ON THE CLASSIFICATION OF HOMOGENEOUS MULTIPLIERS BOUNDED ON $H^1(\mathbb{R}^2)$

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Abstract. Necessary and sufficient conditions for Calderon-Zygmund singular integral operators to be bounded operators on $H^1(\mathbb{R}^2)$ are investigated. Let $m$ be a bounded measurable function on the circle, extended to $\mathbb{R}^2$ by homogeneity ($m(rx) = m(x)$). If the Calderon-Zygmund singular integral operator $T_m$, defined by $T_m f = \mathcal{F}^{-1}(m \mathcal{F}(f))$, is bounded on $H^1(\mathbb{R}^2)$, then it is proved that $S^*m$ has bounded variation on the circle, where the Fourier transform of $S$ on the circle is $\mathcal{S}(n) = (-i \text{sgn}(n))^{n+1}$. This implies that $m$ must have an absolutely convergent Fourier series on the circle, and other relations on the Fourier series of $m$. Partial converses are also given. The problems are formulated in terms of distributions on the circle and on $\mathbb{R}^2$.

1. Introduction

The principal subject of this paper is classification of Calderon-Zygmund singular integrals that are bounded operators on $H^1(\mathbb{R}^2)$. These operators have kernels which are homogeneous of degree $-2$ and multipliers which are homogeneous of degree 0. We treat the problem in the general context of distributions. More precisely, suppose that $m$ is a function or distribution on $\mathbb{R}^2$ which is homogeneous of degree 0 and let $U$ be the distribution on $\mathbb{R}^2$ defined by $\mathcal{F}(U) = m$, where $\mathcal{F}$ denotes the Fourier transform on $\mathbb{R}^2$. We investigate necessary and sufficient conditions on $m$ so that $U$ extends to a bounded operator on $H^1(\mathbb{R}^2)$. We use the definition of $H^1(\mathbb{R}^2)$ in terms of atomic decompositions, as in Coifman and Weiss [2].

A first necessary condition is that $m$ be a bounded measurable function, for if the multiplier operator $T_m(f) = \mathcal{F}^{-1}(m \mathcal{F}(f))$ is a bounded operator on $H^1$, then by duality we have $T_m: \text{BMO} \to \text{BMO}$ continuously. By interpolation it follows that $T_m: L^2 \to L^2$ continuously, and hence that $m$ must be a bounded measurable function. The problem is what additional conditions

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upon \( m \) force \( T_m \in B(H^1) \), the bounded operators on \( H^1(\mathbb{R}^2) \). We obtain necessary conditions in terms of the bounded variation of certain convolutions with \( m \). We also prove that these necessary conditions are sufficient for \( T_m \) to map \( L^p \) boundedly to \( L^p \) for \( p > 1 \). In [12], Taibleson and Weiss have shown that if \( T_m \) is in \( B(H^p) \) for \( p \leq 1 \) then \( m \) is continuous on \( \mathbb{R}^2 \setminus \{0\} \). The replacement of \( L^1 \) by \( H^1 \) for this study is natural as it is well known that \( T_m \) is not bounded on \( L^1 \) if \( m \) is not constant. In fact, \( T_m(L^1) \) is not contained in \( L^1 \). See Stein [10, p. 42] for specifics.

In related work, Daly [4] provided a complete classification of those homogeneous multiplier operators that send atoms to molecules boundedly in \( H^p(\mathbb{R}^n) \) for \( 0 < p \leq 1 \). This classification, the \( L^p \) result indicated above (see Theorem 7) and conditions found by Taibleson and Weiss [12] are the most complete sufficiency results for \( \mathbb{R}^n \).

The corresponding extension and classification problem for homogeneous multipliers for local fields is solved in [5].

In §2 we analyze homogeneous distributions and obtain a polar decomposition for them. The results will be given for \( \mathbb{R}^n \), although in this paper we will only use the results for \( n = 2 \). The results in §2 are of independent interest, and will provide a basis for the corresponding \( H^1 \) problem in \( \mathbb{R}^n \) for \( n > 2 \).

For notational purposes we will follow Hörmander [8] as much as possible. Both [8] and Donoghue [6] contain results on homogeneous distributions. For \( H^p \) theory we follow Coifman, Taibleson, and Weiss [2, and 12].

### 2. Homogeneous Distributions

Three subsets of \( \mathbb{R}^n \) play a role in our analysis: \( \mathbb{R}^n \), \( \mathbb{R}^n \setminus \{0\} \), and the unit sphere \( \sum_{n-1} \). If \( X \) is one of the three, let \( C_0^\infty(X) \) denote the infinitely differentiable functions on \( X \) having compact support. For a compact subset \( K \) of \( X \) and a nonnegative integer \( k \), let

\[
\|\varphi\|_{K,k} = \sum_{|a| \leq k} \sup\{|D^a\varphi(x)| : x \in K\} .
\]

These seminorms define the topology of \( C_0^\infty(X) \); see [8, p. 34]. The space \( D'(X) \) of distributions on \( X \) is the dual of \( C_0^\infty(X) \) with this topology. So-called test functions are elements of \( C_0^\infty(X) \). In the case \( X = \sum_{n-1} \), the space \( X \) is itself compact and so the only \( K \) needed in the above definition is \( \sum_{n-1} \); and, \( C_0^\infty(\sum_{n-1}) = C_0^\infty(\sum_{n-1}) \).

The value of \( U \) in \( D' \) at \( \phi \) is denoted \( \langle U, \phi \rangle \). A distribution \( U \) in \( D'(\mathbb{R}^n) \) or \( D'(\mathbb{R}^n \setminus \{0\}) \) is said to be homogeneous of degree \( a \), for \( a \in \mathbb{R} \), if

\[
\langle U, \phi \rangle = t^{a+n} \langle U, \phi_t \rangle
\]

where \( \phi_t(x) = \phi(tx) \) for \( t > 0 \).

If \( u \) is a distribution in \( D'(\sum_{n-1}) \), then \( u \) can be used to obtain a distribution \( U_a \) on \( \mathbb{R}^n \setminus \{0\} \) that is homogeneous of degree \( a \), in the following manner.
For $a \in \mathbb{R}$ and $\varphi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$, define $U_a$ by

$$\langle U_a, \varphi \rangle = \int_0^\infty r^{n-1+a} \langle u, \varphi_r \rangle \, dr$$

where $\varphi_r$ is the function defined on $\sum_{n-1}$ by $\varphi_r(x') = \varphi(rx')$. As $\varphi$ has compact support, the integral above converges absolutely and defines a linear functional $U_a$ which is in $D'(\mathbb{R}^n \setminus \{0\})$. The distribution $u$ is said to be the kernel of the distribution $U_a$.

If $a = -(n + k)$ and $k$ is not a nonnegative integer, then $U_a$ has a unique extension to a distribution $\tilde{U}_a$ in $D'(\mathbb{R}^n)$. See Hörmander [8], Theorem 3.2.3. If $k$ is a nonnegative integer, then $U_a$ has an extension $\tilde{U}_a$ in $D'(\mathbb{R}^n)$ if and only if $\langle U_a, x^\alpha \psi \rangle = 0$ for radial $\psi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ and all nonnegative integer multi-indices $\alpha$ such that $|\alpha| = k$. To obtain this condition and hence the extension to $\mathbb{R}^n$, it is easy to see that we must have $\langle u, x^\alpha \rangle = 0$ for $|\alpha| = k$. If $k = 0$, this condition reduces to $\langle u, 1 \rangle = 0$. The homogeneous extension $\tilde{U}_a$ in this case is only unique up to a linear combination of the derivatives of order $k$ of the Dirac measure $\delta$. This is because derivatives of order $k$ of $\delta$ are homogeneous of order $-(n + k)$ and are supported at zero. See Theorem 3.2.4 of Hörmander [8]. For the purposes of this paper, we will be interested in the case $(n, k, a) = (2, 0, -2)$ and the extension where the contribution of $\delta$ is zero.

Suppose next that $U \in D'(\mathbb{R}^n)$ is homogeneous of degree $a$. From Donoghue [6, p. 154], $U$ is a tempered distribution. That is, $U$ has a unique continuous extension to the space $S$ of rapidly decreasing functions on $\mathbb{R}^n$. We will show that there is a $u \in D'(\sum_{n-1})$ for which $U = U_a$. We need the following two lemmas.

**Lemma 1.** If $U \in D'(\mathbb{R}^n)$ is homogeneous of degree $a \geq -n$, then $U$ has finite order.

**Proof.** If $\varphi \in C_0^\infty(\mathbb{R}^n)$ and has support in the unit ball $B^1$, there exists a constant $c$ and an integer $N$ such that

$$(i) \quad |\langle U, \varphi \rangle| \leq c \cdot \|\varphi\|_{B^1, N}.$$

For any $\varphi \in C_0^\infty(\mathbb{R}^n)$, there exists $r \geq 1$ such that $\text{sup}(\varphi) \subset B^r$, the ball of radius $r$. Then $\varphi_{1/r}$ has support in $B^1$. Thus

$$|\langle U, \varphi_{1/r} \rangle| \leq c \cdot \|\varphi_{1/r}\|_{B^1, N} \leq c \cdot \sum_{k=0}^N r^{-k} \sum_{|\alpha| = k} \|D^\alpha \varphi\|_\infty \leq c \cdot \|\varphi\|_{B^r, N}.$$

As $U$ is homogeneous of degree $a$, $\langle U, \varphi_{1/r} \rangle = r^{a+n} \langle U, \varphi \rangle$. So we have

$$|\langle U, \varphi \rangle| \leq r^{-a-n} c \cdot \|\varphi\|_{B^r, N}.$$

Thus $U$ has finite order if $a \geq -n$. (If $N$ is the smallest integer for which (i) holds, then the order of $U$ is $N$.)
Lemma 2. If the distribution $U$ is homogeneous of degree $a \geq -n$, then there exists a continuous function $g$ and a multi-index $\alpha$ such that, for $\varphi \in C^\infty_0(\mathbb{R}^n)$,

\[
\langle U, \varphi \rangle = \langle D^\alpha g, \varphi \rangle = \langle g, (-1)^{|\alpha|} D^\alpha \varphi \rangle.
\]

Proof. From Donoghue [6, p. 106], a distribution of finite order $N$ is an $(N+2)$ derivative of a continuous function. □

We can now state and prove the main theorem on the correspondence between homogeneous distributions on $\mathbb{R}^n$ and $\mathbb{R}^n \setminus \{0\}$ and distributions on $\sum_{n-1}$.

Theorem 3. Let $a \in \mathbb{R}$.

(1) If $u \in D'(\sum_{n-1})$ and $a \neq -(n+k)$ where $k$ is a nonnegative integer, define $U$ for $\varphi \in C^\infty_0(\mathbb{R}^n \setminus \{0\})$ by

\[
\langle U, \varphi \rangle = \int_0^\infty r^{n-1+a} \langle u, \varphi_r \rangle \, dr.
\]

Then $U$ is in $D'(\mathbb{R}^n \setminus \{0\})$ and has a unique extension to a distribution in $D'(\mathbb{R}^n)$ that is homogeneous of degree $a$.

(2) If $U \in D'(\mathbb{R}^n)$ is homogeneous of degree $a$ and $a$ is not of the form $-(n+k)$ for a nonnegative integer $k$, then there exists $u \in D'(\sum_{n-1})$ such that

\[
\langle U, \varphi \rangle = \int_0^\infty r^{n-1+a} \langle u, \varphi_r \rangle \, dr.
\]

for $\varphi \in C^\infty_0(\mathbb{R}^n \setminus \{0\})$.

(3) If $a = -(n+k)$ and $k$ is a nonnegative integer, then (1) holds if $\langle u, x^\alpha \rangle = 0$ for each nonnegative integer multi-index $\alpha$ with $|\alpha| = k$. In this case the extension to $D'(\mathbb{R}^n)$ is unique only up to addition of a linear combination of derivatives of order $k$ of $\delta$. If $U$ in $D'(\mathbb{R}^n)$ is homogeneous of degree $a$ as in (2), then $u$ exists and $\langle u, x^\alpha \rangle = 0$ must hold for $|\alpha| = k$.

Proof. We have proved (1). To prove (2), first suppose that $a > -n$. Then Lemma 2 applies and equality (ii) can be rewritten as

\[
\langle U, \varphi \rangle = \int_{\mathbb{R}^n} (-1)^{|\alpha|} g(x) D^\alpha(\varphi)(x) \, dx
\]

\[
= (-1)^{|\alpha|} \int_0^\infty r^{n-1} \int_{\sum_{n-1}} g(rx') D^\alpha(\varphi)(rx') \, dx' \, dr
\]

\[
= (-1)^{|\alpha|} \int_0^\infty r^{n-1+a} \int_{\sum_{n-1}} (r^{-a-|\alpha|} g(rx')) D^\alpha(\varphi_r)(x') \, dx' \, dr.
\]

Consider the function $h(rx') = r^{-(a+|\alpha|)} g(rx')$. The function $h$ is continuous on $\mathbb{R}^n \setminus \{0\}$ and, as $U$ is homogeneous of degree $a$, $h$ is homogeneous of degree 0. Thus $U$ can be written as

\[
\langle U, \varphi \rangle = \int_0^\infty r^{n-1+a} \langle u, \varphi_r \rangle \, dr
\]
where $u$ is defined by

$$(iv) \quad \langle u, \sigma \rangle = \int \sum_{n=1} h(x')(\cdot)^{\alpha} D^{\alpha}(\sigma)(x') \, dx'$$

for $\sigma \in C_0^\infty(\mathbb{R}^n)$. The derivative $D^{\alpha}(\sigma)$ is defined by extending $\sigma$ radially. Explicitly, choose a radial $\psi \in C_0^\infty(\mathbb{R}^n\setminus\{0\})$ that is identically equal to 1 on a neighborhood of $\sum_{n=1}$, and let $D^{\alpha}(\sigma)(x') = D^{\alpha}(\psi \sigma)(x')$. Then $u \in D'(\sum_{n=1})$ and $U$ is the $U_a$ defined by $u$. Note that for given $\phi \in C_0^\infty(\mathbb{R}^n)$, the function $\langle u, \phi \rangle$ has compact support on $\mathbb{R}^+$ and is a bounded function of $r$, so the function $\{r^{-(n-1+a)} \langle u, \phi \rangle \}$ is in $L^1(\mathbb{R})$. Hence the integral in (iii) is finite.

If $a < -n$, then $|\cdot|^{-a-n} U$ is a distribution in $D'(\mathbb{R}^n\setminus\{0\})$ which is homogeneous of degree $-n$. Hence by Lemma 2 and the above argument there is a continuous function $g$ on $\mathbb{R}^n$ and a multi-index $\alpha$ such that the function $h$ defined by

$$h(x) = |x|^{-(n+|\alpha|)} g(x)$$

and the distribution $u$ defined by

$$\langle u, \sigma \rangle = \int \sum_{n=1} h(x')(\cdot)^{\alpha} D^{\alpha}(\sigma)(x') \, dx'$$

satisfy

$$\langle |\cdot|^{-a-n} U, \phi \rangle = \int_0^\infty r^{-1} \langle u, \phi \rangle \, dr$$

for $\phi \in C_0^\infty(\mathbb{R}^n\setminus\{0\})$. The polar decomposition equality (iii) follows for $U$.

If $a = -(n+k)$ for a nonnegative integer $k$ and if $u \in D'(\sum_{n=1})$, then the formula in (1) defines an element $U$ of $D'(\mathbb{R}^n\setminus\{0\})$ which is homogeneous of degree $a$. The extension to $\mathbb{R}^n$ exists if $\langle u, x^{\alpha} \rangle = 0$ for all $\alpha$ satisfying $|\alpha| = k$. Conversely, if $U$ is in $D'(\mathbb{R}^n\setminus\{0\})$ and is homogeneous of degree $a = -(n+k)$, then the definition of $u$ in (iv) is the same as before and (iii) holds for $\phi \in C_0^\infty(\mathbb{R}^n\setminus\{0\})$. If $U$ extends to $D'(\mathbb{R}^n\setminus\{0\})$, then $\langle U, x^{\alpha} \phi \rangle = 0$ must hold for radial $\psi$. It follows easily that $\langle u, x^{\alpha} \rangle = 0$ must hold. Hence we have proved (3). □

Further remarks. The formulas in Theorem 3 hold for $\phi \in C_0^\infty(\mathbb{R}^n)$ if $a > -n$ but the extensions from $\mathbb{R}^n\setminus\{0\}$ to $\mathbb{R}^n$ are given by limits if $a \leq -n$. Since $U$ is homogeneous it is also tempered and so in fact $U$ extends continuously to $S$ as well. It is easy to see that the formulas hold for those $\phi$ in $S$ for which $\phi$ is zero on a neighborhood of $0$ and for all $\phi$ in $S$ if $a > -n$.

If $U$ is homogeneous of degree $a$, then the Fourier transform $\mathcal{F}(U)$ is homogeneous of degree $n+a$; it too is tempered. In the case $a = -n$, $\mathcal{F}(U)$ is homogeneous of degree 0. The element $m$ of $D'(\sum_{n=1})$ corresponding to $\mathcal{F}(U)$ as in Theorem 3, is called the multiplier of $U$. The distribution $U$ defines in the usual way an operator $T_m$ on $C_0^\infty(\mathbb{R}^n)$ by

$$T_m(\phi)(x) = \langle U, \tau_x \phi \rangle$$
where $\tau_x$ is translation by $-x$ and $\phi(x) = \phi(-x)$. Since $U$ is tempered, this formula extends to $\varphi \in S$. This paper concerns the extension of $T_m$ from $S$ as an operator on Hardy spaces.

3. Multiplier operators

In the remainder of this paper we will be concerned only with distributions on $\mathbb{R}^2$ that are homogeneous of degree $-2$. The distribution $u$ in $D'(\mathbb{R}^2)$ plays the role of the usual kernel in the construction of Calderon-Zygmund singular integrals. We let $\mathbb{S} = T$, the circle. We will always assume that $U$ has no delta measure component (recall that $\delta$ is homogeneous of degree $-2$). The Fourier transform of $U$ is homogeneous of degree 0 and so there is a unique distribution $m$ on $T$ satisfying

$$\langle \mathcal{F}(U), \varphi \rangle = \int_{0}^{\infty} \langle m, \varphi_r \rangle r \, dr.$$ 

In this case the relationship between $u$ and $m$ can be stated explicitly, in terms of the Fourier transform on the circle $T$, as

$$u(n) = n \text{sgn}(n) n^{-1} \cdot m(n)/2\pi i$$ 

for $n \neq 0$, where $\text{sgn}(n)$ denotes the sign of $n$.

We give two proofs of (1). From Stein [10], for dimension 2, $m$ can be computed from $u$ on $T$ by

$$m = -[(i\pi/2) \text{sgn}(|\cos(\cdot)|) + \log |\cos(\cdot)|] * u.$$ 

A straightforward computation gives, for $n \neq 0$,

$$-[(i\pi/2) \text{sgn}(|\cos(\cdot)|) + \log |\cos(\cdot)|] \hat{u}(n) = 2\pi i (\text{sgn}(n))^{-(n+1)}/n.$$ 

Expression (1) follows immediately.

The formula (1) can also be derived by directly computing the Fourier transform of the distribution $e^{inx}/|x|^2$ in $D'(\mathbb{R}^2 \setminus \{0\})$ using the methods of the proof of Theorem 5, infra. We give this derivation, which is heuristic in that existence of certain integrals needs to be established.

Let $\chi_n(\theta) = e^{inx}$, $n \neq 0$. Each $\chi_n \cdot |\cdot|^2$ for a real is locally integrable on $\mathbb{R}^2 \setminus \{0\}$ and so defines a distribution. Let $\varphi \in C_0^\infty (\mathbb{R}^2 \setminus \{0\})$. Using Theorem 3 and letting $y = se^{i\beta}$, we have

$$\langle \mathcal{F}(\chi_n \cdot |\cdot|^2), \varphi \rangle = \langle \chi_n, |\cdot|^2, \mathcal{F}(\varphi) \rangle$$ 

$$= \int_{0}^{\infty} \int_{0}^{2\pi} \frac{1}{r} e^{in\theta} \int_{\mathbb{R}^2} \varphi(se^{i\beta}) e^{-i < se^{i\beta}, re^{i\theta}>} dy \, d\theta \, dr$$ 

$$= \int_{\mathbb{R}^2} \varphi(se^{i\beta}) \int_{0}^{2\pi} \frac{1}{r} e^{in\theta} e^{-i < se^{i\beta}, re^{i\theta}>} d\theta \, dr \, dy.$$ 

The inner-product in $\mathbb{R}^2$ can be rewritten using $\langle e^{i\beta}, e^{i\theta} \rangle = \cos(\theta - \beta)$. The $\theta$-integral is a Bessel function, namely

$$\int_{0}^{2\pi} e^{in\theta} e^{-isr \cos(\theta - \beta)} d\theta = 2\pi e^{in\theta} (-i)^n J_n(rs).$$
where \( J_n \) is the \( n \)th-Bessel function. Using the multiplicative invariance of the measure \( r^{-1} \, dr \) on \( \mathbb{R}^+ \), the equality \( J_{-n}(r) = (-1)^n J_n(r) \) for \( n > 0 \), and integration formulas from Watson [13, p. 391, no. 385], the \( r \)-integral becomes

\[
\int_0^\infty \frac{1}{r} J_n(rs) \, dr = (\text{sgn}(n))^{n+1}/n.
\]

Thus

\[
\langle \mathcal{F}(\chi_n) \cdot |\cdot|^{-2}, \varphi \rangle = (2\pi i)^{-n}(\text{sgn}(n))^{n+1}/n \langle \chi_n, \varphi \rangle.
\]

We have shown that if we take \( u = \chi_n \), then \( m = c_n \chi_n \) where \( c_n \) is the constant on the right. The relationship (1) between the Fourier transforms of \( u \) and \( m \) on \( T \) follows for any \( m \) and \( u \).

Since the degree of \( U \) is \(-2\), for the extension of \( U \) to \( \mathbb{R}^2 \) it is necessary that \( \hat{u}(0) = 0 \). Since we have assumed that \( U \) has no delta measure component, it is also true that \( \hat{m}(0) = 0 \).

Define the convolution operator \( S \) on \( C^\infty(T) \) by

\[
S(n) = 2\pi(-i\text{sgn}(n))^{n+1}.
\]

The operator \( S \) will play a central role in the classification of those homogeneous multipliers that give rise to bounded operators on \( H^1(\mathbb{R}^2) \). If we let \( \tau_\theta \) denote translation by \( \theta \), then a straightforward computation shows that \( S \) is invertible with inverse

\[
S^{-1} = (1/4\pi^2)\tau_\pi S.
\]

In fact, \( S \) can be written as

\[
S = (1/2)\tau_{\pi/2}\{H(I + \tau_\pi) + i(I - \tau_\pi)\}
\]

where \( H \) is the Hilbert transform on the circle, defined by \( \hat{H}(n) = -i\text{sgn}(n) \).

In terms of \( S \), the relationship (1) between \( u \) and \( m \) is

\[
\hat{u}(n) = (2\pi)^{-2}i(-1)^n \cdot \hat{S}(n)\hat{m}(n).
\]

As we observed in the introduction, if \( T_m \) extends to a bounded operator on \( L^2(\mathbb{R}^2) \), then \( m \) must be a bounded measurable function on \( T \). Thus \( S \ast m \) is a function and we have

\[
(2\pi)^2i(-1)^{n+1}\hat{u}(n)/n = (S \ast m)^\wedge(n).
\]

Denoting the distributional derivative by \( D \), it follows that we may write \( u \) in \( D'(T) \) as

\[
u = -D(S^{-1} \ast m).
\]

From Edwards [7, Theorem 12.5.16], \( u \) is a measure if and only if \( \hat{u}(n)/n \) is the Fourier transform of a function of bounded variation. Hence we have the following lemma.

**Lemma 4.** The function \( S \ast m \) has bounded variation if and only if \( u \) is a measure.

We will now prove the main result of the paper.
Theorem 5. Suppose \( m \) is a bounded measurable function on \( \mathbb{R}^2 \) which is homogeneous of degree 0. If the singular integral operator \( T_m \) is in \( B(H^1) \), then the functions \( S \ast m \), \( S \ast (\sin(\cdot)m) \), and \( S \ast (\cos(\cdot)m) \) have bounded variation on \( T \).

Proof. Assume \( T_m \) is in \( B(H^1) \). Let \( \eta \) be a radial function in \( C^\infty(\mathbb{R}^2) \) that is supported on \( \{ x : 1/4 \leq |x| \leq 4 \} \) and is identically equal to 1 on \( \{ x : 1/2 \leq x \leq 2 \} \). Then \( M = F^{-1}(\eta) \) is in \( H^1(\mathbb{R}^2) \); see Taibleson and Weiss [12, p. 137]. Thus \( T_m(M) \) is in \( L^1(\mathbb{R}^2) \). For \( a > 0 \), we have

\[
\| T_m(M) \|_1 \geq \int_{\mathbb{R}^2} e^{-|z|} |T_m(M)| (z) \, dz
\]

(2)

If \( m \) is expanded in a Fourier series, \( \{ A_n \} \) is any decomposition of \( \mathbb{R}^2 \), \( x^\prime \) is replaced by \( e^{i\theta} \), and \( z \) equals \( se^{i\theta} \), then (2) becomes

\[
\| T_m(M) \|_1 \geq \sum_n \int_{A_n} e^{-as} \left| \int_0^\infty \eta(r) \sum_k \hat{m}(k) 2\pi (-i)^k e^{ik\theta} s \int_0^r J_k(rs) s \, dr \right| \, dz \]

Performing the \( \theta \)-integration, we obtain

(3) \[
\| T_m(M) \|_1 \geq \sum_n \left| \int_{A_n} e^{-as} \int_0^\infty \eta(r) \sum_k \hat{m}(k) 2\pi (-i)^k e^{ik\theta} J_k(rs) s \, dr \right| \, dz
\]

where \( J_k \) is the \( k \)th-Bessel function. If we let \( \{ A_n \} \) be the conical decomposition of \( \mathbb{R}^2 \) induced by a decomposition \( 0 = b_0 < b_1 < \cdots < b_N = 2\pi \) for the circle, (3) becomes

(4) \[
\| T_m(M) \|_1 \geq \sum_n \left| \int_0^\infty \eta(r) \sum_k \hat{m}(k) 2\pi (-i)^k e^{ik\theta} J_k(rs) s \, dr \right| \, dz
\]

Using \( J_k = (\text{sgn}(k))^k J_{|k|} \) and an integration formula from Watson [13, p. 385], the \( s \)-integral can be explicitly evaluated as

(5) \[
\frac{(\text{sgn}(k))^k (r/2)^k (1 + |k|) \, 2F_1((|k| + 2)/2, (|k| - 1)/2; |k| + 1; r^2/(a^2 + r^2)))}{(a^2 + r^2)^{|k|/2}}
\]

where \( 2F_1 \) is a generalized hypergeometric function. Call the function displayed in (5) \( C(k, a, r) \). For \( a > 0 \), \( r\eta(r)C(k, a, r) \) is integrable and is dominated by \( r\eta(r)C(k, 0, r) \). The function \( C(k, 0, r) \) is given by

\[
C(k, 0, r) = \frac{(\text{sgn}(k))^k (1 + |k|) \, 2F_1((|k| + 2)/2, (|k| - 1)/2; |k| + 1; 1)}{2^k r^2 |k|^2}
\]

(6) \[
= (\text{sgn}(k))^k |k|r^{-2}
\]
by using Gauss’s Formula for \( _2F_1(\alpha, \beta; \gamma; 1) \) (see Bateman Manuscript [1, p. 61]). Substituting this into (4) and performing the \( \beta \)-integration, we obtain the bound

\[
\|T_m(M)\|_1 \geq C_\eta \sum_n \left| \sum_k 2\pi (-i)^{k+1} \text{sgn}(k)^{k+1} \hat{m}(k)(e^{-ikb_n} - e^{-ikb_{n-1}}) \right|
\]

\[
= C_\eta \sum_n |S * m(e^{ib_n}) - S * m(e^{ib_{n-1}})|.
\]

As \( \|T_m(M)\|_1 \) is finite, \( S * m \) is of bounded variation on \( T \).

To show that \( S*(\sin(\cdot)z)z \) and \( S*(\cos(\cdot)z)z \) must also be of bounded variation, we consider the two Riesz transforms \( R_1 \) and \( R_2 \), defined by \( \mathcal{F}(R_1)(re^{i\theta}) = i \cdot \sin(\theta) \) and \( \mathcal{F}(R_2)(re^{i\theta}) = i \cdot \cos(\theta) \). These operators are bounded on \( H^1 \), and hence, \( R_1T_m \) and \( R_2T_m \) are bounded on \( H^1 \) if \( T_m \) is bounded. Applying the preceding argument to these operators, we see that, \( S * (\sin(\cdot)z)z \) and \( S * (\cos(\cdot)z)z \) are of bounded variation. \( \square \)

**Theorem 6.** Suppose \( m \) is a bounded measurable function on \( \mathbb{R}^2 \) which is homogeneous of degree 0 and such that \( T_m \) is in \( B(H^1) \). Then \( m \) has an absolutely convergent Fourier series as a function on \( T \).

**Proof.** From Theorem 5, if \( T_m \in B(H^1) \), then \( S * m \), \( S * (\sin(\cdot)z)z \), and \( S * (\cos(\cdot)z)z \) are of bounded variation on \( T \). Thus \( (-\text{sgn}(n))^{n+1}\hat{m}(n) \), \( (-\text{sgn}(n))^{n+1}(\hat{m}(n+1) + \hat{m}(n-1)) \), and \( (-\text{sgn}(n))^{n+1}(\hat{m}(n+1) - \hat{m}(n-1)) \) are the Fourier series of functions of bounded variation; and consequently, \( (-\text{sgn}(n))^{n+1}\hat{m}(n+1) \) is the Fourier transform of a function of bounded variation. Taken together, all this implies that the Fourier transform

\[
(H * S * m)^{(n)} = (-\text{sgn}(n)) \cdot (-\text{sgn}(n))^{n+1}\hat{m}(n)
\]

is a finite sum of transforms of functions of bounded variation. From Zygmund [14, v. 1, p. 242], as both \( S * m \) and \( H*S*m \) are of bounded variation, \( S * m \) has an absolutely convergent Fourier series. However, \( |(S*m)^{(n)}| = 2\pi|\hat{m}(n)| \) and so \( m \) has an absolutely convergent Fourier series. \( \square \)

**Theorem 7.** Suppose \( m \) is a bounded measurable function on \( \mathbb{R}^2 \) which is homogeneous of degree 0 and \( T_m \) is in \( B(H^1) \). Then \( u \) is a measure on \( T \) and \( u \) is absolutely continuous with respect to Lebesgue measure.

**Proof.** Theorem 5 and Lemma 4 imply that \( u \) is a measure. The kernel of \( H*m \) is \( H*u \), because both \( H \) and the correspondence between kernels and multipliers are linear. In the proof of Theorem 6 it was shown that \( H*S*m \) has finite variation. Since \( H*S = S*H \), it follows that \( S*H*m \) has finite variation. Hence, by Lemma 4, \( H*u \) is also a measure. By the F. Riesz and M. Riesz Theorem, \( u \) is absolutely continuous. See Zygmund [14, v. 1, p. 285], and also Edwards [7, p. 99]. \( \square \)

Theorem 6 can be phrased for kernels \( u \). If \( u \) is a kernel in \( D'(T) \) and \( T_m \in B(H^1) \), then by Theorem 7 \( u \) is in \( L^1(T) \). By Theorem 6, \( m \) has an
absolutely convergent Fourier series as a function on $T$. However, $|\hat{m}(n)| = |\hat{n}(n)|/|n|$ if $n \neq 0$, and so $\sum_{n \neq 0} |\hat{u}(n)|/|n| < \infty$. We have proved:

**Corollary 8.** If $u$ is in $L^1(T)$ and the corresponding singular integral $T_m$ is in $B(H^1)$, then $\sum_{n \neq 0} |\hat{u}(n)|/|n| < \infty$.

We now provide a partial converse to Theorem 5.

**Theorem 9.** Suppose $m$ is a bounded measurable function on $\mathbb{R}^2$ which is homogeneous of degree 0. If $S \ast m$, $S \ast (\cos(\cdot)m)$, $S \ast (\sin(\cdot)m)$ are of bounded variation on $T$, then $T_m$ is in $B(L^p)$ for $1 < p < \infty$.

**Proof.** Let $BV(T)$ be the functions of bounded variation and suppose $S \ast m$, $S \ast (\cos(\cdot)m)$, $S \ast (\sin(\cdot)m)$ are in $BV(T)$. A periodic function $f$ on $\mathbb{R}$ is even or odd as a function on the circle according as $f(t) = f(t+\pi)$ or $f(t) = -f(t+\pi)$. Note that $\cos(t)$ as a function on $T$ is odd, not even. The even and odd parts of a function $f$ are given by

$$f_e(\theta) = \frac{(f(\theta) + f(\theta + \pi))}{2}, \quad f_o(\theta) = \frac{(f(\theta) - f(\theta + \pi))}{2}.$$ 

Clearly $f \in BV(T)$ if and only if $f_0$ and $f_e \in BV(T)$. As $S$ is a multiplier operator, $(S \ast m)_0 = S \ast (m_0)$ and $(S \ast m)_e = S \ast (m_e)$. For the odd function $m_0$, we have $m_0 \in BV(T)$ as $S$ reduces to a translation operator for odd functions. By Lemma 4, the corresponding kernel $u_0$ is a measure. As a measure on $[-\pi, \pi]$, it satisfies

$$\int g(t + \pi) \, du_0(t) = -\int g(t) \, du_0(t)$$

for periodic functions $g$. That is, $u_0$ is an odd measure. A straightforward adaptation of the “method of rotations” shows that $T_{m_0} \in B(L^p)$ for $1 < p < \infty$. See Theorem (2.6) and the proof preceding it in Chapter VI of [11], where the kernel is an odd function in $L^1(T)$. For the even multiplier $m_e$, the functions $\sin(\cdot)m_e$, $\cos(\cdot)m_e$ are both odd functions. We have

$$S \ast (\cos(\cdot)m_e) = (S \ast (\cos(\cdot)m))_0, \quad S \ast (\sin(\cdot)m_e) = (S \ast (\sin(\cdot)m))_0,$$

and so $S \ast (\cos(\cdot)m_e)$ and $S \ast (\cos(\cdot)m_e)$ are in $BV(T)$. Hence $\sin(\cdot)m_e$ and $\cos(\cdot)m_e$ are also in $BV(T)$ and so are multipliers of kernels which are odd measures. As before, the corresponding operators are in $B(L^p)$; that is, the Riesz transforms

$$R_1 T_{m_e}, \quad R_2 T_{m_e} \in B(L^p).$$

Each $R_j$ is also in $B(L^p)$, and so also is each $R_j^2 T_{m_e}$. So $T_{m_e} \in B(L^p)$ as $R_1^2 + R_2^2 = -I$. As $T_m = T_{m_0} + T_{m_e}$, we have $T_m \in B(L^p)$ for $1 < p < \infty$. □

In [9], Ricci and Weiss give a characterization of $H^1(\mathbb{R}^n)$ that is particularly suitable to our presentation here. They consider Calderon-Zygmund singular operators (distributions) $T$ with kernel $k(x) = u(x^\prime)/|x|^n$, where
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$u \in L^1(\sum_{n=1}^\infty)$ and has integral zero. By composing $T$ with the Riesz transforms $R_j$ ($1 \leq j \leq n$), one obtains $n$ distributions that are each homogeneous of degree $-n$. In general, these distributions may or may not be given by kernels in $L^1(\sum_{n=1}^\infty)$. Ricci and Weiss characterize $H^1(\sum_{n=1}^\infty)$ by showing that $u \in H^1(\sum_{n=1}^\infty)$ if and only if $R_j \circ T$ has a kernel $u_j \in L^1(\sum_{n=1}^\infty)$. In the case of dimension 2, we know from Theorem 7 that the operators $R_i \circ U$ and $R_2 \circ U$ have kernels $u_1$ and $u_2$ in $L^1(T)$ if $U(=T_m)$ is in $B(H^1)$. Hence the following theorem results from the Ricci-Weiss result and our Theorem 7.

**Theorem 10.** If $U$ is a distribution that is homogeneous of degree $-2$ and has an extension in $B(H^1(\mathbb{R}^2))$, then the kernel $u$ of $U$ is in $H^1(T)$.

Note that Corollary 8 follows immediately from Theorem 10.

It is well known that if $u \in H^1(T)$, then the Calderon-Zygmund singular integral $T_m$ with kernel $u$ is bounded on $L^p$ for $1 < p < \infty$, and of course, $\sum_{n \neq 0} \frac{|\hat{u}(n)|}{|n|} < \infty$ holds. See for example Connett [3]. Thus $u \in H^1(T)$ is a sufficient condition for boundedness of $T_m$ on $L^p$ for $1 < p < \infty$. Theorem 10 gives a corresponding necessary condition, with $H^1$ replacing $L^p$.

**Remarks on sufficient conditions for $T_m$ to be in $B(H^1)$**. The condition that $m$ have an absolutely convergent Fourier series is not sufficient to guarantee that $T_m$ extends to an operator on $H^1$. This is easily seen by considering lacunary series and using Theorem 7. For example, if $m(\theta) = \sum e^{it2^n}/n^2$, then $|\hat{u}(2^n)| = 2^n/n^2$ and $u$ cannot be a measure. In fact, by considering similar examples, one sees that conditions of the form

$$\sum n^\delta |\hat{m}(n)| < \infty$$

for $0 \leq \delta < 1$ are insufficient to guarantee that $T_m \in B(H^1)$. Conversely, Taibleson and Weiss [12] show that the condition

$$\sum |\hat{m}(n)|^2 n^4 < \infty$$

implies that $T_m$ is bounded on $H^p$ for $2/3 < p \leq 1$, by showing that $T_m$ sends atoms to molecules. Daly [4] has shown that if $T_m$ sends atoms to molecules boundedly, then condition (6) is satisfied. Taken together, the state of results on sufficient conditions for $T_m \in B(H^p)$ in terms of $m$ appears to be these results by Daly, Taibleson, and Weiss and our Theorem 9.

The condition that $S \ast m$ is of bounded variation is not sufficient to imply that $T_m$ is bounded on $H^p(0 < p \leq 1)$. This can be seen by considering any odd function $m$ that is of bounded variation, but not continuous, on $T$.

**REFERENCES**


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