ON THE CLASSIFICATION OF HOMOGENEOUS MULTIPLIERS BOUNDED ON $H^1(\mathbb{R}^2)$

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Abstract. Necessary and sufficient conditions for Calderon-Zygmund singular integral operators to be bounded operators on $H^1(\mathbb{R}^2)$ are investigated. Let $m$ be a bounded measurable function on the circle, extended to $\mathbb{R}^2$ by homogeneity ($m(rx) = m(x)$). If the Calderon-Zygmund singular integral operator $T_m$, defined by $T_m f = \mathcal{F}^{-1}(m \mathcal{F}(f))$, is bounded on $H^1(\mathbb{R}^2)$, then it is proved that $S^*m$ has bounded variation on the circle, where the Fourier transform of $S$ on the circle is $\hat{S}(n) = (-\text{sgn}(n))^{n+1}$. This implies that $m$ must have an absolutely convergent Fourier series on the circle, and other relations on the Fourier series of $m$. Partial converses are also given. The problems are formulated in terms of distributions on the circle and on $\mathbb{R}^2$.

1. Introduction

The principal subject of this paper is classification of Calderon-Zygmund singular integrals that are bounded operators on $H^1(\mathbb{R}^2)$. These operators have kernels which are homogeneous of degree $-2$ and multipliers which are homogeneous of degree 0. We treat the problem in the general context of distributions. More precisely, suppose that $m$ is a function or distribution on $\mathbb{R}^2$ which is homogeneous of degree 0 and let $U$ be the distribution on $\mathbb{R}^2$ defined by $\mathcal{F}(U) = m$, where $\mathcal{F}$ denotes the Fourier transform on $\mathbb{R}^2$. We investigate necessary and sufficient conditions on $m$ so that $U$ extends to a bounded operator on $H^1(\mathbb{R}^2)$. We use the definition of $H^1(\mathbb{R}^2)$ in terms of atomic decompositions, as in Coifman and Weiss [2].

A first necessary condition is that $m$ be a bounded measurable function, for if the multiplier operator $T_m(f) = \mathcal{F}^{-1}(m \mathcal{F}(f))$ is a bounded operator on $H^1$, then by duality we have $T_m : BMO \to BMO$ continuously. By interpolation it follows that $T_m : L^2 \to L^2$ continuously, and hence that $m$ must be a bounded measurable function. The problem is what additional conditions...
upon \( m \) force \( T_m \in B(H^1) \), the bounded operators on \( H^1(\mathbb{R}^2) \). We obtain necessary conditions in terms of the bounded variation of certain convolutions with \( m \). We also prove that these necessary conditions are sufficient for \( T_m \) to map \( L^p \) boundedly to \( L^p \) for \( p > 1 \). In [12], Taibleson and Weiss have shown that if \( T_m \) is in \( B(H^p) \) for \( p \leq 1 \) then \( m \) is continuous on \( \mathbb{R}^2 \setminus \{0\} \). The replacement of \( L \) by \( H^1 \) for this study is natural as it is well known that \( T_m \) is not bounded on \( L^1 \) if \( m \) is not constant. In fact, \( T_m(L^1) \) is not contained in \( L^1 \). See Stein [10, p. 42] for specifics.

In related work, Daly [4] provided a complete classification of those homogeneous multiplier operators that send atoms to molecules boundedly in \( H^p(\mathbb{R}^n) \) for \( 0 < p \leq 1 \). This classification, the \( L^p \) result indicated above (see Theorem 7) and conditions found by Taibleson and Weiss [12] are the most complete sufficiency results for \( R^n \).

The corresponding extension and classification problem for homogeneous multipliers for local fields is solved in [5].

In §2 we analyze homogeneous distributions and obtain a polar decomposition for them. The results will be given for \( \mathbb{R}^n \), although in this paper we will only use the results for \( n = 2 \). The results in §2 are of independent interest, and will provide a basis for the corresponding \( H^1 \) problem in \( \mathbb{R}^n \) for \( n > 2 \).

For notational purposes we will follow Hörmander [8] as much as possible. Both [8] and Donoghue [6] contain results on homogeneous distributions. For \( H^p \) theory we follow Coifman, Taibleson, and Weiss [2, and 12].

2. **Homogeneous distributions**

Three subsets of \( \mathbb{R}^n \) play a role in our analysis: \( \mathbb{R}^n \), \( \mathbb{R}^n \setminus \{0\} \), and the unit sphere \( \sum_{n-1} \). If \( X \) is one of the three, let \( C_0^\infty(X) \) denote the infinitely differentiable functions on \( X \) having compact support. For a compact subset \( K \) of \( X \) and a nonnegative integer \( k \), let

\[
\|\phi\|_{K,k} = \sum_{|\alpha| \leq k} \sup\{|D^\alpha \phi(x)| : x \in K\}.
\]

These seminorms define the topology of \( C_0^\infty(X) \); see [8, p. 34]. The space \( D'(X) \) of distributions on \( X \) is the dual of \( C_0^\infty(X) \) with this topology. So-called test functions are elements of \( C_0^\infty(X) \). In the case \( X = \sum_{n-1} \), the space \( X \) is itself compact and so the only \( K \) needed in the above definition is \( \sum_{n-1} \); and, \( C_0^\infty(\sum_{n-1}) = C_0^\infty(\sum_{n-1}) \).

The value of \( U \) in \( D' \) at \( \phi \) is denoted \( \langle U, \phi \rangle \). A distribution \( U \) in \( D'(\mathbb{R}^n) \) or \( D'(\mathbb{R}^n \setminus \{0\}) \) is said to be homogeneous of degree \( a \), for \( a \in \mathbb{R} \), if

\[
\langle U, \phi \rangle = t^{a+n} \langle U, \phi_t \rangle
\]

where \( \phi_t(x) = \phi(tx) \) for \( t > 0 \).

If \( u \) is a distribution in \( D'(\sum_{n-1}) \), then \( u \) can be used to obtain a distribution \( U_a \) on \( \mathbb{R}^n \setminus \{0\} \) that is homogeneous of degree \( a \), in the following manner.
For $a \in \mathbb{R}$ and $\varphi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$, define $U_a$ by

$$\langle U_a, \varphi \rangle = \int_0^\infty r^{n-1+a} \langle u, \varphi_r \rangle \, dr$$

where $\varphi_r$ is the function defined on $\sum_{n-1}$ by $\varphi_r(x') = \varphi(rx')$. As $\varphi$ has compact support, the integral above converges absolutely and defines a linear functional $U_a$ which is in $D'(\mathbb{R}^n \setminus \{0\})$. The distribution $u$ is said to be the kernel of the distribution $U_a$.

If $a = -(n + k)$ and $k$ is not a nonnegative integer, then $U_a$ has a unique extension to a distribution $\hat{U}_a$ in $D'(\mathbb{R}^n)$. See Hörmander [8], Theorem 3.2.3. If $k$ is a nonnegative integer, then $U_a$ has an extension $\hat{U}_a$ in $D'(\mathbb{R}^n)$ if and only if $\langle U_a, x^\alpha \psi \rangle = 0$ for radial $\psi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ and all nonnegative integer multi-indices $\alpha$ such that $|\alpha| = k$. To obtain this condition and hence the extension to $\mathbb{R}^n$, it is easy to see that we must have $\langle u, x^\alpha \rangle = 0$ for $|\alpha| = k$. If $k = 0$, this condition reduces to $\langle u, 1 \rangle = 0$. The homogeneous extension $\hat{U}_a$ in this case is only unique up to a linear combination of the derivatives of order $k$ of the Dirac measure $\delta$. This is because derivatives of order $k$ of $\delta$ are homogeneous of order $-(n + k)$ and are supported at zero. See Theorem 3.2.4 of Hörmander [8]. For the purposes of this paper, we will be interested in the case $(n, k, a) = (2, 0, -2)$ and the extension where the contribution of $\delta$ is zero.

Suppose next that $U \in D'(\mathbb{R}^n)$ is homogeneous of degree $a$. From Donoghue [6, p. 154], $U$ is a tempered distribution. That is, $U$ has a unique continuous extension to the space $S$ of rapidly decreasing functions on $\mathbb{R}^n$. We will show that there is a $u \in D'(\sum_{n-1})$ for which $U = U_a$. We need the following two lemmas.

**Lemma 1.** If $U \in D'(\mathbb{R}^n)$ is homogeneous of degree $a \geq -n$, then $U$ has finite order.

**Proof.** If $\varphi \in C_0^\infty(\mathbb{R}^n)$ and has support in the unit ball $B^1$, there exists a constant $c$ and an integer $N$ such that

$$|\langle U, \varphi \rangle| \leq c \cdot \|\varphi\|_{B^{1,N}}.$$

For any $\varphi \in C_0^\infty(\mathbb{R}^n)$, there exists $r \geq 1$ such that $\text{sup}(\varphi) \subset B^r$, the ball of radius $r$. Then $\varphi_{1/r}$ has support in $B^1$. Thus

$$|\langle U, \varphi_{1/r} \rangle| \leq c \cdot \|\varphi_{1/r}\|_{B^{1,N}} \leq c \cdot \sum_{k=0}^N r^{-k} \sum_{|\alpha|=k} \|D^\alpha \varphi\|_{\infty} \leq c \cdot \|\varphi\|_{B^r,N}.$$

As $U$ is homogeneous of degree $a$, $\langle U, \varphi_{1/r} \rangle = r^{-a+n} \langle U, \varphi \rangle$. So we have

$$|\langle U, \varphi \rangle| \leq r^{-a-n} c \cdot \|\varphi\|_{B^r,N}.$$

Thus $U$ has finite order if $a \geq -n$. (If $N$ is the smallest integer for which (i) holds, then the order of $U$ is $N$.) \qed
Lemma 2. If the distribution $U$ is homogeneous of degree $a \geq -n$, then there exists a continuous function $g$ and a multi-index $\alpha$ such that, for $\varphi \in C_0^\infty (\mathbb{R}^n)$,

$$(U, \varphi) = (D^\alpha g, \varphi) = (g, (-1)^{|\alpha|}D^\alpha \varphi).$$

Proof. From Donoghue [6, p. 106], a distribution of finite order $N$ is an $(N+2)$ derivative of a continuous function. □

We can now state and prove the main theorem on the correspondence between homogeneous distributions on $\mathbb{R}^n$ and $\mathbb{R}^n \setminus \{0\}$ and distributions on $\sum_{n-1}$.

Theorem 3. Let $a \in \mathbb{R}$.

(1) If $u \in D'(\sum_{n-1})$ and $a \neq -(n + k)$ where $k$ is a nonnegative integer, define $U$ for $\varphi \in C_0^\infty (\mathbb{R}^n \setminus \{0\})$ by

$$<U, \varphi> = \int_0^\infty r^{n+a} <u, \varphi> \, dr.$$ 

Then $U$ is in $D'(\mathbb{R}^n \setminus \{0\})$ and has a unique extension to a distribution in $D'(\mathbb{R}^n)$ that is homogeneous of degree $a$.

(2) If $U \in D'(\mathbb{R}^n)$ is homogeneous of degree $a$ and $a$ is not of the form $-(n + k)$ for a nonnegative integer $k$, then there exists $u \in D'(\sum_{n-1})$ such that

$$<U, \varphi> = \int_0^\infty r^{n+a} <u, \varphi> \, dr.$$ 

for $\varphi \in C_0^\infty (\mathbb{R}^n \setminus \{0\})$.

(3) If $a = -(n+k)$ and $k$ is a nonnegative integer, then (1) holds if $<u, x^\alpha> = 0$ for each nonnegative integer multi-index $\alpha$ with $|\alpha| = k$. In this case the extension to $D'(\mathbb{R}^n)$ is unique only up to addition of a linear combination of derivatives of order $k$ of $\delta$. If $U$ in $D'(\mathbb{R}^n)$ is homogeneous of degree $a$ as in (2), then $u$ exists and $<u, x^\alpha> = 0$ must hold for $|\alpha| = k$.

Proof. We have proved (1). To prove (2), first suppose that $a > -n$. Then Lemma 2 applies and equality (ii) can be rewritten as

$$<U, \varphi> = \int_{\mathbb{R}^n} (-1)^{|\alpha|} g(x)D^\alpha(\varphi)(x) \, dx$$

$$= (-1)^{|\alpha|} \int_0^\infty r^{n-1} \int_{\sum_{n-1}} g(rx')D^\alpha(\varphi)(rx') \, dx' \, dr$$

$$= (-1)^{|\alpha|} \int_0^\infty r^{n-1+a} \int_{\sum_{n-1}} (r^{-a-|\alpha|} g(rx'))D^\alpha(\varphi)(x') \, dx' \, dr.$$ 

Consider the function $h(rx') = r^{-(a+|\alpha|)} g(rx')$. The function $h$ is continuous on $\mathbb{R}^n \setminus \{0\}$ and, as $U$ is homogeneous of degree $a$, $h$ is homogeneous of degree 0. Thus $U$ can be written as

(iii) $$<U, \varphi> = \int_0^\infty r^{n-1+a} <u, \varphi> \, dr$$
where \( u \) is defined by

\[
\langle u, \sigma \rangle = \int_{\sum_{n-1}} h(x') (-1)^{\alpha} D^\alpha(\sigma)(x') \, dx'
\]

for \( \sigma \in C^\infty(\sum_{n-1}) \). The derivative \( D^\alpha(\sigma) \) is defined by extending \( \sigma \) radially. Explicitly, choose a radial \( \psi \in C^\infty(\mathbb{R}^n \setminus \{0\}) \) that is identically equal to 1 on a neighborhood of \( \sum_{n-1} \), and let \( D^\alpha(\sigma)(x') = D^\alpha(\psi \sigma)(x') \). Then \( u \in D'(\sum_{n-1}) \) and \( U \) is the \( U_a \) defined by \( u \). Note that for given \( \varphi \in C^\infty_0(\mathbb{R}^n) \), the function \( \langle u, \varphi \rangle \) has compact support on \( \mathbb{R}^+ \) and is a bounded function of \( r \), so the function \( \{ r^{n-1+a} \langle u, \varphi \rangle \} \) is in \( L^1(\mathbb{R}^+) \). Hence the integral in (iii) is finite.

If \( a < -n \), then \( | \cdot |^{-a-n} U \) is a distribution in \( D'(\mathbb{R}^n \setminus \{0\}) \) which is homogeneous of degree \( -n \). Hence by Lemma 2 and the above argument there is a continuous function \( g \) on \( \mathbb{R}^n \) and a multi-index \( \alpha \) such that the function \( h \) defined by

\[
h(x) = |x|^{-(n+|\alpha|)} g(x)
\]

and the distribution \( u \) defined by

\[
\langle u, \sigma \rangle = \int_{\sum_{n-1}} h(x') (-1)^{\alpha} D^\alpha(\sigma)(x') \, dx'
\]

satisfy

\[
\langle | \cdot |^{-a-n} U, \varphi \rangle = \int_0^\infty r^{-1} \langle u, \varphi \rangle \, dr
\]

for \( \varphi \in C^\infty_0(\mathbb{R}^n \setminus \{0\}) \). The polar decomposition equality (iii) follows for \( U \).

If \( a = -(n+k) \) for a nonnegative integer \( k \) and if \( u \in D'(\sum_{n-1}) \), then the formula in (1) defines an element \( U \) of \( D'(\mathbb{R}^n \setminus \{0\}) \) which is homogeneous of degree \( a \). The extension to \( \mathbb{R}^n \) exists if \( \langle u, x^\alpha \rangle = 0 \) for all \( \alpha \) satisfying \( |\alpha| = k \). Conversely, if \( U \) is in \( D'(\mathbb{R}^n \setminus \{0\}) \) and is homogeneous of degree \( a = -(n+k) \), then the definition of \( u \) in (iv) is the same as before and (iii) holds for \( \varphi \in C^\infty_0(\mathbb{R}^n \setminus \{0\}) \). If \( U \) extends to \( D'(\mathbb{R}^n \setminus \{0\}) \), then \( \langle U, x^\alpha \varphi \rangle = 0 \) must hold for radial \( \psi \). It follows easily that \( \langle u, x^\alpha \rangle = 0 \) must hold. Hence we have proved (3).

Further remarks. The formulas in Theorem 3 hold for \( \varphi \in C^\infty_0(\mathbb{R}^n) \) if \( a > -n \) but the extensions from \( \mathbb{R}^n \setminus \{0\} \) to \( \mathbb{R}^n \) are given by limits if \( a \leq -n \). Since \( U \) is homogeneous it is also tempered and so in fact \( U \) extends continuously to \( S \) as well. It is easy to see that the formulas hold for those \( \varphi \) in \( S \) for which \( \varphi \) is zero on a neighborhood of \( 0 \) and for all \( \varphi \) in \( S \) if \( a > -n \).

If \( U \) is homogeneous of degree \( a \), then the Fourier transform \( \mathcal{F}(U) \) is homogeneous of degree \( n + a \); it too is tempered. In the case \( a = -n \), \( \mathcal{F}(U) \) is homogeneous of degree 0. The element \( m \) of \( D'(\sum_{n-1}) \) corresponding to \( \mathcal{F}(U) \) as in Theorem 3, is called the multiplier of \( U \). The distribution \( U \) defines in the usual way an operator \( T_m \) on \( C^\infty_0(\mathbb{R}^n) \) by

\[
T_m(\varphi)(x) = \langle U, \tau_x \varphi \rangle
\]
where $\tau_x$ is translation by $-x$ and $\phi(x) = \phi(-x)$. Since $U$ is tempered, this formula extends to $\varphi \in S$. This paper concerns the extension of $T_m$ from $S$ as an operator on Hardy spaces.

3. Multiplier operators

In the remainder of this paper we will be concerned only with distributions on $\mathbb{R}^2$ that are homogeneous of degree $-2$. The distribution $u$ in $D'(\sum_1)$ plays the role of the usual kernel in the construction of Calderon-Zygmund singular integrals. We let $\sum_1 = T$, the circle. We will always assume that $U$ has no delta measure component (recall that $\delta$ is homogeneous of degree $-2$). The Fourier transform of $U$ is homogeneous of degree $0$ and so there is a unique distribution $m$ on $T$ satisfying

$$\langle \mathcal{F}(U), \varphi \rangle = \int_0^\infty \langle m, \varphi_r \rangle r \, dr.$$ 

In this case the relationship between $u$ and $m$ can be stated explicitly, in terms of the Fourier transform on the circle $T$, as

$$u(n) = \frac{n \text{sgn}(n)}{2\pi i} m(n),$$

for $n \neq 0$, where $\text{sgn}(n)$ denotes the sign of $n$.

We give two proofs of (1). From Stein [10], for dimension 2, $m$ can be computed from $u$ on $T$ by

$$m = -[(i\pi/2) \text{sgn}(|\cdot|) + \log|\cos(\cdot)|] * u.$$ 

A straightforward computation gives, for $n \neq 0$,

$$-[(i\pi/2) \text{sgn}(|\cdot|) + \log|\cos(\cdot)|] u(n) = 2\pi i \text{sgn}(n) \frac{\cos(n) - (n+1)}{n}.$$ 

Expression (1) follows immediately.

The formula (1) can also be derived by directly computing the Fourier transform of the distribution $e^{in\theta} / |\cdot|^2$ in $D'(\mathbb{R}^2\setminus 0)$ using the methods of the proof of Theorem 5, infra. We give this derivation, which is heuristic in that existence of certain integrals needs to be established.

Let $\chi_n(\theta) = e^{in\theta}$, $n \neq 0$. Each $\chi_n \cdot |\cdot|^2$ for a real is locally integrable on $\mathbb{R}^2\setminus\{0\}$ and so defines a distribution. Let $\varphi \in C_0^\infty(\mathbb{R}^2\setminus 0)$. Using Theorem 3 and letting $y = se^{i\theta}$, we have

$$\langle \mathcal{F}(\chi_n |\cdot|^2), \varphi \rangle = \langle \chi_n |\cdot|^2, \mathcal{F}(\varphi) \rangle = \int_0^{2\pi} \int_0^{2\pi} \int_{\mathbb{R}^2} \varphi(se^{i\theta}) e^{-i <se^{i\theta},re^{i\theta}>} \, dy \, d\theta \, dr \int_0^{2\pi} \int_0^{2\pi} \int_{\mathbb{R}^2} \varphi(se^{i\theta}) e^{-i <se^{i\theta},re^{i\theta}>} \, d\theta \, dr \, dy.$$ 

The inner-product in $\mathbb{R}^2$ can be rewritten using $\langle e^{i\beta}, e^{i\theta} \rangle = \cos(\theta - \beta)$. The $\theta$-integral is a Bessel function, namely

$$\int_0^{2\pi} e^{in\theta} e^{-i rs \cos(\theta - \beta)} \, d\theta = 2\pi e^{in\beta} (-i)^n J_n(rs).$$
where \( J_n \) is the \( n \)th-Bessel function. Using the multiplicative invariance of the measure \( r^{-1} \, dr \) on \( \mathbb{R}^+ \), the equality \( J_{-n}(r) = (-1)^n J_n(r) \) for \( n > 0 \), and integration formulas from Watson [13, p. 391, no. 385], the \( r \)-integral becomes

\[
\int_0^\infty \frac{1}{r} J_n(rs) \, dr = (\text{sgn}(n))^{n+1} / n .
\]

Thus

\[
\langle \mathcal{F}(\chi_n, \cdot)^{-1}, \varphi \rangle = (2\pi i^{-n}(\text{sgn}(n))^{n+1} / n) \langle \chi_n, \varphi \rangle.
\]

We have shown that if we take \( u = \chi_n \), then \( m = c_n \chi_n \) where \( c_n \) is the constant on the right. The relationship (1) between the Fourier transforms of \( u \) and \( m \) on \( \mathbb{T} \) follows for any \( m \) and \( u \).

Since the degree of \( U \) is \(-2\), for the extension of \( U \) to \( \mathbb{R}^2 \) it is necessary that \( \hat{u}(0) = 0 \). Since we have assumed that \( U \) has no delta measure component, it is also true that \( \hat{m}(0) = 0 \).

Define the convolution operator \( S \) on \( C^\infty(\mathbb{T}) \) by

\[
\hat{S}(n) = 2\pi (-\text{sgn}(n))^{n+1}.
\]

The operator \( S \) will play a central role in the classification of those homogeneous multipliers that give rise to bounded operators on \( H^1(\mathbb{R}^2) \). If we let \( \tau_\theta \) denote translation by \( \theta \), then a straightforward computation shows that \( S \) is invertible with inverse

\[
S^{-1} = (1/4\pi^2)\tau^- S .
\]

In fact, \( S \) can be written as

\[
S = (1/2)\tau_{\pi/2}\{H(I + \tau_n) + i(I - \tau_n)\}
\]

where \( H \) is the Hilbert transform on the circle, defined by \( \hat{H}(n) = -\text{sgn}(n) \).

In terms of \( S \), the relationship (1) between \( u \) and \( m \) is

\[
\hat{u}(n) = (2\pi)^{-2} i^n(-1)^n \cdot \hat{S}(n) \hat{m}(n) .
\]

As we observed in the introduction, if \( T_m \) extends to a bounded operator on \( L^2(\mathbb{R}^2) \), then \( m \) must be a bounded measurable function on \( \mathbb{T} \). Thus \( S \ast m \) is a function and we have

\[
(2\pi)^2 i(-1)^{n+1} \hat{u}(n)/n = (S \ast m)^\wedge(n) .
\]

Denoting the distributional derivative by \( D \), it follows that we may write \( u \) in \( D'(\mathbb{T}) \) as

\[
u = -D(S^{-1} \ast m).
\]

From Edwards [7, Theorem 12.5.16], \( u \) is a measure if and only if \( \hat{u}(n)/n \) is the Fourier transform of a function of bounded variation. Hence we have the following lemma.

**Lemma 4.** The function \( S \ast m \) has bounded variation if and only if \( u \) is a measure.

We will now prove the main result of the paper.
Theorem 5. Suppose \( m \) is a bounded measurable function on \( \mathbb{R}^2 \) which is homogeneous of degree 0. If the singular integral operator \( T_m \) is in \( B(H^1) \), then the functions \( S \ast m \), \( S \ast (\sin(\cdot)m) \), and \( S \ast (\cos(\cdot)m) \) have bounded variation on \( T \).

Proof. Assume \( T_m \) is in \( B(H^1) \). Let \( \eta \) be a radial function in \( C^\infty(\mathbb{R}^2) \) that is supported on \( \{x : 1/4 \leq |x| \leq 4\} \) and is identically equal to 1 on \( \{x : 1/2 \leq x \leq 2\} \). Then \( M = \mathcal{F}^{-1}(\eta) \) is in \( H^1(\mathbb{R}^2) \); see Taibleson and Weiss [12, p. 137]. Thus \( T_m(M) \) is in \( L^1(\mathbb{R}^2) \). For \( a > 0 \), we have

\[
\|T_m(M)\|_1 \geq \int_{\mathbb{R}^2} e^{-a|z|} |T_m(M)(z)| \, dz
\]

(2)

If \( m \) is expanded in a Fourier series, \( \{A_n\} \) is any decomposition of \( \mathbb{R}^2 \), \( x' \) is replaced by \( e^{i\theta} \), and \( z \) equals \( se^{i\beta} \), then (2) becomes

\[
\|T_m(M)\|_1 \geq \sum_n \int_{A_n} e^{-as} \left| \int_0^\infty \eta(r) \sum_k \hat{m}(k) \int_0^{2\pi} e^{ik\theta} e^{irs\cos(\theta-\beta)} d\theta \, r \, dr \right| \, dz.
\]

Performing the \( \theta \)-integration, we obtain

(3) \[
\|T_m(M)\|_1 \geq \sum_n \left| \int_{A_n} e^{-as} \int_0^\infty \eta(r) \sum_k \hat{m}(k) 2\pi(-i)^k e^{ik\beta} J_k(rs) r \, dr \, dz \right|
\]

where \( J_k \) is the \( k \)th-Bessel function. If we let \( \{A_n\} \) be the conical decomposition of \( \mathbb{R}^2 \) induced by a decomposition \( 0 = b_0 < b_1 < \ldots < b_N = 2\pi \) for the circle, (3) becomes

(4) \[
\|T_m(M)\|_1 \geq \sum_n \left| \int_0^\infty \eta(r) \sum_k \hat{m}(k) 2\pi(-i)^k e^{ik\beta} J_k(rs) s \, ds \, r \, dr \right|.
\]

Using \( J_k = (\text{sgn}(k))^k J_k \) and an integration formula from Watson [13, p. 385], the \( s \)-integral can be explicitly evaluated as

(5) \[
\left( \frac{\text{sgn}(k)}{a^2 + r^2} \right)^k \frac{(1 + |k|)}{(2 + |k|)^{2k}} \cdot \phantom{2}_2F_1 \left( \frac{|k| + 2}{2}, \left( \frac{|k| - 1}{2}, \left( \frac{|k| + 1}{2} + \frac{r^2}{(a^2 + r^2)} \right) \right) \right)
\]

where \( \phantom{2}_2F_1 \) is a generalized hypergeometric function. Call the function displayed in (5) \( C(k, a, r) \). For \( a > 0 \), \( r\eta(r)C(k, a, r) \) is integrable and is dominated by \( r\eta(r)C(k, 0, r) \). The function \( C(k, 0, r) \) is given by

\[
C(k, 0, r) = \left( \frac{\text{sgn}(k)}{a^2 + r^2} \right)^k \frac{(1 + |k|)}{(2 + |k|)^{2k}} \cdot \phantom{2}_2F_1 \left( \frac{|k| + 2}{2}, \left( \frac{|k| - 1}{2}, \left( \frac{|k| + 1}{2} + \frac{r^2}{(a^2 + r^2)} \right) \right) \right)
\]

(5) \[
C(k, 0, r) = \left( \frac{\text{sgn}(k)}{2^k r^2} \right)^k \frac{(1 + |k|)}{(2 + |k|)^{2k}} \cdot \phantom{2}_2F_1 \left( \frac{|k| + 2}{2}, \left( \frac{|k| - 1}{2}, \left( \frac{|k| + 1}{2} + 1 \right) \right) \right)
\]
by using Gauss's Formula for \( \frac{\Gamma}{2F_1(\alpha, \beta; \gamma; 1)} \) (see Bateman Manuscript [1, p. 61]). Substituting this into (4) and performing the \( \beta \)-integration, we obtain the bound
\[
\| T_m(\mathcal{M}) \|_1 \geq C_\eta \cdot \sum_n \left| \sum_k 2\pi (-i)^{k+1} \text{sgn}(k) \hat{m}(k)(e^{-ikb_n} - e^{-ikb_{n-1}}) \right|
\]
\[
= C_\eta \cdot \sum_n \left| S \ast m(e^{ib_n}) - S \ast m(e^{ib_{n-1}}) \right|.
\]
As \( \| T_m(\mathcal{M}) \|_1 \) is finite, \( S \ast m \) is of bounded variation on \( T \).

To show that \( S \ast (\sin(\cdot)\tau) \) and \( S \ast (\cos(\cdot)\tau) \) must also be of bounded variation, we consider the two Riesz transforms \( R_1 \) and \( R_2 \), defined by \( \mathcal{F}(R_1)(re^{i\theta}) = i \cdot \sin(\theta) \) and \( \mathcal{F}(R_2)(re^{i\theta}) = i \cdot \cos(\theta) \). These operators are bounded on \( H^1 \); and hence, \( R_1 T_m \) and \( R_2 T_m \) are bounded on \( H^1 \) if \( T_m \) is bounded. Applying the preceding argument to these operators, we see that, \( S \ast (\sin(\cdot)\tau) \) and \( S \ast (\cos(\cdot)\tau) \) are of bounded variation. \( \square \)

**Theorem 6.** Suppose \( m \) is a bounded measurable function on \( \mathbb{R}^2 \) which is homogeneous of degree 0 and such that \( T_m \) is in \( B(H^1) \). Then \( m \) has an absolutely convergent Fourier series as a function on \( T \).

**Proof.** From Theorem 5, if \( T_m \in B(H^1) \), then \( S \ast m \), \( S \ast (\sin(\cdot)\tau) \), and \( S \ast (\cos(\cdot)\tau) \) are of bounded variation on \( T \). Thus \( (-\text{sgn}(n))^{n+1} \hat{m}(n) \), \( (-\text{sgn}(n))^{n+1} (\hat{m}(n+1) + \hat{m}(n-1)) \), and \( (-\text{sgn}(n))^{n+1} (\hat{m}(n+1) - \hat{m}(n-1)) \) are the Fourier series of functions of bounded variation; and consequently, \( (-\text{sgn}(n))^{n+1} \hat{m}(n+1) \) is the Fourier transform of a function of bounded variation. Taken together, all this implies that the Fourier transform
\[
(H \ast S \ast m)^\wedge(n) = (-\text{sgn}(n)) \cdot (-\text{sgn}(n))^{n+1} \hat{m}(n)
\]
is a finite sum of transforms of functions of bounded variation. From Zygmund [14, v. 1, p. 242], as both \( S \ast m \) and \( H \ast S \ast m \) are of bounded variation, \( S \ast m \) has an absolutely convergent Fourier series. However, \( \| (S \ast m)^\wedge(n) \| = 2\pi |\hat{m}(n)| \) and so \( m \) has an absolutely convergent Fourier series. \( \square \)

**Theorem 7.** Suppose \( m \) is a bounded measurable function on \( \mathbb{R}^2 \) which is homogeneous of degree 0 and \( T_m \) is in \( B(H^1) \). Then \( u \) is a measure on \( T \) and \( u \) is absolutely continuous with respect to Lebesgue measure.

**Proof.** Theorem 5 and Lemma 4 imply that \( u \) is a measure. The kernel of \( H \ast m \) is \( H \ast u \), because both \( H \) and the correspondence between kernels and multipliers are linear. In the proof of Theorem 6 it was shown that \( H \ast S \ast m \) has finite variation. Since \( H \ast S = S \ast H \), it follows that \( S \ast H \ast m \) has finite variation. Hence, by Lemma 4, \( H \ast u \) is also a measure. By the F. Riesz and M. Riesz Theorem, \( u \) is absolutely continuous. See Zygmund [14, v. 1, p. 285], and also Edwards [7, p. 99]. \( \square \)

Theorem 6 can be phrased for kernels \( u \). If \( u \) is a kernel in \( D'(T) \) and \( T_m \in B(H^1) \), then by Theorem 7 \( u \) is in \( L^1(T) \). By Theorem 6, \( m \) has an
Corollary 8. If \( u \) is in \( L^1(T) \) and the corresponding singular integral \( T_m \) is in \( B(H^1) \), then \( \sum_{n \neq 0} \frac{|\hat{u}(n)|}{|n|} < \infty \).

We now provide a partial converse to Theorem 5.

Theorem 9. Suppose \( m \) is a bounded measurable function on \( \mathbb{R}^2 \) which is homogeneous of degree 0. If \( S \ast m \), \( S \ast (\cos(\cdot)m) \), \( S \ast (\sin(\cdot)m) \) are of bounded variation on \( T \), then \( T_m \) is in \( B(L^p) \) for \( 1 < p < \infty \).

Proof. Let \( BV(T) \) be the functions of bounded variation and suppose \( S \ast m \), \( S \ast (\cos(\cdot)m) \), \( S \ast (\sin(\cdot)m) \) \( \in BV(T) \). A periodic function \( f \) on \( \mathbb{R} \) is even or odd as a function on the circle according as \( f(t) = f(t+\pi) \) or \( f(t) = -f(t+\pi) \). Note that \( \cos(t) \) as a function on \( T \) is odd, not even. The even and odd parts of a function \( f \) are given by

\[
f_e(\theta) = \frac{f(\theta) + f(\theta + \pi)}{2}, \quad f_o(\theta) = \frac{f(\theta) - f(\theta + \pi)}{2}.
\]

Clearly \( f \in BV(T) \) if and only if \( f_e \) and \( f_o \) \( \in BV(T) \). As \( S \) is a multiplier operator, \( (S \ast m)_o = S \ast (m_o) \) and \( (S \ast m)_e = S \ast (m_e) \). For the odd function \( m_o \), we have \( m_o \in BV(T) \) as \( S \) reduces to a translation operator for odd functions. By Lemma 4, the corresponding kernel \( u_0 \) is a measure. As a measure on \([-\pi, \pi]\), it satisfies

\[
\int g(t + \pi) \, du_0(t) = -\int g(t) \, du_0(t)
\]

for periodic functions \( g \). That is, \( u_0 \) is an odd measure. A straightforward adaptation of the “method of rotations” shows that \( T_{m_o} \in B(L^p) \) for \( 1 < p < \infty \). See Theorem (2.6) and the proof preceding it in Chapter VI of [11], where the kernel is an odd function in \( L^1(T) \). For the even multiplier \( m_e \), the functions \( \sin(\cdot)m_e \), \( \cos(\cdot)m_e \) are both odd functions. We have

\[
S \ast (\cos(\cdot)m_e) = (S \ast (\cos(\cdot)m)_0, \quad S \ast (\sin(\cdot)m_e) = (S \ast (\sin(\cdot)m)_0,
\]

and so \( S \ast \sin(\cdot)m_e \) and \( S \ast (\cos(\cdot)m_e) \) are in \( BV(T) \). Hence \( \sin(\cdot)m_e \) and \( \cos(\cdot)m_e \) are also in \( BV(T) \) and so are multipliers of kernels which are odd measures. As before, the corresponding operators are in \( B(L^p) \); that is, the Riesz transforms

\[
R_1 T_{m_e}, \quad R_2 T_{m_e} \in B(L^p).
\]

Each \( R_j \) is also in \( B(L^p) \), and so also is each \( R_j^2 T_{m_e} \). So \( T_{m_e} \in B(L^p) \) as \( R_1^2 + R_2^2 = -I \). As \( T_m = T_{m_0} + T_{m_e} \), we have \( T_m \in B(L^p) \) for \( 1 < p < \infty \). \( \square \)

In [9], Ricci and Weiss give a characterization of \( H^1(\sum_{n=1}^\infty) \) that is particularly suitable to our presentation here. They consider Calderon-Zygmund singular operators (distributions) \( T \) with kernel \( k(x) = u(x')/|x|^n \), where
If \( u \in L^1(\sum_{n=1}^\infty) \) and has integral zero. By composing \( T \) with the Riesz transforms \( R_j \) (\( 1 \leq j \leq n \)), one obtains \( n \) distributions that are each homogeneous of degree \(-n\). In general, these distributions may or may not be given by kernels in \( L^1(\sum_{n=1}^\infty) \). Ricci and Weiss characterize \( H^1(\sum_{n=1}^\infty) \) by showing that \( u \in H^1(\sum_{n=1}^\infty) \) if and only if \( R_j \circ T \) has a kernel \( u_j \in L^1(\sum_{n=1}^\infty) \). In the case of dimension 2, we know from Theorem 7 that the operators \( R_i \circ U \) and \( R_2 \circ U \) have kernels \( u_1 \) and \( u_2 \) in \( L^1(T) \) if \( U = T^m \) is in \( B(H^1) \). Hence the following theorem results from the Ricci-Weiss result and our Theorem 7.

**Theorem 10.** If \( U \) is a distribution that is homogeneous of degree \(-2\) and has an extension in \( B(H^1(\mathbb{R}^2)) \), then the kernel \( u \) of \( U \) is in \( H^1(T) \).

Note that Corollary 8 follows immediately from Theorem 10.

It is well known that if \( u \in H^1(T) \), then the Calderon-Zygmund singular integral \( T^m \) with kernel \( u \) is bounded on \( L^p \) for \( 1 < p < \infty \), and of course, \( \sum_{n \neq 0} |\hat{u}(n)| < \infty \) holds. See for example Connett [3]. Thus \( u \in H^1(T) \) is a sufficient condition for boundedness of \( T^m \) on \( L^p \) for \( 1 < p < \infty \). Theorem 10 gives a corresponding necessary condition, with \( H^1 \) replacing \( L^p \).

**Remarks on sufficient conditions for \( T^m \) to be in \( B(H^1) \).** The condition that \( m \) have an absolutely convergent Fourier series is not sufficient to guarantee that \( T^m \) extends to an operator on \( H^1 \). This is easily seen by considering lacunary series and using Theorem 7. For example, if \( m(\theta) = \sum e^{i2^n \theta}/n^2 \), then \( |\hat{u}(2^n)| = 2^n / n^2 \) and \( u \) cannot be a measure. In fact, by considering similar examples, one sees that conditions of the form

\[
\sum n^\delta |\hat{m}(n)| < \infty
\]

for \( 0 \leq \delta < 1 \) are insufficient to guarantee that \( T^m \in B(H^1) \). Conversely, Taibleson and Weiss [12] show that the condition

\[
\sum |\hat{m}(n)|^2 n^4 < \infty
\]

implies that \( T^m \) is bounded on \( H^p \) for \( 2/3 < p \leq 1 \), by showing that \( T^m \) sends atoms to molecules. Daly [4] has shown that if \( T^m \) sends atoms to molecules boundedly, then condition (6) is satisfied. Taken together, the state of results on sufficient conditions for \( T^m \in B(H^p) \) in terms of \( m \) appears to be these results by Daly, Taibleson, and Weiss and our Theorem 9.

The condition that \( S \ast m \) is of bounded variation is not sufficient to imply that \( T^m \) is bounded on \( H^p(0 < p \leq 1) \). This can be seen by considering any odd function \( m \) that is of bounded variation, but not continuous, on \( T \).

**References**


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