ON THE CLASSIFICATION OF HOMOGENEOUS MULTIPLIERS BOUNDED ON $H^1(\mathbb{R}^2)$

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Abstract. Necessary and sufficient conditions for Calderon-Zygmund singular integral operators to be bounded operators on $H^1(\mathbb{R}^2)$ are investigated. Let $m$ be a bounded measurable function on the circle, extended to $\mathbb{R}^2$ by homogeneity ($m(rx) = m(x)$). If the Calderon-Zygmund singular integral operator $T_m$, defined by $T_m f = \mathcal{F}^{-1}(m\mathcal{F}(f))$, is bounded on $H^1(\mathbb{R}^2)$, then it is proved that $S^*m$ has bounded variation on the circle, where the Fourier transform of $S$ on the circle is $S(n) = (-\text{sgn}(n))^{s+1}$. This implies that $m$ must have an absolutely convergent Fourier series on the circle, and other relations on the Fourier series of $m$. Partial converses are also given. The problems are formulated in terms of distributions on the circle and on $\mathbb{R}^2$.

1. Introduction

The principal subject of this paper is classification of Calderon-Zygmund singular integrals that are bounded operators on $H^1(\mathbb{R}^2)$. These operators have kernels which are homogeneous of degree $-2$ and multipliers which are homogeneous of degree 0. We treat the problem in the general context of distributions. More precisely, suppose that $m$ is a function or distribution on $\mathbb{R}^2$ which is homogeneous of degree 0 and let $U$ be the distribution on $\mathbb{R}^2$ defined by $\mathcal{F}(U) = m$, where $\mathcal{F}$ denotes the Fourier transform on $\mathbb{R}^2$. We investigate necessary and sufficient conditions on $m$ so that $U$ extends to a bounded operator on $H^1(\mathbb{R}^2)$. We use the definition of $H^1(\mathbb{R}^2)$ in terms of atomic decompositions, as in Coifman and Weiss [2].

A first necessary condition is that $m$ be a bounded measurable function, for if the multiplier operator $T_m(f) = \mathcal{F}^{-1}(m\mathcal{F}(f))$ is a bounded operator on $H^1$, then by duality we have $T_m: BMO \rightarrow BMO$ continuously. By interpolation it follows that $T_m: L^2 \rightarrow L^2$ continuously, and hence that $m$ must be a bounded measurable function. The problem is what additional conditions...
upon \( m \) force \( T_m \in B(H^1) \), the bounded operators on \( H^1(\mathbb{R}^2) \). We obtain necessary conditions in terms of the bounded variation of certain convolutions with \( m \). We also prove that these necessary conditions are sufficient for \( T_m \) to map \( L^p \) boundedly to \( L^p \) for \( p > 1 \). In [12], Taibleson and Weiss have shown that if \( T_m \) is in \( B(H^p) \) for \( p \leq 1 \) then \( m \) is continuous on \( \mathbb{R}^2 \setminus \{0\} \). The replacement of \( L^1 \) by \( H^1 \) for this study is natural as it is well known that \( T_m \) is not bounded on \( L^1 \) if \( m \) is not continuous. In fact, \( T_m(L^1) \) is not contained in \( L^1 \). See Stein [10, p. 42] for specifics.

In related work, Daly [4] provided a complete classification of those homogeneous multiplier operators that send atoms to molecules boundedly in \( H^p(\mathbb{R}^n) \) for \( 0 < p \leq 1 \). This classification, the \( L^p \) result indicated above (see Theorem 7) and conditions found by Taibleson and Weiss [12] are the most complete sufficiency results for \( \mathbb{R}^2 \).

The corresponding extension and classification problem for homogeneous multipliers for local fields is solved in [5].

In §2 we analyze homogeneous distributions and obtain a polar decomposition for them. The results will be given for \( \mathbb{R}^n \), although in this paper we will only use the results for \( n = 2 \). The results in §2 are of independent interest, and will provide a basis for the corresponding \( H^1 \) problem in \( \mathbb{R}^n \) for \( n > 2 \).

For notational purposes we will follow Hörmander [8] as much as possible. Both [8] and Donoghue [6] contain results on homogeneous distributions. For \( H^p \) theory we follow Coifman, Taibleson, and Weiss [2, and 12].

2. Homogeneous distributions

Three subsets of \( \mathbb{R}^n \) play a role in our analysis: \( \mathbb{R}^n, \mathbb{R}^n \setminus \{0\} \), and the unit sphere \( \sum_{n-1} \). If \( X \) is one of the three, let \( C_0^\infty(X) \) denote the infinitely differentiable functions on \( X \) having compact support. For a compact subset \( K \) of \( X \) and a nonnegative integer \( k \), let

\[
\| \varphi \|_{K,k} = \sum_{|a| \leq k} \sup \{|D^a \varphi(x)| : x \in K\}.
\]

These seminorms define the topology of \( C_0^\infty(X) \); see [8, p. 34]. The space \( D'(X) \) of distributions on \( X \) is the dual of \( C_0^\infty(X) \) with this topology. So-called test functions are elements of \( C_0^\infty(X) \). In the case \( X = \sum_{n-1} \), the space \( X \) is itself compact and so the only \( K \) needed in the above definition is \( \sum_{n-1} \); and, \( C_0^\infty(\sum_{n-1}) = C^\infty(\sum_{n-1}) \).

The value of \( U \) in \( D' \) at \( \varphi \) is denoted \( \langle U, \varphi \rangle \). A distribution \( U \) in \( D'(\mathbb{R}^n) \) or \( D'(\mathbb{R}^n \setminus \{0\}) \) is said to be homogeneous of degree \( a \), for \( a \in \mathbb{R} \), if

\[
\langle U, \varphi \rangle = t^{a+n} \langle U, \varphi_t \rangle
\]

where \( \varphi_t(x) = \varphi(tx) \) for \( t > 0 \).

If \( u \) is a distribution in \( D'(\sum_{n-1}) \), then \( u \) can be used to obtain a distribution \( U_a \) on \( \mathbb{R}^n \setminus \{0\} \) that is homogeneous of degree \( a \), in the following manner.
For $a \in \mathbb{R}$ and $\varphi \in C^\infty_0(\mathbb{R}^n \setminus \{0\})$, define $U_a$ by
\[
\langle U_a, \varphi \rangle = \int_0^\infty r^{n-1+a} \langle u, \varphi_r \rangle \, dr
\]
where $\varphi_r$ is the function defined on $\sum_{n-1}$ by $\varphi_r(x^i) = \varphi(rx^i)$. As $\varphi$ has compact support, the integral above converges absolutely and defines a linear functional $U_a$ which is in $D'(\mathbb{R}^n \setminus \{0\})$. The distribution $u$ is said to be the kernel of the distribution $U_a$.

If $a = -(n + k)$ and $k$ is not a nonnegative integer, then $U_a$ has a unique extension to a distribution $\tilde{U}_a$ in $D'(\mathbb{R}^n)$. See Hörmander [8], Theorem 3.2.3. If $k$ is a nonnegative integer, then $U_a$ has an extension $\tilde{U}_a$ in $D'(\mathbb{R}^n)$ if and only if $\langle U_a, x^\alpha \psi \rangle = 0$ for radial $\psi \in C^\infty_0(\mathbb{R}^n \setminus \{0\})$ and all nonnegative integer multi-indices $\alpha$ such that $|\alpha| = k$. To obtain this condition and hence the extension to $\mathbb{R}^n$, it is easy to see that we must have $\langle u, x^\alpha \rangle = 0$ for $|\alpha| = k$. If $k = 0$, this condition reduces to $\langle u, 1 \rangle = 0$. The homogeneous extension $\tilde{U}_a$ in this case is only unique up to a linear combination of the derivatives of order $k$ of the Dirac measure $\delta$. This is because derivatives of order $k$ of $\delta$ are homogeneous of order $-(n + k)$ and are supported at zero. See Theorem 3.2.4 of Hörmander [8]. For the purposes of this paper, we will be interested in the case $(n, k, a) = (2, 0, -2)$ and the extension where the contribution of $\delta$ is zero.

Suppose next that $U \in D'(\mathbb{R}^n)$ is homogeneous of degree $a$. From Donoghue [6, p. 154], $U$ is a tempered distribution. That is, $U$ has a unique continuous extension to the space $S$ of rapidly decreasing functions on $\mathbb{R}^n$. We will show that there is a $u \in D'(\sum_{n-1})$ for which $U = U_a$. We need the following two lemmas.

**Lemma 1.** If $U \in D'(\mathbb{R}^n)$ is homogeneous of degree $a \geq -n$, then $U$ has finite order.

**Proof.** If $\varphi \in C^\infty_0(\mathbb{R}^n)$ and has support in the unit ball $B^1$, there exists a constant $c$ and an integer $N$ such that
\[
\langle U, \varphi \rangle \leq c \cdot \|\varphi\|_{B^1,N}.
\]
For any $\varphi \in C^\infty_0(\mathbb{R}^n)$, there exists $r \geq 1$ such that $\text{sup}(\varphi) \subset B^r$, the ball of radius $r$. Then $\varphi_{1/r}$ has support in $B^1$. Thus
\[
\langle U, \varphi_{1/r} \rangle \leq c \cdot \|\varphi_{1/r}\|_{B^1,N} \leq c \cdot \sum_{k=0}^{N} r^{-k} \sum_{|\alpha|=k} \|D^\alpha \varphi\|_\infty \leq c \cdot \|\varphi\|_{B^r,N}.
\]
As $U$ is homogeneous of degree $a$, $\langle U, \varphi_{1/r} \rangle \sim r^{a+n} \langle U, \varphi \rangle$. So we have
\[
\langle U, \varphi \rangle \leq r^{-a-n} c \cdot \|\varphi\|_{B^r,N}.
\]
Thus $U$ has finite order if $a \geq -n$. (If $N$ is the smallest integer for which (i) holds, then the order of $U$ is $N$.) $$

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Lemma 2. If the distribution $U$ is homogeneous of degree $a \geq -n$, then there exists a continuous function $g$ and a multi-index $\alpha$ such that, for $\varphi \in C_0^\infty(\mathbb{R}^n)$,

\[ \langle U, \varphi \rangle = \langle D^\alpha g, \varphi \rangle = \langle g, (-1)^{|\alpha|} D^\alpha \varphi \rangle. \]

Proof. From Donoghue [6, p. 106], a distribution of finite order $N$ is an $(N+2)$ derivative of a continuous function. $\square$

We can now state and prove the main theorem on the correspondence between homogeneous distributions on $\mathbb{R}^n$ and $\mathbb{R}^n \{0\}$ and distributions on $\sum_{n-1}$.

Theorem 3. Let $a \in \mathbb{R}$.

1. If $u \in D'(\sum_{n-1})$ and $a \neq -(n+k)$ where $k$ is a nonnegative integer, define $U$ for $\varphi \in C_0^\infty(\mathbb{R}^n \{0\})$ by

\[ \langle U, \varphi \rangle = \int_0^\infty r^{n-1+a} \langle u, \varphi_r \rangle \, dr. \]

Then $U$ is in $D'(\mathbb{R}^n \{0\})$ and has a unique extension to a distribution in $D'(\mathbb{R}^n)$ that is homogeneous of degree $a$.

2. If $U \in D'(\mathbb{R}^n)$ is homogeneous of degree $a$ and $a$ is not of the form $-(n+k)$ for a nonnegative integer $k$, then there exists $u \in D'(\sum_{n-1})$ such that

\[ \langle U, \varphi \rangle = \int_0^\infty r^{n-1+a} \langle u, \varphi_r \rangle \, dr. \]

for $\varphi \in C_0^\infty(\mathbb{R}^n \{0\})$.

3. If $a = -(n+k)$ and $k$ is a nonnegative integer, then (1) holds if $\langle u, \varphi^\alpha \rangle = 0$ for each nonnegative integer multi-index $\alpha$ with $|\alpha| = k$. In this case the extension to $D'(\mathbb{R}^n)$ is unique only up to addition of a linear combination of derivatives of order $k$ of $\delta$. If $U$ in $D'(\mathbb{R}^n)$ is homogeneous of degree $a$ as in (2), then $u$ exists and $\langle u, \varphi^\alpha \rangle = 0$ must hold for $|\alpha| = k$.

Proof. We have proved (1). To prove (2), first suppose that $a > -n$. Then Lemma 2 applies and equality (ii) can be rewritten as

\[ \langle U, \varphi \rangle = \int_{\mathbb{R}^n} (-1)^{|\alpha|} g(x)D^\alpha(\varphi)(x) \, dx \]

\[ = (-1)^{|\alpha|} \int_0^\infty r^{n-1} \int_{\sum_{n-1}} g(rx')D^\alpha(\varphi)(rx') \, dx' \, dr \]

\[ = (-1)^{|\alpha|} \int_0^\infty r^{n-1+a} \int_{\sum_{n-1}} (r^{-a-|\alpha|} g(rx'))D^\alpha(\varphi_r)(x') \, dx' \, dr. \]

Consider the function $h(rx') = r^{-(a+|\alpha|)} g(rx')$. The function $h$ is continuous on $\mathbb{R}^n \{0\}$ and, as $U$ is homogeneous of degree $a$, $h$ is homogeneous of degree 0. Thus $U$ can be written as

\[ \langle U, \varphi \rangle = \int_0^\infty r^{n-1+a} \langle u, \varphi_r \rangle \, dr \]

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where \( u \) is defined by

\[
(u, \sigma) = \int_{\sum_{n-1}} h(x') (-1)^{|\alpha|} D^\alpha(\sigma)(x') \, dx'
\]

for \( \sigma \in C_0^\infty(\sum_{n-1}) \). The derivative \( D^\alpha(\sigma) \) is defined by extending \( \sigma \) radially. Explicitly, choose a radial \( \psi \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) \) that is identically equal to 1 on a neighborhood of \( \sum_{n-1} \), and let \( D^\alpha(\sigma)(x') = D^\alpha(\psi \sigma)(x') \). Then \( u \in D'(\sum_{n-1}) \) and \( U \) is the \( U_\alpha \) defined by \( u \). Note that for given \( \varphi \in C_0^\infty(\mathbb{R}^n) \), the function \( \langle u, \varphi \rangle \) has compact support on \( \mathbb{R}^+ \) and is a bounded function of \( r \), so the function \( \{ r^{n-1+a} \langle u, \varphi \rangle \} \) is in \( L^1(\mathbb{R}^1) \). Hence the integral in (iii) is finite.

If \( a < -n \), then \( | \cdot |^{-a-n} U \) is a distribution in \( D'(\mathbb{R}^n \setminus 0) \) which is homogeneous of degree \( -n \). Hence by Lemma 2 and the above argument there is a continuous function \( g \) on \( \mathbb{R}^n \) and a multi-index \( \alpha \) such that the function \( h \) defined by

\[
h(x) = |x|^{-(-n+|\alpha|)} g(x)
\]

and the distribution \( u \) defined by

\[
(u, \sigma) = \int_{\sum_{n-1}} h(x') (-1)^{|\alpha|} D^\alpha(\sigma)(x') \, dx'
\]

satisfy

\[
\langle | \cdot |^{-a-n} U, \varphi \rangle = \int_0^\infty r^{-1} \langle u, \varphi \rangle \, dr
\]

for \( \varphi \in C_0^\infty(\mathbb{R}^n \setminus 0) \). The polar decomposition equality (iii) follows for \( U \).

If \( a = -(n + k) \) for a nonnegative integer \( k \) and if \( u \in D'(\sum_{n-1}) \), then the formula in (1) defines an element \( U \) of \( D'(\mathbb{R}^n \setminus 0) \) which is homogeneous of degree \( a \). The extension to \( \mathbb{R}^n \) exists if \( \langle u, x^\alpha \rangle = 0 \) for all \( \alpha \) satisfying \( |\alpha| = k \). Conversely, if \( U \) is in \( D'(\mathbb{R}^n \setminus 0) \) and is homogeneous of degree \( a = -(n + k) \), then the definition of \( u \) in (iv) is the same as before and (iii) holds for \( \varphi \in C_0^\infty(\mathbb{R}^n \setminus 0) \). If \( U \) extends to \( D'(\mathbb{R}^n \setminus 0) \), then \( \langle U, x^\alpha \varphi \rangle = 0 \) must hold for radial \( \psi \). It follows easily that \( \langle u, x^\alpha \rangle = 0 \) must hold. Hence we have proved (3). \qed

**Further remarks.** The formulas in Theorem 3 hold for \( \varphi \in C_0^\infty(\mathbb{R}^n) \) if \( a > -n \) but the extensions from \( \mathbb{R}^n \setminus 0 \) to \( \mathbb{R}^n \) are given by limits if \( a \leq -n \). Since \( U \) is homogeneous it is also tempered and so in fact \( U \) extends continuously to \( S \) as well. It is easy to see that the formulas hold for those \( \varphi \) in \( S \) for which \( \varphi \) is zero on a neighborhood of \( 0 \) and for all \( \varphi \) in \( S \) if \( a > -n \).

If \( U \) is homogeneous of degree \( a \), then the Fourier transform \( \mathcal{F}(U) \) is homogeneous of degree \( n + a \); it too is tempered. In the case \( a = -n \), \( \mathcal{F}(U) \) is homogeneous of degree 0. The element \( m \) of \( D'(\sum_{n-1}) \) corresponding to \( \mathcal{F}(U) \) as in Theorem 3, is called the multiplier of \( U \). The distribution \( U \) defines in the usual way an operator \( T_m \) on \( C_0^\infty(\mathbb{R}^n) \) by

\[
T_m(\varphi)(x) = \langle U, \tau_x \varphi \rangle
\]
where $\tau_x$ is translation by $-x$ and $\phi(x) = \phi(-x)$. Since $U$ is tempered, this formula extends to $\varphi \in S$. This paper concerns the extension of $T_m$ from $S$ as an operator on Hardy spaces.

3. Multiplier operators

In the remainder of this paper we will be concerned only with distributions on $\mathbb{R}^2$ that are homogeneous of degree $-2$. The distribution $u$ in $D'(\sum_1)$ plays the role of the usual kernel in the construction of Calderon-Zygmund singular integrals. We let $\sum_1 = T$, the circle. We will always assume that $U$ has no delta measure component (recall that $\delta$ is homogeneous of degree $-2$). The Fourier transform of $U$ is homogeneous of degree 0 and so there is a unique distribution $m$ on $T$ satisfying

$$\langle \mathcal{F}(U), \varphi \rangle = \int_0^\infty \langle m, \varphi_r \rangle r \, dr.$$ 

In this case the relationship between $u$ and $m$ can be stated explicitly, in terms of the Fourier transform on the circle $T$, as

$$u(n) = n(\text{sgn}(n))^{n+1} \cdot m(n)/2\pi i$$

for $n \neq 0$, where $\text{sgn}(n)$ denotes the sign of $n$.

We give two proofs of (1). From Stein [10], for dimension 2, $u(n)$ can be computed from $u$ on $T$ by

$$m = -[(i\pi/2) \text{sgn}(\cos(\cdot)) + \log |\cos(\cdot)|] \ast u.$$ 

A straightforward computation gives, for $n \neq 0$,

$$-[(i\pi/2) \text{sgn}(\cos(\cdot)) + \log |\cos(\cdot)|]^\wedge(n) = 2\pi (\text{sgn}(n))^{-(n+1)}/n.$$ 

Expression (1) follows immediately.

The formula (1) can also be derived by directly computing the Fourier transform of the distribution $e^{in\theta}/|\cdot|^2$ in $D'(\mathbb{R}^2 \setminus 0)$ using the methods of the proof of Theorem 5, infra. We give this derivation, which is heuristic in that existence of certain integrals needs to be established.

Let $\chi_n(\theta) = e^{in\theta}$, $n \neq 0$. Each $\chi_n \cdot |\cdot|^2$ for a real is locally integrable on $\mathbb{R}^n \setminus \{0\}$ and so defines a distribution. Let $\varphi \in C^\infty_0(\mathbb{R}^2 \setminus 0)$. Using Theorem 3 and letting $y = se^{i\beta}$, we have

$$\langle \mathcal{F}(\chi_n \cdot |\cdot|^2), \varphi \rangle = \langle \chi_n \cdot |\cdot|^2, \mathcal{F}(\varphi) \rangle$$

$$= \int_0^\infty \int_0^{2\pi} \frac{1}{r} e^{in\theta} \int_{\mathbb{R}^2} \varphi(se^{i\beta}) e^{-i<se^{i\beta},re^{i\theta}>} \, dy \, d\theta \, dr$$

$$= \int_{\mathbb{R}^2} \varphi(se^{i\beta}) \int_0^{2\pi} \frac{1}{r} e^{in\theta} e^{-i<se^{i\beta},re^{i\theta}>} \, d\theta \, dr \, dy.$$ 

The inner-product in $\mathbb{R}^2$ can be rewritten using $\langle e^{i\beta}, e^{i\theta} \rangle = \cos(\theta - \beta)$. The $\theta$-integral is a Bessel function, namely

$$\int_0^{2\pi} e^{in\theta} e^{-i\alpha \cos(\theta - \beta)} \, d\theta = 2\pi e^{in\beta} (-i)^n J_n(rs)$$
where $J_n$ is the $n$th-Bessel function. Using the multiplicative invariance of the measure $r^{-1} \, dr$ on $\mathbb{R}^+$, the equality $J_{-n}(r) = (-1)^n J_n(r)$ for $n > 0$, and integration formulas from Watson [13, p. 391, no. 385], the $r$-integral becomes

$$
\int_0^\infty \frac{1}{r} J_n(rs) \, dr = (\text{sgn}(n))^{n+1}/n .
$$

Thus

$$
\langle \mathcal{F}(\chi_n \cdot |^{-2}, \varphi) = (2\pi i)^{-n} (\text{sgn}(n))^{n+1}/n \rangle \langle \chi_n, \varphi \rangle .
$$

We have shown that if we take $u = \chi_n$, then $m = c_n \chi_n$ where $c_n$ is the constant on the right. The relationship (1) between the Fourier transforms of $u$ and $m$ on $T$ follows for any $m$ and $u$.

Since the degree of $U$ is $-2$, for the extension of $U$ to $\mathbb{R}^2$ it is necessary that $\hat{u}(0) = 0$. Since we have assumed that $U$ has no delta measure component, it is also true that $\hat{m}(0) = 0$.

Define the convolution operator $S$ on $C^\infty(T)$ by

$$
S(n) = 2\pi i^{-n} (\text{sgn}(n))^{n+1} .
$$

The operator $S$ will play a central role in the classification of those homogeneous multipliers that give rise to bounded operators on $H^1(\mathbb{R}^2)$. If we let $\tau_\theta$ denote translation by $\theta$, then a straightforward computation shows that $S$ is invertible with inverse

$$
S^{-1} = (1/4\pi^2)\tau_n S .
$$

In fact, $S$ can be written as

$$
S = (1/2)\tau_{n/2} \{ H(I + \tau_n) + i(I - \tau_n) \}
$$

where $H$ is the Hilbert transform on the circle, defined by $\hat{H}(n) = - \text{sgn}(n)$.

In terms of $S$, the relationship (1) between $u$ and $m$ is

$$
\hat{u}(n) = (2\pi)^{-2} i^{-n} (-1)^n \cdot \hat{S}(n) \hat{m}(n) .
$$

As we observed in the introduction, if $T_m$ extends to a bounded operator on $L^2(\mathbb{R}^2)$, then $m$ must be a bounded measurable function on $T$. Thus $S \ast m$ is a function and we have

$$
(2\pi)^2 i(-1)^{n+1} \hat{u}(n)/n = (S \ast m)\hat{}(n) .
$$

Denoting the distributional derivative by $D$, it follows that we may write $u$ in $D'(T)$ as

$$
u = -D(S^{-1} \ast m).
$$

From Edwards [7, Theorem 12.5.16], $u$ is a measure if and only if $\hat{u}(n)/n$ is the Fourier transform of a function of bounded variation. Hence we have the following lemma.

**Lemma 4.** The function $S \ast m$ has bounded variation if and only if $u$ is a measure.

We will now prove the main result of the paper.
Theorem 5. Suppose \( m \) is a bounded measurable function on \( \mathbb{R}^2 \) which is homogeneous of degree 0. If the singular integral operator \( T_m \) is in \( B(H^1) \), then the functions \( S \ast m \), \( S \ast (\sin(\cdot)m) \), and \( S \ast (\cos(\cdot)m) \) have bounded variation on \( T \).

Proof. Assume \( T_m \) is in \( B(H^1) \). Let \( \eta \) be a radial function in \( C^\infty(\mathbb{R}^2) \) that is supported on \( \{x : 1/4 \leq |x| \leq 4\} \) and is identically equal to 1 on \( \{x : 1/2 \leq x \leq 2\} \). Then \( M = \mathcal{F}^{-1}(\eta) \) is in \( H^1(\mathbb{R}^2) \); see Taibleson and Weiss [12, p. 137]. Thus \( T_m(M) \) is in \( L^1(\mathbb{R}^2) \). For \( a > 0 \), we have

\[
\|T_m(M)\|_1 \geq \int_{\mathbb{R}^2} e^{-|a|z} |T_m(M)(z)| \, dz 
\]

(2)

If \( m \) is expanded in a Fourier series, \( \{A_n\} \) is any decomposition of \( \mathbb{R}^2 \), \( x' \) is replaced by \( e^{i\theta} \), and \( z \) equals \( se^{i\beta} \), then (2) becomes

\[
\|T_m(M)\|_1 \geq \sum_n \int_{A_n} e^{-as} \left| \int_0^\infty \eta(r) \sum_k \hat{m}(k) \int_0^{2\pi} e^{ik\theta} e^{irs\cos(\theta-\beta)} \, d\theta \, dr \right| dz .
\]

Performing the \( \theta \)-integration, we obtain

(3) \[
\|T_m(M)\|_1 \geq \sum_n \int_{A_n} e^{-as} \left| \int_0^\infty \eta(r) \sum_k \hat{m}(k) 2\pi(-i)^k e^{ik\beta} J_k(rs) \, dr \right| dz .
\]

where \( J_k \) is the \( k \)th-Bessel function. If we let \( \{A_n\} \) be the conical decomposition of \( \mathbb{R}^2 \) induced by a decomposition \( 0 = b_0 < b_1 < \cdots < b_N = 2\pi \) for the circle, (3) becomes

\[
\|T_m(M)\|_1 \geq \sum_n \int_0^\infty \eta(r) \sum_k \hat{m}(k) 2\pi(-i)^k \int_{b_{n-1}}^{b_n} e^{ik\beta} J_k(rs) \, ds \, dr \, dz .
\]

Using \( J_k = (\text{sgn}(k))^k J_{|k|} \) and an integration formula from Watson [13, p. 385], the \( s \)-integral can be explicitly evaluated as

(5) \[
\frac{\text{sgn}(k)^k (r/2)^k (1 + |k|)}{(a^2 + r^2)^{(2+|k|)/2}} \, _2F_1\left(\frac{|k|+2}{2}, \frac{|k|-1}{2}; |k|+1; \frac{r^2}{(a^2 + r^2)}\right)
\]

where \( _2F_1 \) is a generalized hypergeometric function. Call the function displayed in (5) \( C(k, a, r) \). For \( a > 0 \), \( r\eta(r)C(k, a, r) \) is integrable and is dominated by \( r\eta(r)C(k, 0, r) \). The function \( C(k, 0, r) \) is given by

\[
C(k, 0, r) = \frac{\text{sgn}(k)^k (1 + |k|)}{2^k r^2} \, _2F_1\left(\frac{|k|+2}{2}, \frac{|k|-1}{2}; |k|+1; 1\right)
\]

(0)
by using Gauss’s Formula for $\,_{2}F_{1}(\alpha, \beta; \gamma; 1)$ (see Bateman Manuscript [1, p. 61]). Substituting this into (4) and performing the $\beta$-integration, we obtain the bound
\[
\|T_{m}(M)\|_{1} \geq C_{n} \sum_{n} \left| \frac{2\pi}{k} (-i)^{k+1} \text{sgn}(k)k^{k+1} \hat{m}(k)(e^{-ikb_{n}} - e^{-ikb_{n-1}}) \right|
\]
\[
= C_{n} \sum_{n} \left| S * m(e^{ib_{n}}) - S * m(e^{ib_{n-1}}) \right|
\]
As $\|T_{m}(M)\|_{1}$ is finite, $S * m$ is of bounded variation on $\mathbb{T}$.

To show that $S * (\sin(\cdot)z)$ and $S * (\cos(\cdot)z)$ must also be of bounded variation, we consider the two Riesz transforms $R_{1}$ and $R_{2}$, defined by $\mathcal{F}(R_{1})(re^{i\theta}) = i \cdot \sin(\theta)$ and $\mathcal{F}(R_{2})(re^{i\theta}) = i \cdot \cos(\theta)$. These operators are bounded on $L^{1}$; and hence, $R_{1}T_{m}$ and $R_{2}T_{m}$ are bounded on $H^{1}$ if $T_{m}$ is bounded. Applying the preceding argument to these operators, we see that $S * (\sin(\cdot)z)$ and $S * (\cos(\cdot)z)$ are of bounded variation.

**Theorem 6.** Suppose $m$ is a bounded measurable function on $\mathbb{R}^{2}$ which is homogeneous of degree 0 and such that $T_{m}$ is in $B(H^{1})$. Then $m$ has an absolutely convergent Fourier series as a function on $\mathbb{T}$.

**Proof.** From Theorem 5, if $T_{m} \in B(H^{1})$, then $S * m$, $S * (\sin(\cdot)z)$, and $S * (\cos(\cdot)z)$ are of bounded variation on $\mathbb{T}$. Thus $(-\text{sgn}(n))^{n+1}\hat{m}(n)$, $(-\text{sgn}(n))^{n+1}(\hat{m}(n+1) + \hat{m}(n-1))$, and $(-\text{sgn}(n))^{n+1}(\hat{m}(n+1) - \hat{m}(n-1))$ are the Fourier series of functions of bounded variation; and consequently, $(-\text{sgn}(n))^{n+1}\hat{m}(n+1)$ is the Fourier transform of a function of bounded variation. Taken together, all this implies that the Fourier transform
\[
(H * S * m)^{(n)} = (-\text{sgn}(n)) \cdot (-\text{sgn}(n))^{n+1}\hat{m}(n)
\]
is a finite sum of transforms of functions of bounded variation. From Zygmund [14, v. 1, p. 242], as both $S * m$ and $H * S * m$ are of bounded variation, $S * m$ has an absolutely convergent Fourier series. However, $|(S * m)^{(n)}| = 2\pi|\hat{m}(n)|$ and so $m$ has an absolutely convergent Fourier series.

**Theorem 7.** Suppose $m$ is a bounded measurable function on $\mathbb{R}^{2}$ which is homogeneous of degree 0 and $T_{m}$ is in $B(H^{1})$. Then $u$ is a measure on $\mathbb{T}$ and $u$ is absolutely continuous with respect to Lebesgue measure.

**Proof.** Theorem 5 and Lemma 4 imply that $u$ is a measure. The kernel of $H * m$ is $H * u$, because both $H$ and the correspondence between kernels and multipliers are linear. In the proof of Theorem 6 it was shown that $H * S * m$ has finite variation. Since $H * S = S * H$, it follows that $S * H * m$ has finite variation. Hence, by Lemma 4, $H * u$ is also a measure. By the F. Riesz and M. Riesz Theorem, $u$ is absolutely continuous. See Zygmund [14, v. 1, p. 285], and also Edwards [7, p. 99].

Theorem 6 can be phrased for kernels $u$. If $u$ is a kernel in $D'(\mathbb{T})$ and $T_{m} \in B(H^{1})$, then by Theorem 7 $u$ is in $L^{1}(\mathbb{T})$. By Theorem 6, $m$ has an
absolutely convergent Fourier series as a function on $T$. However, $|\hat{m}(n)| = |\hat{u}(n)|/|n|$ if $n \neq 0$, and so $\sum_{n \neq 0} \frac{|\hat{u}(n)|}{|n|} < \infty$. We have proved:

**Corollary 8.** If $u$ is in $L^1(T)$ and the corresponding singular integral $T_m$ is in $B(H^1)$, then $\sum_{n \neq 0} \frac{|\hat{u}(n)|}{|n|} < \infty$.

We now provide a partial converse to Theorem 5.

**Theorem 9.** Suppose $m$ is a bounded measurable function on $\mathbb{R}^2$ which is homogeneous of degree 0. If $S \ast m$, $S \ast (\cos(\cdot)m)$, $S \ast (\sin(\cdot)m)$ are of bounded variation on $T$, then $T_m$ is in $B(L^p)$ for $1 < p < \infty$.

**Proof.** Let $BV(T)$ be the functions of bounded variation and suppose $S \ast m$, $S \ast (\cos(\cdot)m)$, $S \ast (\sin(\cdot)m) \in BV(T)$. A periodic function $f$ on $\mathbb{R}$ is even or odd as a function on the circle according as $f(t) = f(t+\pi)$ or $f(t) = -f(t+\pi)$. Note that $\cos(t)$ as a function on $T$ is odd, not even. The even and odd parts of a function $f$ are given by

$$f_0(\theta) = (f(\theta) + f(\theta + \pi))/2, \quad f_0(\theta) = (f(\theta) - f(\theta + \pi))/2.$$ 

Clearly $f \in BV(T)$ if and only if $f_0$ and $f_0 \in BV(T)$. As $S$ is a multiplier operator, $(S \ast m)_0 = S \ast (m_0)$ and $(S \ast m)_e = S \ast (m_e)$. For the odd function $m_0$, we have $m_0 \in BV(T)$ as $S$ reduces to a translation operator for odd functions. By Lemma 4, the corresponding kernel $u_0$ is a measure. As a measure on $[-\pi, \pi]$, it satisfies

$$\int g(t + \pi) \, du_0(t) = -\int g(t) \, du_0(t)$$

for periodic functions $g$. That is, $u_0$ is an odd measure. A straightforward adaptation of the “method of rotations” shows that $T_{m_0} \in B(L^p)$ for $1 < p < \infty$. See Theorem (2.6) and the proof preceding it in Chapter VI of [11], where the kernel is an odd function in $L^1(T)$. For the even multiplier $m_e$, the functions $\sin(\cdot)m_e$, $\cos(\cdot)m_e$ are both odd functions. We have

$$S \ast (\cos(\cdot)m_e) = (S \ast (\cos(\cdot)m))_0, \quad S \ast (\sin(\cdot)m_e) = (S \ast (\sin(\cdot)m))_0,$$

and so $S \ast (\sin(\cdot)m_e)$ and $S \ast (\cos(\cdot)m_e)$ are in $BV(T)$. Hence $\sin(\cdot)m_e$ and $\cos(\cdot)m_e$ are also in $BV(T)$ and so are multipliers of kernels which are odd measures. As before, the corresponding operators are in $B(L^p)$; that is, the Riesz transforms

$$R_1T_m, \quad R_2T_m \in B(L^p).$$

Each $R_j$ is also in $B(L^p)$, and so also is each $R_j^2T_m$. So $T_m \in B(L^p)$ as $R_1^2 + R_2^2 = -I$. As $T_m = T_{m_0} + T_{m_e}$, we have $T_m \in B(L^p)$ for $1 < p < \infty$. □

In [9], Ricci and Weiss give a characterization of $H^1(\sum_{n=1}^\infty)$ that is particularly suitable to our presentation here. They consider Calderon-Zygmund singular operators (distributions) $T$ with kernel $k(x) = u(x')/|x|^n$, where
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u \in L^1(\sum_{n-1}) and has integral zero. By composing \( T \) with the Riesz transforms \( R_j \) \((1 \leq j \leq n)\), one obtains \( n \) distributions that are each homogeneous of degree \(-n\). In general, these distributions may or may not be given by kernels in \( L^1(\sum_{n-1}) \). Ricci and Weiss characterize \( H^1(\sum_{n-1}) \) by showing that \( u \in H^1(\sum_{n-1}) \) if and only if \( R_j \circ T \) has a kernel \( u_j \in L^1(\sum_{n-1}) \). In the case of dimension 2, we know from Theorem 7 that the operators \( R_1 \circ U \) and \( R_2 \circ U \) have kernels \( u_1 \) and \( u_2 \) in \( L^1(\mathbb{T}) \) if \( U(= T_m) \) is in \( B(H^1) \). Hence the following theorem results from the Ricci-Weiss result and our Theorem 7.

**Theorem 10.** If \( U \) is a distribution that is homogeneous of degree \(-2\) and has an extension in \( B(H^1(\mathbb{R}^2)) \), then the kernel \( u \) of \( U \) is in \( H^1(\mathbb{T}) \).

Note that Corollary 8 follows immediately from Theorem 10.

It is well known that if \( u \in H^1(\mathbb{T}) \), then the Calderon-Zygmund singular integral \( T_m \) with kernel \( u \) is bounded on \( L^p \) for \( 1 < p < \infty \), and of course, \( \sum_{n \neq 0} |\hat{u}(n)|^p < \infty \) holds. See for example Connett [3]. Thus \( u \in H^1(\mathbb{T}) \) is a sufficient condition for boundedness of \( T_m \) on \( L^p \) for \( 1 < p < \infty \). Theorem 10 gives a corresponding necessary condition, with \( H^1 \) replacing \( L^p \).

**Remarks on sufficient conditions for \( T_m \) to be in \( B(H^1) \).** The condition that \( m \) have an absolutely convergent Fourier series is not sufficient to guarantee that \( T_m \) extends to an operator on \( H^1 \). This is easily seen by considering lacunary series and using Theorem 7. For example, if \( m(\theta) = \sum e^{i\theta n^2} / n^2 \), then \( |\hat{u}(2^n)| = 2^n / n^2 \) and \( u \) cannot be a measure. In fact, by considering similar examples, one sees that conditions of the form

\[
\sum n^\delta |\hat{u}(n)| < \infty
\]

for \( 0 \leq \delta < 1 \) are insufficient to guarantee that \( T_m \in B(H^1) \). Conversely, Taibleson and Weiss [12] show that the condition

\[
\sum |\hat{m}(n)|^2 n^4 < \infty
\]

implies that \( T_m \) is bounded on \( H^p \) for \( 2/3 < p < 1 \), by showing that \( T_m \) sends atoms to molecules. Daly [4] has shown that if \( T_m \) sends atoms to molecules boundedly, then condition (6) is satisfied. Taken together, the state of results on sufficient conditions for \( T_m \in B(H^p) \) in terms of \( m \) appears to be these results by Daly, Taibleson, and Weiss and our Theorem 9.

The condition that \( S \ast m \) is of bounded variation is not sufficient to imply that \( T_m \) is bounded on \( H^p(0 < p \leq 1) \). This can be seen by considering any odd function \( m \) that is of bounded variation, but not continuous, on \( \mathbb{T} \).

**References**


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