COMPOSITION OPERATORS INDUCED
BY FUNCTIONS WITH SUPREMUM STRICTLY SMALLER THAN 1

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Abstract. We give some partial solutions to the following problem. Does a function analytic in the unit disc \( D \) with supremum strictly smaller than 1, induce a bounded composition operator on all weighted Hardy spaces \( H^2(\beta) \).

For \( H \) a Hilbert space of functions on a set \( X \), and \( \phi \) a function that maps \( X \) into itself we define the composition operator \( C_\phi \) on \( H \) by \( C_\phi f = f \circ \phi \).

Let \( \beta = \{\beta_n\}_{n=0}^\infty \) be a sequence of positive numbers such that \( \beta_0 = 1 \) and
\[
\lim_{n \to \infty} \frac{\beta_{n+1}}{\beta_n} = 1.
\]

Define the set \( H^2(\beta) \) to be the set of all complex formal power series \( f(z) = \sum_{n=0}^\infty a_n z^n \) with \( \sum_{n=0}^\infty |a_n|^2 \beta_n^2 < \infty \). Then \( H^2(\beta) \) is a Hilbert space of functions analytic in the unit disc \( D \) with the inner product
\[
(f, g)_\beta = \sum_{n=0}^\infty a_n \overline{b_n} \beta_n^2
\]
for \( f(z) = \sum_{n=0}^\infty a_n z^n \) and \( g(z) = \sum_{n=0}^\infty b_n z^n \) (for details see [11]). If \( \beta_n = 1 \) for all \( n \), then \( H^2(\beta) \) is the classical Hardy space \( H^2 \), and some general properties of composition operators on \( H^2 \) are known (see for example [8, 10, 4, 2, and 5]). There are few results in the case of more general \( H^2(\beta) \) spaces (see for example [1, 6, 3, 9, and 12]) and it is interesting to see how they can differ from the "\( H^2 \) case," or to see how difficult even the basic question about boundedness of composition operators can become.

We can hope to get some results about boundedness of composition operators on general \( H^2(\beta) \) spaces if they are induced by some special types of functions. Some interesting examples are the functions that have sup norm strictly smaller than 1. In the "\( H^2 \) case" it is not too hard to see that they induce trace class
composition operators (see [10]) and we shall see below that getting such a nice result is not accidental.

For the definitions of trace class and Schatten p-class for $p > 0$ see [7, Chapter III].

Let $\|\phi\|_\infty$ denote the supremum of $\{\phi(z): z \in D\}$.

**Proposition 1.** Let the space $H^2(\beta)$ be such that every function in $H^2(\beta)$ with sup norm strictly smaller than 1 induces a bounded composition operator on $H^2(\beta)$. Then, if $\phi \in H^2(\beta)$ and $\|\phi\|_\infty < 1$, $C_\phi$ is in every Schatten p-class, $p > 0$, of $H^2(\beta)$.

**Proof.** Let $\|\phi\|_\infty = r < 1$ and $r < r_1 < 1$. Let $\psi(z) = r_1 z$. We shall use the fact that for every $H^2(\beta)$ space $\psi \in H^2(\beta)$. Also, $C_\psi$ is in every Schatten p-class of $H^2(\beta)$ because

$$C_\psi \frac{z^n}{\beta_n} = \frac{\Psi^n}{\beta_n} = r_1^n \frac{z^n}{\beta_n}$$

and for $p > 0$

$$\sum_{n=0}^{\infty} r_1^{pn} < \infty.$$ 

Let $\phi_1 = 1/r_1 \cdot \phi$. Then $\phi_1$ belongs to the given space $H^2(\beta)$ and $\|\phi_1\|_\infty = (1/r_1) \cdot \|\phi\|_\infty = r/r_1 < 1$. So the operator $C_{\phi_1}$ is bounded on $H^2(\beta)$ and because $\phi = \psi \circ \phi_1$, we have $C_\phi = C_{\phi_1} \cdot C_\psi$. But $C_{\phi_1}$ is bounded, $C_\psi$ is in every Schatten p-class of $H^2(\beta)$ and so $C_\phi$ is also in every Schatten p-class of $H^2(\beta)$. \qed

We can distinguish two different types of spaces $H^2(\beta)$: one when the sequence $\beta$ is bounded and the other when the sequence $\beta$ is unbounded. The following proposition shows an interesting correlation between the space $H^2(\beta)$ and the space $H^\infty$ (the space of functions bounded and analytic in the unit disc) in each of the cases.

**Proposition 2.** $H^\infty \subset H^2(\beta)$ if and only if the sequence $\beta$ is bounded.

**Proof.** Suppose that there exists an $M > 0$ such that $\beta_n \leq M$ for all $n$, and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. If $f \in H^2$ then $\sum_{n=0}^{\infty} |a_n|^2 < \infty$, and

$$\sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 \leq M^2 \sum_{n=0}^{\infty} |a_n|^2 < \infty,$$

i.e. $f \in H^2(\beta)$. Thus $H^\infty \subset H^2 \subset H^2(\beta)$.

Suppose now that the sequence $\beta$ is unbounded. We are going to show that then there is a function $\phi \in H^\infty$ with $\phi \notin H^2(\beta)$. Let the subsequence
\{\beta_n\}_{k=1}^{\infty} \text{ be such that } \beta_{n_k} \geq k \text{ and } n_{k+1} > n_k, \ k \geq 1 \text{ and let } 1 < b < \frac{3}{2}. \text{ Let}
\begin{align*}
c &= \sum_{n=1}^{\infty} \frac{1}{k^b} \\
f(z) &= \frac{1}{2c} \sum_{n=0}^{\infty} a_n z^n
\end{align*}
where
\begin{align*}
a_n &= \begin{cases} 
\frac{1}{k^b}, & n = n_k, \\
0, & n \neq n_k.
\end{cases}
\end{align*}
Then
\begin{align*}
\|f\|_\infty &\leq \frac{1}{2c} \sum_{k=1}^{\infty} \frac{1}{k^b} = \frac{1}{2} \\
\|f\|_\beta &\geq \frac{1}{4c^2} \sum_{k=1}^{\infty} \frac{1}{k^2} k^2 \beta_{n_k}^2 \\
&\geq \frac{1}{4c^2} \sum_{k=1}^{\infty} \frac{1}{k^2} k^2 = \frac{1}{4c^2} \sum_{k=1}^{\infty} \frac{1}{k^{2(b-1)}} = \infty,
\end{align*}
since \(0 < 2(b-1) < 1\). So \(f \in H^\infty\), and \(f \notin H^2(\beta)\). \qed

Note that in the case when the sequence \(\beta\) is bounded and 0 is not an accumulation point for \(\beta\), then the \(\beta\)-norm is actually equivalent to the \(H^2\)-norm and so \(H^2(\beta) = H^2\).

Also note that if the sequence \(\beta\) is bounded by \(M\) and \(f(z) = \sum_{n=0}^{\infty} a_n z^n\) and \(f \in H^\infty\), then
\begin{align*}
\|f\|_\beta^2 &= \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 \\
&\leq M^2 \sum_{n=0}^{\infty} |a_n|^2 = M^2 \|f\|_2^2 \leq M^2 \|f\|_\infty^2.
\end{align*}

Now, we have the following result.

**Theorem 1.** If \(H^\infty \subset H^2(\beta)\) and \(\|\phi\|_\infty < 1\), then \(C_\phi\) is in every Schatten \(p\)-class, \(p > 0\), of \(H^2(\beta)\).

**Proof.** Using Proposition 1 the only thing that we have to show is that if \(H^\infty \subset H^2(\beta)\), then \(C_\phi\) is bounded on \(H^2(\beta)\) for any \(\phi\) with \(\|\phi\|_\infty < 1\). This is rather trivial because for any \(f \in H^2(\beta)\) we actually have that \(f\) is continuous on \(\overline{\phi(D)} \subset D\), i.e., \(f \circ \phi \in H^\infty \subset H^2(\beta)\), and this implies that \(C_\phi\) is bounded on \(H^2(\beta)\). \qed

There is one special case when the sequence \(\beta\) is unbounded for which we can prove that every function \(\phi\) in \(H^2(\beta)\) with sup norm strictly smaller than 1 induces a bounded composition operator on \(H^2(\beta)\), i.e., an operator in every Schatten \(p\)-class of \(H^2(\beta)\). For that, we need the following.

For a given space \(H^2(\beta)\), we define \(H^\infty(\beta)\) to be the set
\[\{f : fg \in H^2(\beta), \forall g \in H^2(\beta)\}.$$
Because the function $e_0(z) = 1$ for $z \in D$ belongs to $H^2(\beta)$, we have that $H^\infty(\beta) \subset H^2(\beta)$. We define an "$\infty, \beta$" norm on $H^\infty(\beta)$ using multiplication operators on $H^2(\beta)$. For $\phi$ in $H^\infty(\beta)$, we have
\[ \|\phi\|_{\infty, \beta} = \|M_\phi\| \]
where $M_\phi$ is the multiplication operator on $H^2(\beta)$. In the case of the Hardy space $H^2$, the space $H^\infty(\beta)$ is $H^\infty$, the space of bounded analytic functions on the unit disc $D$, and the $\infty, \beta$-norm of $H^\infty$ is the usual sup norm.

In some special cases (as for example the spaces $S_a$ when $a > \frac{1}{2}$ and $Q_a$ when $0 < a < 1$) the $\beta$ norm of $H^2(\beta)$ and $\infty, \beta$-norm of $H^\infty(\beta)$ are equivalent, i.e. $H^\infty(\beta) = H^2(\beta)$. Then we say that the space $H^2(\beta)$ is strictly cyclic. For details see [11].

**Theorem 2.** Let the sequence $\beta$ be such that $H^2(\beta) = H^\infty(\beta)$, and the function $\phi$ in $H^2(\beta)$ be such that $\|\phi\|_{\infty} < 1$. Then the operator $C_\phi$ is in every Schatten p-class, $p > 0$, of $H^2(\beta)$.

**Proof.** Let $H^2(\beta) = H^\infty(\beta)$. By Corollary 1 to Proposition 31 in [11], we have that the spectrum of each element of the Banach algebra $H^\infty(\beta)$ is its range on $D$. By Proposition 20 in [11], the spectrum of $\phi$ as an element of $H^\infty(\beta)$ is the same as the spectrum of the multiplication operator $M_\phi$ on $H^2(\beta)$.

Suppose now that $\phi \in H^2(\beta)$ and $\|\phi\|_{\infty} < 1$. Then for the spectral radius $r(M_\phi)$ we have
\[ r(M_\phi) = \lim_{n \to \infty} \|M_\phi^n\|^{1/n} < 1. \]
We use this fact to prove first that $C_\phi$ is in the trace class of $H^2(\beta)$. Let $f_n = z^n/\beta_n$, where $n \geq 0$. The sequence $\{f_n\}$ is orthonormal in $H^2(\beta)$, and
\[ \sum_{n=0}^{\infty} \|C_\phi f_n\| = \sum_{n=0}^{\infty} \|\phi^n\|/\beta_n. \]
For any $f$ in $H^\infty(\beta)$, we have that
\[ \|f\|_\beta = \|M_f e_0\|_\beta \leq \|M_f\| \|e_0\|_\beta = \|M_f\| \]
and so $\|f\|_\beta \leq \|f\|_{\infty, \beta}$. But then
\[ \sum_{n=0}^{\infty} \|C_\phi f_n\| \leq \sum_{n=0}^{\infty} \|\phi^n\|/\beta_n = \sum_{n=0}^{\infty} \|M_\phi^n\|/\beta_n < \infty \]
because
\[ \lim_{n \to \infty} \left( \frac{\|M_\phi^n\|}{\beta_n} \right)^{1/n} < \lim_{n \to \infty} \left( \frac{1}{\beta_n} \right)^{1/n} = 1. \]
So every function in $H^2(\beta)$ with sup norm strictly smaller than 1 induces a composition operator that is in the trace class of $H^2(\beta)$ and by Proposition 1 it induces a composition operator that is in every Schatten $p$-class of $H^2(\beta)$. \[\square\]
An interesting result that we would like to mention is a consequence of a more general consideration in [9]. If the sequence $\beta$ is such that the functions in $H^2(\beta)$ are continuous on $\overline{D}$ and disc automorphisms induce bounded composition operators, then $C_\phi$ compact on $H^2(\beta)$ implies that $\|\phi\|_\infty < 1$ (as for example in the case of spaces $S_\alpha$, $\alpha > \frac{1}{2}$).

Placing more restrictions on the function $\phi$, we can get another result. Let $A_D^D = \{f: f$ is analytic in some neighborhood of $D\}$.

**Theorem 3.** Let $\phi \in A_D^D$ and $\|\phi\|_\infty < 1$. Then the operator $C_\phi$ is bounded on all $H^2(\beta)$ spaces.

**Proof.** If $\|\phi\|_\infty < 1$, then there exists a disc $D_1 \supset D$ such that $\phi(D_1) \subset D$, since $\phi \in A_D^D$. But then for $f$ in $H^2(\beta)$, $f \circ \phi$ is analytic on $D_1$, i.e., $f \circ \phi \in A_D^D$. The result follows since $A_D^D \subset H^2(\beta)$. □

**Corollary 1.** If $\phi \in A_D^D$ and $\|\phi\|_\infty < 1$, then $C_\phi$ is in every Schatten $p$-class, $p > 0$, of all $H^2(\beta)$ spaces.

**Proof.** Let $\|\phi\|_\infty = r < 1$ and $r < r_1 < 1$. Then the function $\psi(z) = r_1 z$ induces a composition operator that is in every Schatten $p$-class of any $H^2(\beta)$. The function $\phi_1 = (1/r_1)\phi$ belongs to $A_D^D$ because $\phi$ does, and also $\|\phi_1\|_\infty = r/r_1 < 1$. By Theorem 3 the operator $C_{\phi_1}$ is bounded on all $H^2(\beta)$ spaces, and the same idea as in the proof of Proposition 1 leads us to the conclusion that $C_\phi = C_{\phi_1} \cdot C_\psi$ is in every Schatten $p$-class of any $H^2(\beta)$. □

**References**


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