,k=1}^n F_jAF_kF_i\right) = \sum_{j=1}^n \omega(F_jAF_i) = \sum_{j=1}^n \omega(UE_jU^*AUE_iU^*) \\ &= \omega(UE_iU^*AUE_iU^*) = \omega(F_iAF_i).\end{aligned}$$

Similarly $\omega(F_iA) = \omega(F_iAF_i)$ for each i and A. Thus, the family F_i $(i = 1, \dots, n)$ belongs to $\mathscr{C}_\omega$ and has the asserted properties.

Theorem 3.2. *If $\mathscr{R}$ is a semifinite countably decomposable von Neumann algebra and ω is a positive normal linear functional on $\mathrm{M}_n(\mathscr{R})$, then ω is unitarily diagonalizable.*

Proof. The von Neumann algebra $\mathrm{M}_n(\mathscr{R})$ is semifinite and countably decomposable. Hence it admits a faithful normal semifinite tracial weight τ. From the version of Radon-Nikodym theorem in [3, 9.2.19] there is a positive operator K in the unit ball of $\mathrm{M}_n(\mathscr{R})$ such that $\tau(I - K) < \infty$ and

$$(*) \qquad \tau((I-K)A) = \omega(KA) = \omega(AK) \qquad \text{for all } A \in \mathrm{M}_n(\mathscr{R}).$$

We shall show first that the relative commutant $\{K\}^c = \{K\}' \cap \mathrm{M}_n(\mathscr{R})$ belongs to the centralizer $\mathscr{C}_\omega$. For this we note that K is a one-to-one map. Indeed, if P is the null-projection of K, then $KP = 0$ and from $(*)$, $\tau(P) = \tau((I-K)P) = \omega(KP) = 0$. Since τ is faithful, this implies $P = 0$. Thus, the inverse K^{-1} is a closed densely defined positive operator affiliated with the abelian von Neumann algebra $\mathscr{M}_0$ generated by K ([3, 5.6.12]). If E_n denotes the spectral projection of K^{-1} corresponding to the interval $[1/n, n]$, then $E_n \underset{n}{\to} I$ strongly and $K^{-1}E_n$ belongs to $\mathscr{M}_0$. Consequently if $H \in \{K\}^c$, then $\omega(AH) = \lim_n \omega(KK^{-1}E_nAH) = \lim_n \tau((I-K)K^{-1}E_nAH) = \lim_n \tau(H(I-K)K^{-1}E_nA) = \lim_n \tau((I-K)K^{-1}E_nHA) = \lim_n \omega(E_nAH) = \omega(HA)$, from $(*)$. Hence $\{K\}^c \subseteq \mathscr{C}_\omega$. In particular, if $\mathscr{M}$ is a maximal abelian subalgebra of $\mathrm{M}_n(\mathscr{R})$ containing K, then $\mathscr{M} \subseteq \mathscr{C}_\omega$. From [2, Theorem 3.18] $\mathscr{M}$ contains n orthogonal equivalent projections F_i $(i = 1, \dots, n)$ with sum I. By comparison theory F_i is equivalent to E_i for each i. Therefore ω is unitarily diagonalizable, by Proposition 3.1.

Remark. The possibility of unitary diagonalization of normal states on matrix algebras over semi-finite countably decomposable von Neumann algebras is due to the fact that the centralizers are always relatively large. This need not be the case if the von Neumann algebra is of type III. In [1] the example was shown of a faithful normal state on a type III factor, whose centralizer consists of the scalar multiples of the identity. This indicates a counterexample illustrating the failure of unitary diagonalization in the type III case.

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