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SOUSLIN SUBSETS OF $P(\omega)$-SPACES

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ABSTRACT. A space X is a $P(\omega)$-space if and only if X is countably θ-refinable and every Souslin subset of X is a generalized F_σ-set in X.

R. W. Hansell investigated in [1] the problem of when a topological property is hereditary with respect to all Souslin subsets (i.e., subsets generated from the closed sets via the (A)-operation), and he proved that if X is a subparacompact $P(\omega)$-space, then every Souslin subset of X is a generalized F_σ-set in X. In this paper we will show that the condition of subparacompactness can be removed and the proof can be simplified.

Let N be the set of the positive integers. If $\tau = (\sigma_n) \in N^N$, denote $(\tau|n) = (\sigma_1, \dots, \sigma_n)$.

Following [2], we call a subset S of a topological space X a generalized F_σ-set, if whenever $S \subset U$ and U is open in X, then $S \subset F \subset U$ for some F_σ-set F in X.

Definition 1 [3]. A space X is said to be a $P(\omega)$-space if, given any open collection of X of the form

$$\{G(\sigma_1, \dots, \sigma_n) \colon \sigma_i \text{ in } N, n = 1, 2, \dots\}$$

where $G(\sigma_1, \dots, \sigma_n) \subset G(\sigma_1, \dots, \sigma_n, \sigma_{n+1})$ whenever $\sigma_1, \dots, \sigma_{n+1}$ belong to N, then there is a closed collection $\{H(\sigma_1, \dots, \sigma_n) \colon \sigma_i \text{ in } N,\ n = 1, 2 \dots\}$ of X such that

(1) $H(\sigma_1, \dots, \sigma_n) \subset G(\sigma_1, \dots, \sigma_n)$, for all $\sigma_1, \dots, \sigma_n$ in N, and

(2) if (σ_n) is any sequence in N, then $\bigcup_{n=1}^\infty G(\sigma_1, \dots, \sigma_n) = X$ implies $\bigcup_{n=1}^\infty H(\sigma_1, \dots, \sigma_n) = X$.

Remark. If in the definition of a $P(\omega)$-space we assume only that the sets $H(\sigma_1, \dots, \sigma_n)$ are F_σ-sets in X rather than closed sets, the resulting class of spaces is the same [3, Lemma 3.1].

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Lemma 1 [4, THEOREM 2.2]. *For a space X, the following conditions are equivalent*:

(a) *X is countably metacompact.*

(b) *X is countably θ-refinable.*

(c) *If $\{F_n: n \in N\}$ is a decreasing sequence of closed subsets of X with $\bigcap_{n=1}^{\infty} F_n = \varnothing$, there is a sequence $\{G_n: n \in N\}$ of G_δ-sets in X such that $G_n \supset F_n$ for all $n \in N$ and $\bigcap_{n=1}^{\infty} G_n = \varnothing$.*

(d) *If $\{F_n: n \in N\}$ is a decreasing sequence of closed subsets of X with $\bigcap_{n=1}^{\infty} F_n = \varnothing$, there is a sequence $\{U_n: n \in N\}$ of open sets in X such that $U_n \supset F_n$ for all $n \in N$ and $\bigcap_{n=1}^{\infty} U_n = \varnothing$.*

Definition 2. A subset S of topological space X is said to be *countably θ-refinable* in X, if for any open collection $\{G_n: n \in N\}$ in X, $G_n \subset G_{n+1}$, $S \subset \bigcup_{n=1}^{\infty} G_n$, there exists a closed collection $\{F_n: n \in N\}$ in X such that $F_n \subset F_{n+1}$, $F_n \subset G_n$ for each $n \in N$, and $S \subset \bigcup_{n=1}^{\infty} F_n$.

Remark. (a) In view of the dual of Lemma 1 part (d), if S is countably θ-refinable in some containing space, then S is countably θ-refinable in its relative topology.

(b) In Definition 2, it is equivalent for the sets F_n to be F_σ-sets in X.

Theorem. *For a topological space X, the following three conditions are mutually equivalent*:

(1) *X is a $P(\omega)$-space.*

(2) *Every Souslin subset $S \subset X$ is countably θ-refinable in X.*

(3) *X is countably θ-refinable and every Souslin subset of X is a generalized F_σ-set in X.*

Proof. (1) $\Rightarrow$ (2) Let $S = \bigcup_{(\sigma_n) \in N^N} \bigcap_{n=1}^{\infty} F(\sigma_1, \ldots, \sigma_n)$ be a Souslin subset of X where each $F(\sigma_1, \ldots, \sigma_n)$ is a closed subset and $F(\sigma_1, \ldots, \sigma_n) \subset F(\sigma_1, \ldots, \sigma_n, \sigma_{n+1})$. Suppose $\{G_n: n \in N\}$ is an open collection in X such that $G_n \subset G_{n+1}$, and $S \subset \bigcup_{n=1}^{\infty} G_n$. Let $G(\sigma_1, \ldots, \sigma_n) = G_n \bigcup (X - F(\sigma_1, \ldots, \sigma_n))$ for each $(\sigma_1, \ldots, \sigma_n) \in N^n$. Then $\{G(\sigma_1, \ldots, \sigma_n): \sigma_i \text{ in } N,\ n = 1, 2, \ldots\}$ is an open collection of X with $G(\sigma_1, \ldots, \sigma_n) \subset G(\sigma_1, \ldots, \sigma_n, \sigma_{n+1})$. Since X is a $P(\omega)$-space, there exists a closed collection $\{H(\sigma_1, \ldots, \sigma_n): \sigma_i \text{ in } N,\ n = 1, 2, \ldots\}$ satisfying the conditions (1) and (2) in Definition 1.

For each $(\sigma_n) \in N^N$, if $x \in S$, then there exists an n_0 such that $x \in G_{n_0} \subset G(\sigma_1, \ldots, \sigma_{n_0})$. If $x \in X - S$, then $x \notin \bigcap_{n=1}^{\infty} F(\sigma_1, \ldots, \sigma_n)$ and there exists an n_1 such that $x \in X - F(\sigma_1, \ldots, \sigma_{n_1}) \subset G(\sigma_1, \ldots, \sigma_{n_1})$. So $\bigcup_{n=1}^{\infty} G(\sigma_1, \ldots, \sigma_n) = X$ and also $\bigcup_{n=1}^{\infty} H(\sigma_1, \ldots, \sigma_n) = X$ for each $(\sigma_n) \in N^N$.

Let $F_n = \bigcup_{i=1}^{n} \bigcup_{(\sigma_1, \ldots, \sigma_i) \in N^i} H(\sigma_1, \ldots, \sigma_i) \cap F(\sigma_1, \ldots, \sigma_i)$; then F is obviously a F_σ-set in X, $F_n \subset F_{n+1}$, and for each $(\sigma_1, \ldots, \sigma_i) \in N^i$, $H(\sigma_1, \ldots, \sigma_i) \cap F(\sigma_1, \ldots, \sigma_i) \subset G(\sigma_1, \ldots, \sigma_i) \cap F(\sigma_1, \ldots, \sigma_i) \subset G_i$, so $F_n \subset G_n$. If

$x \in S$, there exists a $(\sigma_n) \in N^N$ such that $x \in \bigcap_{n=1}^{\infty} F(\sigma_1, \dots, \sigma_n)$, as $\bigcup_{n=1}^{\infty} H(\sigma_1, \dots, \sigma_n) = X$, we have $x \in H(\sigma_1 \dots, \sigma_{n_0}) \cap F(\sigma_1, \dots, \sigma_{n_0})$, for some n_0, thus $S \subset \bigcup_{n=1}^{\infty} F_n$.

(2) $\Rightarrow$ (3) This follows immediately from our remarks after Definition 2.

(3) $\Rightarrow$ (1) Assume given any open collection of X of the form

$$\{G(\sigma_1, \dots, \sigma_n): \sigma_i \text{ in } N,\ n = 1, 2 \dots\}$$

where $G(\sigma_1, \dots, \sigma_n) \subset G(\sigma_1, \dots, \sigma_n, \sigma_{n+1})$. Let $D = \{\tau = (\sigma_n) \in N^N : \bigcup_{n=1}^{\infty} G(\sigma_1, \dots, \sigma_n) = X\}$. Since X is countably θ-refinable, for each $\tau = (\sigma_n) \in D$, we can choose a closed collection $\{F_\tau(\sigma_1, \dots, \sigma_n): n \in N\}$ such that

$$F_\tau(\sigma_1, \dots, \sigma_n) \subset G(\sigma_1, \dots, \sigma_n),$$

$$F_\tau(\sigma_1, \dots, \sigma_n) \subset F_\tau(\sigma_1, \dots, \sigma_n, \sigma_{n+1}), \quad \bigcup_{n=1}^{\infty} F_\tau(\sigma_1, \dots, \sigma_n) = X.$$

For each $(\sigma_1, \dots, \sigma_n) \in N^n$, let $S(\sigma_1, \dots, \sigma_n) = \bigcup_{\tau \in D,\ (\tau|n) = (\sigma_1, \dots, \sigma_n)} F_\tau(\sigma_1, \dots, \sigma_n)$. Then $S(\sigma_1, \dots, \sigma_n)$ is a Souslin subset of X, $S(\sigma_1, \dots, \sigma_n) \subset G(\sigma_1, \dots, \sigma_n)$, so there exists a F_σ-set $H(\sigma_1, \dots, \sigma_n)$ such that $S(\sigma_1, \dots, \sigma_n) \subset H(\sigma_1, \dots, \sigma_n) \subset G(\sigma_1, \dots, \sigma_n)$. It is easy to check that the collection $\{H(\sigma_1, \dots, \sigma_n): \sigma_i \text{ in } N,\ n = 1, 2, \dots\}$ satisfies the conditions in Definition 1, so this shows X is a $P(\omega)$-space.

Corollary 1. (1) *A space X is perfect if and only if X is a $P(\omega)$-space and every open subset of X is a Souslin subset in X.*

(2) *If the Borel and Baire sets coincide in a normal $P(\omega)$-space X, then X is perfect.*

Corollary 2. *If X is a normal $P(\omega)$-space, and S is a Souslin subset of X, then the zero-sets (Baire sets) of S are precisely the intersections of zero-sets (Baire sets) of X with S, in particular, S is z-embedded in X.*

Corollary 3. *Let X be a $P(\omega)$-space. If P is any topological property that is hereditary with respect to generalized F_σ-sets, and X belongs to P, then every Souslin subset of X belongs to P.*

Remark. The following topological properties can serve as P in Corollary 3, Lindelöf, paracompact, normal, subparacompact, metacompact, submetacompact, normal and countably paracompact, normal and paraLindelöf, and collectionwise normal.

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