ON THE SECOND DERIVATIVES OF CONVEX FUNCTIONS
ON HILBERT SPACES

NOBUYUKI KATO

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Abstract. Let \( \phi \) be a proper l.s.c. convex function on a real Hilbert space \( H \). We show that if \( H \) is separable, then \( \phi \) is twice differentiable in some sense on a dense subset of the graph of \( \partial \phi \).

1. INTRODUCTION

Let \( H \) be a (real) Hilbert space. The inner product and associated norm will be denoted by \( \langle \cdot, \cdot \rangle \) and \( |\cdot| \), respectively. Let \( \phi : H \to (-\infty, +\infty] \) be a proper lower semicontinuous (l.s.c.) convex function. Then subdifferential \( \partial \phi \) is defined by

\[
\partial \phi(x) := \{ y \in H | \phi(z) - \phi(x) \geq \langle y, z - x \rangle \ \text{for all} \ z \in H \}.
\]

We may regard \( \partial \phi(x) \) as a generalized first derivative of \( \phi \) at \( x \). Actually, if \( \phi \) is Gâteaux differentiable at \( x \) and has a continuous Gâteaux derivative \( \nabla \phi(x) \), then \( \partial \phi(x) = \{ \nabla \phi(x) \} \).

Our purpose is to show that \( \phi \) is twice differentiable in some sense on a dense subset of the graph of \( \partial \phi \). Indeed, we show the existence of an operator which has nice properties and may be considered as a generalized second derivative of \( \phi \). Our main result is an infinite-dimensional version of an interesting work of Rockafellar [9].

2. Preliminaries

To formulate our result, we will give some preparation in this section. Given a multi-valued operator \( T \) from \( H \) to \( 2^H \), we define \( D(T) := \{ x \in H | Tx \neq \emptyset \} \), \( R(T) := \bigcup_{x \in D(T)} Tx \). The graph of \( T \) is denoted by \( G(T) \).

An operator \( T \) is said to be linear if its graph \( G(T) \) is a linear subspace of \( H \times H \). (\( T \) may be multi-valued.) For any linear operator \( T \), we define its
adjoint $T^*$ by

$$G(T^*) := \{(v, g) \in H \times H \mid \langle f, v \rangle = \langle u, g \rangle \text{ for all } (u, f) \in G(T)\}$$

(cf. [1]). We say that $T$ is self-adjoint if $T = T^*$, and $T$ is positive if $\langle x, y \rangle \geq 0$ for all $(x, y) \in G(T)$.

We say that an operator $T: H \to 2^H$ is monotone if its graph $G(T)$ is a monotone set in $H \times H$, i.e.,

$$\langle x - y, x' - y' \rangle \geq 0 \text{ for all } (x, x'), (y, y') \in G(T).$$

$T$ is said to be maximal monotone if it is monotone and there is no monotone extension of $T$. It is well known that $T$ is maximal monotone iff $T$ is monotone and the range condition $R(I + T) = H$ is satisfied. The subdifferential $\partial \phi$ is an important example of maximal monotone operator. For a maximal monotone operator $T$, we define its minimal section $T^0$ by

$$T^0 x := \{y \in Tx \mid \|y\| = \inf\{|z| : z \in Tx\}\}.$$ 

In Hilbert space setting, it is known that $T^0$ is a well defined single-valued monotone operator in $H$, with domain $D(T^0) = D(T)$. (See e.g. [5,6].)

Let $\{S_n\}$ be a sequence of sets in a Banach space $X$. Then we define the notion of set convergence as follows. (See e.g. [4,9].)

$$s-(w-) \lim_{n \to \infty} S_n := \{v \in X \mid \text{there exist } \{n_k\} \subset \{n\} \text{ and } v_{n_k} \in S_{n_k} \text{ such that } s-(w-) \lim_{k \to \infty} v_{n_k} = v\},$$

$$s-(w-) \lim_{n \to \infty} S_n := \{v \in X \mid \text{there exist } v_n \in S_n \text{ such that } s-(w-) \lim_{n \to \infty} v_n = v\},$$

$$s-(w-) \lim_{n \to \infty} S_n = S \text{ if } s-(w-) \lim_{n \to \infty} S_n = s-(w-) \lim_{n \to \infty} S_n = S,$$

where $s-(w-)$ means the strong (or weak) topology, respectively.

Let $\phi^n$ ($n = 0, 1, \ldots$) be proper l.s.c. convex functions on $H$. Then we say that $\phi^n$ converges to $\phi^0$ in the sense of Mosco if

$$w- \lim_{n \to \infty} \text{Epi } \phi^n \subset \text{Epi } \phi^0 \subset s- \lim_{n \to \infty} \text{Epi } \phi^n,$$

where $\text{Epi } \phi^n$ is the epi-graph of $\phi^n$, which is defined by

$$\text{Epi } \phi^n = \{(x, \lambda) \in H \times \mathbb{R} \mid \phi^n(x) \leq \lambda\}.$$ 

According to Attouch [4, Theorem 1.2], the following conditions are equivalent:

(a) $\phi^n$ converges to $\phi^0$ in the sense of Mosco.

(b) For any $\lambda > 0$ and $x \in H$,

$$(I + \lambda \partial \phi^n)^{-1} x \to (I + \lambda \partial \phi^0)^{-1} x \text{ as } n \to \infty,$$
and there exist \((u_n, v_n) \in G(\partial \phi^n)\) such that
\[
u_n \to u_0, v_n \to v_0 \quad \text{and} \quad \phi^n(u_n) \to \phi^0(u_0).
\]

(c) For any \(\lambda > 0\) and \(x \in H\),
\[
\phi^n_\lambda(x) \to \phi^0_\lambda(x) \quad \text{as} \quad n \to \infty,
\]
where \(\phi^n_\lambda(x)\) is the Yosida regularization of \(\phi^n\) defined by
\[
\phi^n_\lambda(x) = \min_{y \in H} \left\{ \phi^n(y) + \frac{1}{2\lambda} |x-y|^2 \right\}.
\]

3. Main result

Let \(H\) be a Hilbert space and \(\phi: H \to (-\infty, +\infty]\) be a proper l.s.c. convex function. For each \((x,y) \in G(\partial \phi)\) and \(t > 0\), consider the second-order difference quotients
\[
\Delta_{x,y,t}(h) := t^{-2}(\phi(x + th) - \phi(x) - t(y,h)).
\]
It is easily seen that \(\Delta_{x,y,t}: H \to (-\infty, +\infty]\) is proper l.s.c. and convex. Our main theorem is now stated as follows:

**Theorem.** Let \(H\) be a separable Hilbert space and \(\phi\) be as above. Then there exists a dense subset \(E\) of \(G(\partial \phi)\) such that \(\phi\) is twice differentiable on \(E\) in the following sense:

For any \((x,y) \in E\), there exists a proper l.s.c. convex function \(q_{x,y}: H \to (-\infty, +\infty]\) such that

(i) \(q_{x,y} \equiv \partial q_{x,y}\) is a (multi-valued) linear positive self-adjoint operator in \(H\) (in the sense of §2).

(ii) The minimal section \(q^0_{x,y}\) of \(q_{x,y}\) is a single-valued linear positive self-adjoint operator in the closed linear subspace \(D(q^0_{x,y})\) of \(H\), and
\[
a_{x,y}(h) = \begin{cases} 
\frac{1}{2} |q^0_{x,y}|^{1/2}h|^2 & \text{if} \ h \in D((q^0_{x,y})^{1/2}) \\
+ \infty & \text{otherwise},
\end{cases}
\]

(iii) \(\Delta_{x,y,t}\) converges to \(a_{x,y}\) as \(t \downarrow 0\) in the sense of Mosco.

It should be noted that \(q_{x,y}\) may be called the generalized second derivative of \(\phi\) at \((x,y)\) as shown below.

**Remark.** We observe that if \(\phi: U \subset H \to \mathbb{R}\), \(U\) open convex, is of class \(C^2\) and convex, then \(q_{x,y} = d^2 \phi(x)\) for \(x \in U\) satisfying \((x,y) \in E\) with \(y = d\phi(x)\), where \(d\phi\) and \(d^2 \phi\) denote the first and second derivatives of \(\phi\).
Let $h \in H$ be fixed. By Taylor's theorem,
\[ \phi(x + th) = \phi(x) + t\langle d\phi(x), h \rangle + (t^2/2)\langle d^2\phi(x)h, h \rangle + o(t^2) \]
as $t \to 0$. Hence
\[ \Delta_t(h) := t^{-2}(\phi(x + th)\phi(x) - t\langle d\phi(x), h \rangle) \]
\[ \to \frac{1}{2}\langle d^2\phi(x)h, h \rangle \quad \text{as } t \to 0. \]

Let $\overline{q}(h) = \frac{1}{2}\langle d^2\phi(x)h, h \rangle$. Noting that $d^2\phi(x): H \to H$ is self-adjoint, $\overline{q}: H \to \mathbb{R}$ is continuous, convex, and $\partial \overline{q} = d^2\phi(x)$. Since $\Delta_t: H \to \mathbb{R}$ is also continuous, convex, and $D(\Delta_t) = D(\overline{q}) = H$, we have
\[ \Delta_t \to \overline{q} \quad \text{in the sense of Mosco} \]
by virtue of Salinetti and Wets [10, Corollary 2E]. On the other hand, by our theorem,
\[ \Delta_t \to q_{\overline{x}, \overline{y}}, \quad \overline{y} = d\phi(x), \quad \text{in the sense of Mosco}. \]

Then we have $(\overline{q})_\lambda = (q_{\overline{x}, \overline{y}})_\lambda$ by [4, Theorem 1.2]. See the condition (c) of §2.

As a consequence, we obtain $\partial \overline{q} = \partial q_{\overline{x}, \overline{y}}$, i.e. $d^2\phi(x) = Q_{\overline{x}, \overline{y}}$.

4. Proof of Theorem

We begin with the following lemma.

**Lemma 4.1.** Let $F: H \to H$ be a Lipschitz continuous mapping and suppose that $F$ is Gâteaux differentiable at $\overline{u}$ and has a continuous linear Gâteaux derivative $\nabla F(\overline{u})$, i.e.,
\[ (4.1) \quad \nabla F(\overline{u})h = \lim_{t \downarrow 0} t^{-1}[F(\overline{u} + th) - F(\overline{u})] \quad \text{for all } h \in H. \]

Then for any bijective continuous linear mapping $\Psi: H \times H \to H \times H$,
\[ \lim_{t \downarrow 0} t^{-1}[\Psi(G(F)) - \Psi(\overline{u}, F(\overline{u}))] = \Psi(G(\nabla F(\overline{u}))), \]
where $\lim_{t \downarrow 0}$ is taken in the sense of set convergence with respect to the strong topology of $H \times H$ (see §2).

**Proof.** Let $(u, v) \in \Psi(G(\nabla F(\overline{u})))$ and $t_n \downarrow 0$. Setting $(x, y) = \Psi^{-1}(u, v)$, then
\[ y = \nabla F(\overline{u})x = \lim_{n \to \infty} t_n^{-1}[F(\overline{u} + t_nx) - F(\overline{u})]. \]

Putting $x_n = \overline{u} + t_nx$, $y_n = F(\overline{u} + t_nx)$, then the linearity and continuity of $\Psi$ imply that
\[ \Psi(x, t_n^{-1}[F(\overline{u} + t_nx) - F(\overline{u})]) = t_n^{-1}[\Psi(x_n, y_n) - \Psi(\overline{u}, F(\overline{u}))] \]
\[ \in t_n^{-1}[\Psi(G(F)) - \Psi(\overline{u}, F(\overline{u}))] \]
and
\[ \Psi(x, t_n^{-1}[F(\overline{u} + t_nx) - F(\overline{u})]) \to \Psi(x, y) = (u, v). \]
This shows that \( \lim_{t \to 0} t^{-1} [\Psi(G(F)) - \Psi(\bar{u}, F(\bar{u}))] \supset \Psi(G(\nabla F(\bar{u}))). \)

Conversely, let \((u, v) \in \lim_{t \to 0} t^{-1} [\Psi(G(F)) - \Psi(\bar{u}, F(\bar{u}))]. \) Then there exist \( t_n \downarrow 0 \) and \((u_n, v_n) \to (u, v)\) such that \( \Psi(\bar{u}, F(\bar{u})) + t_n(u_n, v_n) \in \Psi(G(F)). \)

Put \((x_n, y_n) = \Psi^{-1}(u_n, v_n)\) and \((x, y) = \Psi^{-1}(u, v)\). Then we have \((x_n, y_n) \to (x, y)\) since \( \Psi^{-1} \) is also continuous by the open mapping theorem. Thus, noting that \((\bar{u} + t_n x_n, F(\bar{u}) + t_n y_n) \in G(F)\), we have

\[
t_n^{-1} [F(\bar{u} + t_n x) - F(\bar{u})] = y_n - y.
\]

But since \( F \) is Lipschitzian, we get

\[
t_n^{-1} [F(\bar{u} + t_n x) - F(\bar{u})] \to y.
\]

By the hypothesis (4.1), we have \( y = \nabla F(\bar{u}) x \) i.e. \((x, y) \in G(\nabla F(\bar{u})). \) Thus \((u, v) \in \Psi(G(\nabla F(\bar{u})))\) and the relation \( \lim_{t \to 0} t^{-1} [\Psi(G(F)) - \Psi(\bar{u}, F(\bar{u}))] \subset \Psi(G(\nabla F(\bar{u})))\) holds. \( \square \)

Next, we investigate the properties of multi-valued linear maximal monotone operators in \( H. \)

**Lemma 4.2.** Let \( Q \) be a multi-valued linear maximal monotone operator in \( H. \) Suppose that \( Q \) is the subdifferential of some proper l.s.c. convex function \( q: H \to (-\infty, +\infty). \) Then:

(i) \( Q \) is positive self-adjoint in the sense of §2;

(ii) the minimal section \( Q^0 \) of \( Q \) is a single-valued positive self-adjoint operator in the closed linear subspace \( Y := D(Q) \) of \( H, \) and

\[
q(h) = \begin{cases} 
\frac{1}{2} |(Q^0)^{1/2} h|^2 + \text{const} & \text{if } h \in D((Q^0)^{1/2}) \subset Y \\
+\infty & \text{otherwise.}
\end{cases}
\]

**Proof.** (i) It follows from [7, Proposition 2.15] that \( Q_\lambda = Q_{\lambda}^*. \) Hence we see that \( J_\lambda = J_\lambda^*. \) Here \( Q_\lambda = (1/\lambda)(I - J_\lambda) \) and \( J_\lambda = (I + \lambda Q)^{-1} \) for \( \lambda > 0. \) Let \((v, g) \in G(Q). \) Then for any \((u, f) \in G(Q), \) we have

\[
\langle f + u, v \rangle = \langle f + u, J_\lambda(g + v) \rangle = \langle J_\lambda(f + u), g + v \rangle = \langle u, g + v \rangle.
\]

Thus \( \langle f, v \rangle = \langle u, g \rangle. \) This shows that \( Q \subset Q^*. \)

On the other hand, \( Q^* \) is monotone by [8, Theorem 2]. Then the maximality of \( Q \) implies that \( Q = Q^*. \) Positivity is obvious.

(ii) It is known [6, IV, Theorem 1.2; 7, Theorem 4.1] that there exists a unique contraction semigroup \( \{S(t)\} \) on \( Y = \overline{D(Q)} \) such that the minimal section \(-Q^0\) of \(-Q\) is the infinitesimal generator of \( \{S(t)\}. \) Notice that \( \{S(t)\} \) is a linear \((C_0)\)-contraction semigroup on the closed linear subspace \( Y. \)

Then \( Q^0 \) is a single-valued linear maximal monotone operator in \( Y \) (densely defined in \( Y). \) Since \( Q^0 \subset Q = \partial q, \) \( Q^0 \) is cyclically monotone by [7, Theorem
Then it follows from [7, Proposition 2.15] that $Q^0$ is positive self-adjoint in $Y$ and $Q^0 = \partial_y \psi$ (the subdifferential considered in $Y$) with

$$
\psi(h) = \begin{cases} 
\frac{1}{2} \|(Q^0)^{1/2}h\|^2 & \text{if } h \in D((Q^0)^{1/2}) \subset Y \\
+\infty & \text{otherwise}.
\end{cases}
$$

$\psi$ is also proper l.s.c. convex in $H$, and it is easily checked that $Q^0 \subset \partial \psi$ and $D(Q) = D(Q^0) \subset D(\partial \psi) \subset D(Q) = Y$. Then [7, Corollary 2.2] yields that $Q = \partial \psi$, and hence we conclude that $q = \psi + \text{const}$ by [7, Corollary 2.10].

**Remark 4.1.** It is shown that $Q = Q^0 + \partial I_Y$, where $I_Y$ is the indicator function of $Y$ (i.e. $I_Y(u) = 0$ if $u \in Y$, $= +\infty$ if $u \notin Y$). In fact, let $f \in Qu$, then $f$ is written as $f = f_1 + f_2$ with $f_1 \in Y$ and $f_2 \in Y^\perp$ since $H = Y \oplus Y^\perp$ ($Y^\perp$ is the orthogonal complement of $Y$). As shown in Lemma 4.2, $Q = \partial \psi$ and so

$$
\psi(z) - \psi(u) \geq \langle f, z - u \rangle = \langle f_1 + f_2, z - u \rangle
$$

for all $z \in Y$. Therefore, $f_1 \in \partial_y \psi(u) = Q^0 u$. Noting that $\partial I_Y(u) = Y^\perp$ for $u \in Y$, we obtain $Q \subset Q^0 + \partial I_Y$. Conversely, $Q^0 + \partial I_Y \subset \partial \psi + \partial I_Y \subset \partial (\psi + I_Y) = \partial \psi = Q$.

**Proof of Theorem.** Put $F := 2(I + \partial \phi)^{-1} - I$, $\Phi(x, y) := (x + y, x - y)$. It is obvious that $F: H \to H$ is Lipschitz continuous, $\Phi: H \times H \to H \times H$ is bijective, continuous linear, and $\Phi^{-1}(u, v) = ((u + v)/2, (u - v)/2)$ is also continuous linear. Furthermore, we have $\Phi(G(\partial \phi)) = G(F)$.

Since $H$ is separable, by the result of Mignot [10, Theorem 1.2], there exists a dense subset $U$ of $H$ on which $F$ is Gâteaux differentiable and has a continuous linear Gâteaux derivative $\nabla F(\bar{u})$ at $\bar{u} \in U$. Thus for $\bar{u} \in U$, (4.1) holds.

Let $E := \Phi^{-1}(G(F|_U)) = \{((u + F(u))/2, (u - F(u))/2)|u \in U\}$, where $F|_U$ is the restriction of $F$ to $U$. Then $E \subset \Phi^{-1}(G(F)) = G(\partial \phi)$ and $E$ is dense in $G(\partial \phi)$ with respect to the product topology of $H \times H$. In fact, noting that $F$ is Lipschitzian, we have $G(F|_U)$ is dense in $G(F)$. Thus by the continuity of $\Phi^{-1}$, it is shown that $E$ is dense in $\Phi^{-1}(G(F)) = G(\partial \phi)$.

Let $(\bar{x}, \bar{y}) \in E$. By the definition, there exists $\bar{u} \in U$ such that $(\bar{x}, \bar{y}) = \Phi^{-1}(\bar{u}, F(\bar{u}))$. Then it follows from Lemma 4.1 with $\Psi = \Phi^{-1}$ that

$$
\lim_{t \to 0^+} t^{-1}[G(\partial \phi) - (\bar{x}, \bar{y})] = \Phi^{-1}(G(\nabla F(\bar{u}))).
$$

Define $Q_{\bar{x}, \bar{y}}$ by $G(Q_{\bar{x}, \bar{y}}) = \Phi^{-1}(G(\nabla F(\bar{u})))$. Then $Q_{\bar{x}, \bar{y}}$ is a linear operator from $H$ to $2^H$ with domain

$$
D(Q_{\bar{x}, \bar{y}}) = \{(x + \nabla F(\bar{u})x)/2 | x \in H\}(\bar{u} \text{ depends on } (\bar{x}, \bar{y}) \in E).
$$
Moreover, \( Q_{\bar{x}, \bar{y}} \) is a maximal monotone operator in \( H \). In fact, the monotonicity is a consequence of (4.3) since \( t^{-1}[G(\partial \phi) - (\bar{x}, \bar{y})] \) is monotone. Let \( x \in H \). Then

\[
\frac{x + \nabla F(\bar{v})x}{2} + Q_{\bar{x}, \bar{y}} \left( \frac{x + \nabla F(\bar{v})x}{2} \right) \ni x
\]

because \( ((x + \nabla F(\bar{v})x)/2 , (x - \nabla F(\bar{v})x)/2 ) \in G(Q_{\bar{x}, \bar{y}}) \). Thus \( R(I + Q_{\bar{x}, \bar{y}}) = H \).

Now, consider the second-order difference quotients

\[
\Delta_{x,y,t}(h) := t^{-2} (\phi(x + th) - \phi(x) - t(y, h))
\]

for \((x, y) \in G(\partial \phi)\) and \( t > 0 \). It is easily verified that \( \Delta_{x,y,t} : H \rightarrow (-\infty, +\infty] \) is a proper l.s.c. convex function with \( \Delta_{x,y,t}(0) = 0 \), and

\[
G(\partial \Delta_{x,y,t}) = t^{-1}[G(\partial \phi) - (x, y)].
\]

Therefore, we obtain from (4.3) that

\[
(4.4) \quad \lim_{t \downarrow 0} G(\partial \Delta_{x,y,t}) = G(Q_{\bar{x}, \bar{y}}).
\]

Let \( t_n \downarrow 0 \). By (4.4), for any \((x, y) \in G(Q_{\bar{x}, \bar{y}})\), there exists \((x_n, y_n) \in G(\partial \Delta_{x,y,t_n})\) such that \( x_n \rightarrow x \) and \( y_n \rightarrow y \). This is equivalent to: for any \( \lambda > 0 \) and \( x \in H \),

\[
(I + \lambda \partial \Delta_{x,y,t_n})^{-1} x \rightarrow (I + \lambda Q_{\bar{x}, \bar{y}})^{-1} x \quad \text{as } n \rightarrow \infty
\]

by [4, Corollary 1.1]. Then it follows from [4, Proposition 1.3] that

\[
Q_{\bar{x}, \bar{y}} = \partial q_{\bar{x}, \bar{y}}
\]

for some proper l.s.c. convex function \( q_{\bar{x}, \bar{y}} \).

Since \((0,0) \in G(\partial q_{\bar{x}, \bar{y}})\), \( q_{\bar{x}, \bar{y}}(y) \geq q_{\bar{x}, \bar{y}}(0) \) for every \( y \in H \). Then setting

\[
q_{\bar{x}, \bar{y}}(x) := q_{\bar{x}, \bar{y}}(x) - q_{\bar{x}, \bar{y}}(0),
\]

we have \( \partial q_{\bar{x}, \bar{y}} = \partial q_{\bar{x}, \bar{y}} = Q_{\bar{x}, \bar{y}} \) and \( q_{\bar{x}, \bar{y}}(0) = 0 \). After all, we obtain:

for any \( \lambda > 0 \) and \( x \in H \), \( (I + \lambda \partial \Delta_{x,y,t_n})^{-1} x \rightarrow (I + \lambda \partial q_{\bar{x}, \bar{y}})^{-1} x \),

\[
(0,0) \in G(\partial q_{\bar{x}, \bar{y}}), \quad (0,0) \in G(\partial \Delta_{x,y,t_n}), \quad \Delta_{x,y,t_n}(0) = q_{\bar{x}, \bar{y}}(0) = 0.
\]

Hence by [4, Theorem 1.2] (see §2(b)), we conclude that \( \Delta_{x,y,t_n} \) converges to \( q_{\bar{x}, \bar{y}} \) in the sense of Mosco. Thus (iii) is established.

The assertions (i) and (ii) are the consequences of Lemma 4.2. Note that the constant appeared in (4.2) becomes zero since \( q_{\bar{x}, \bar{y}}(0) = 0 \). □

Remark 4.2. (1) The above pair of \( F \) and \( \Phi \) was introduced previously in [12] and played an important role. But it is not essential. We may take, for example, \( F := (I + \partial \phi)^{-1} \) and \( \Phi(x, y) := (x + y, x) \) with appropriate changes of proof above.

(2) If \( n := \dim H < +\infty \), \( E \) can be taken as \( G(\partial \phi) \setminus E \) is of measure zero with respect to \( n \)-dimensional Hausdorff measure. Indeed, \( U \) can be taken as \( H \setminus U \) is of measure zero with respect to \( n \)-dimensional Lebesgue measure by the
classical result of Rademacher. See e.g. [14, VIII, 3]. Since $F$ is Lipschitzian, we have $\mathcal{H}^n(G(F|_{H\setminus U})) = 0$, where $\mathcal{H}^n$ is the $n$-dimensional Hausdorff outer measure. Since $\Phi^{-1} : H \times H \to H \times H$ is continuous linear, it is Lipschitz continuous. Denoting by $L$ the Lipschitz constant of $\Phi^{-1}$, \[ \mathcal{H}^n(G(\partial \phi)\setminus E) = \mathcal{H}^n(\Phi^{-1}(G(F|_{H\setminus U}))) \leq L^n \mathcal{H}^n(G(F|_{H\setminus U}))) = 0. \]

**Remark 4.3.**

(1) Concerning (ii) of Theorem, Asakawa [3] previously suggested the expression where the restriction operator $Q|_Y$ (see [2]) is used instead of the minimal section $Q^0$. In our setting, it is shown that both of them coincide. Another expression is possible by using a square root $Q_{1/2}$ defined by the boundary value problem on half-axis of (elliptic) second order differential equations. See [6, V].

(2) After finishing the previous version of this paper, the author was informed by the referee of the J. L. Ndoutoume's work [11], which includes similar results to ours. $G(\partial \phi)$ is said to be smooth at $(x, y) \in G(\partial \phi)$ if $\lim_{t \to 0} t^{-1}[G(\partial \phi) - (x, y)] = S$ for some linear subspace $S$ of $H \times H$ ([12]). [11, Proposition 2.2, Theorem 3.3] shows that if $G(\partial \phi)$ is smooth at $(x, y)$, then there exists a proper l.s.c. convex function $q_{x,y}$ such that (i) the minimal section $Q^0_{x,y}$ of $Q_{x,y} \equiv \partial q_{x,y}$ is a positive symmetric single-valued linear operator, and $q_{x,y}(h) = \frac{1}{2} \langle Q^0_{x,y} h, h \rangle$ for $h \in D(Q_{x,y})$; (ii) $\Delta_{x,y,t}$ converges to $q_{x,y}$ as $t \to 0$ in the sense of Mosco.

The proof of our theorem shows that $G(\partial \phi)$ is smooth at each point in $E$, so that the points at which $G(\partial \phi)$ is smooth are dense in $G(\partial \phi)$ provided $H$ is separable. Our result tells more about $Q^0_{x,y}$, the self-adjointness in $Y$ and the complete expression of $q_{x,y}$ by the square root of $Q^0_{x,y}$. It would be interesting that our proof of the self-adjointness of $Q^0_{x,y}$ is based on the nonlinear as well as linear Hille-Yosida theorem.

The author was also informed by the referee that J.-B. Hiriart-Urruty [9] had defined the second differential for convex functions. Roughly speaking, it corresponds to a convex set $C$ of which support function satisfies $\psi_C^*(h) = |(Q^0_{x,y})^{1/2} h|$. See [9, 11] for further details.

Finally, we note that, by (4.3), our second derivatives are related to the tangent cones (cf. [5]).

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DEPARTMENT OF MATHEMATICS, SCHOOL OF EDUCATION, WASEDA UNIVERSITY, TOKYO 160, JAPAN

Current address: Department of Mathematics, Faculty of Science, Shimane University, Matsue, Shimane 690, Japan