ON THE COHERENCE AND WEAK DIMENSION OF THE RINGS $R(x)$ AND $R(x)$

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Abstract. Let $R$ be a commutative ring. We first derive necessary and sufficient conditions for the rings $R(x)$ and $R(x)$ to be coherent. Next, for stably coherent rings of finite weak dimension exact relations are found between the weak dimension of $R$ and that of $R(x)$ and $R(x)$. These relations are used to determine necessary and sufficient conditions for $R(x)$ and $R(x)$ to be Von Neumann regular or semihereditary.

1. Introduction

Let $R$ be a commutative ring. $R$ is called a regular ring if every finitely generated ideal of $R$ has finite projective dimension. This notion, which has been extensively studied for Noetherian rings, was extended to coherent rings with a considerable degree of success, [8, 15, 16, 17, 29, 33]. For a coherent ring $R$, the regularity condition is closely related to the behaviour of the weak dimension of modules over $R$. In particular, a coherent ring of finite weak dimension is a regular ring, although not every coherent regular ring has finite weak dimension [15]. The class of coherent regular rings includes several of the classical non-Noetherian rings, like Von Neumann regular rings and semihereditary rings.

Let $R$ be a ring and let $S$ be an $R$ algebra. The type of investigation carried out in this paper considers the following kind of questions: Under what conditions will the extension $R \to S$ ascend or descend coherence and regularity? In particular what is the exact relation between the weak dimension of $R$, and that of $S$; and what necessary and sufficient conditions will ascend and descend Von Neumann regularity and semihereditariness?

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The algebras $S$ considered in this paper are two well-known localizations of the polynomial ring in one variable over $R$, $R(x)$ and $R(x)$. For a polynomial $f \in R[x]$, denote by $c(f)$—the so-called content of $f$—the ideal of $R$ generated by the coefficients of $f$. Let

$$U = \{f \in R[x], f \text{ is monic}\}$$

and

$$V = \{f \in R[x], c(f) = R\} = R[x] - \bigcup\{mR[x], m \in \text{Max}(R)\}.$$  

$U$ and $V$ are multiplicatively closed subsets of $R[x]$, and $R(x) = R[x]_U$, $R(x) = R[x]_V$. Note that $R[x] \subset R(x) \subset R(x)$, and $R(x)$ is a localization of $R(x)$.

The ring $R(x)$ is a very useful ring construction in commutative algebra. As a faithfully flat extension of $R$, it shares many of the properties of $R$. In addition it satisfies several other useful properties, which facilitate proving many results on $R$ via passage to $R(x)$. Ascent and descent properties of the extension $R \to R(x)$ have been investigated by a number of authors. In [1], Akiba investigates normality of $R(x)$. In [6], D. D. Anderson, D. F. Anderson and Markanda (and in [4, 5]) conduct a thorough study of the properties of $R(x)$. Between other results they touch on conditions related to semihereditarity, namely that of being a Prüfer ring, a strongly Prüfer ring, and an arithmetical ring. Arnold [7], Hinkle, Huckaba [18], Huckaba, Papick [20, 21] relate the ring $R(x)$ and several other ring constructions to Prüfer and Prüfer like conditions of the ring $R$ and $R(x)$. Ferrand [14], McDonald, Waterhouse [26] investigate projective modules over $R(x)$. Ratliff [31] studies $R(x)$ with regard to certain chain conditions.

The ring $R(x)$ received a considerable amount of attention due to its role in Quillen’s solution to Serre’s Conjecture [30, 19]; and the non-Noetherian extensions of this conjecture [10, 23]. Ascent and descent properties of the extension $R \to R(x)$ have been investigated by a number of authors. In [6, 20, 21] the authors conduct investigations of $R(x)$ analogous and intertwining with the ones of $R(x)$. Brewer, Heinzer [11], determine conditions for $R(x)$ to be a Hilbert ring. Le Riche [24] provides an in-depth study of many of the properties of $R(x)$. Between other results he determines necessary and sufficient conditions for $R(x)$ to be a semihereditary ring.

In this paper we first derive necessary and sufficient conditions for $R(x)$ and $R(x)$ to be coherent rings. This is done in Theorem 1. We next explore the relations between the weak dimension of $R$ and that of $R(x)$ and $R(x)$. In Theorem 2, using the notion of non-Noetherian grade, we pinpoint exact relations between these weak dimensions, provided that $R$ is a stably coherent ring of finite weak dimension. As corollaries, we determine necessary and sufficient conditions for $R(x)$ and $R(x)$ to be Von Neumann regular, for $R(x)$ to be semihereditary; and recapture Le Riche [24] necessary and sufficient conditions for $R(x)$ to be semihereditary.
2. Main results

Using a device of Gr"{u}ison, as in [12], we prove the following.

**Theorem 1.** Let $R$ be a ring and let $x$ be an indeterminate over $R$, then the following conditions are equivalent.

1. $R[x]$ is a coherent ring.
2. $R(x)$ is a coherent ring.
3. $R(x)$ is a coherent ring.

**Proof.** Clearly we need only prove that (3) implies (1). Let $T$ be an arbitrary set, and consider the exact sequence $0 \rightarrow R[x]^T \xrightarrow{\phi} R(x)^T \rightarrow \text{coker} \phi \rightarrow 0$, where $\phi$ is the natural map. According to [13], it suffices to show that $R[x]^T$ is a flat $R[x]$ module. Let $I$ be a finitely generated ideal of $R$, then $IR(x) \cap R[x] = IR[x]$, therefore $R[x]$ is a pure $R$ submodule of $R(x)$ [32, Theorem 3.44], and thus $R[x]^T$ is a pure $R$ submodule of $R(x)^T$. Since $R(x)$ is a coherent ring $R(x)^T$ is a flat $R(x)$ module, therefore both a flat $R$ module and a flat $R[x]$ module. We conclude by [9, p. 18] that coker $\phi$ is a flat $R$ module. Thus w. dim$_{R[x]} \text{coker} \phi \leq 1$ [22, Theorem 3, p. 172], and therefore $R[x]^T$ is a flat $R[x]$ module.

Recall that a ring $R$ is called a *stably coherent ring*, if for every positive integer $n$ the polynomial ring in $n$ variables over $R$ is a coherent ring along with $R$. The class of stably coherent rings includes a wide variety of rings. To name a few: Noetherian rings, Von Neumann regular rings, semihereditary rings, coherent rings of global dimension two, and several others, e.g., [34, 17]. If $R$ is a stably coherent ring, then $R(x)$ and $R(x)$ are coherent rings. Theorem 1, proves that if $R(x)$ or $(R(x))$ is a coherent ring so is $R[x]$. It is still an open question whether the coherence of $R[x]$ suffices to imply the stably coherence of $R$.

We next explore the homological properties of $R(x)$ and $R(x)$ as exhibited in the behaviour of their weak dimensions. Regularity itself is easily disposed of as follows.

**Proposition 1.** Let $R$ be a ring for which $R[x]$ is a coherent ring, then the following conditions are equivalent:

1. $R$ is a regular ring.
2. $R(x)$ is a regular ring.
3. $R(x)$ is a regular ring.

**Proof.** To prove (1) $\rightarrow$ (2) use [16, Proposition 2.5]. To prove (3) $\rightarrow$ (1) use [15, Lemma 2].

We will embark on a brief discussion of non-Noetherian grade as defined by Alfonsi [2, 3], and its relation to the weak dimension for regular coherent rings.
As a definition of grade for a finitely presented module we will adopt its equivalent condition [3, Proposition 1.2]. Let \( R \) be a ring, let \( M \) be a finitely presented \( R \) module, and let \( N \) be an \( R \) module, then \( \text{grade}_R(M, N) \geq n \) if there exists a faithfully flat \( R \) algebra \( S \), which may be taken to be a polynomial extension of \( R \), and elements \( f_1, \ldots, f_n \in (0 :_RM \otimes_RS) \) which form an \( N \otimes_RS \) regular sequence. The largest such integer \( n \) is the \( \text{grade}_R(M, N) \). If no largest integer \( n \) exists put \( \text{grade}_R(M, N) = \infty \).

If \( M \) is a general \( R \) module, then \( \text{grade}_R(M, N) \geq n \) if for every \( y \in M \), \( (0 :_Ry) \) contains a finitely generated ideal \( I_y \) satisfying \( \text{grade}_R(R/I_y, M) \geq n \).

It is clear that if \( M \) is a finitely presented \( R \) module and \( S \) is a faithfully flat \( R \) algebra then \( \text{grade}_R(M, N) = \text{grade}_S(M \otimes S, N \otimes S) \). To show that this conclusion remains valid for any \( R \) module \( M \), we first cite a Lemma proved in [2, Proposition 1.6].

**Lemma 1.** Let \( R \) be a ring, let \( N \) be an \( R \) module, and let \( I \) and \( J \) be two finitely generated ideals of \( R \), then

1. If \( I \subset J \) and \( \text{grade}_R(R/I, N) \geq n \) then \( \text{grade}_R(R/J, N) \geq n \).
2. If \( \text{grade}_R(R/I, N) \geq n \) and \( \text{grade}_R(R/J, N) \geq n \) then \( \text{grade}_R(R/IJ, N) \geq n \).

**Lemma 2.** Let \( R \) be a ring, let \( M \) and \( N \) be two \( R \) modules, and let \( S \) be a faithfully flat \( R \) algebra, then \( \text{grade}_R(M, N) = \text{grade}_S(M \otimes S, N \otimes S) \).

**Proof.** Assume that \( \text{grade}_R(M, N) \geq n \). Let \( y = \sum_{i=1}^k y_i \otimes b_i \in M \otimes S \), and let \( I_{y_i} \subset (0 :_Ry_i) \) be finitely generated ideals satisfying \( \text{grade}_R(R/I_{y_i}, N) \geq n \). Then \( I = \prod I_{y_i} \) satisfies \( \text{grade}_R(R/I, N) \geq n \), and thus \( \text{grade}_S(S/IS, N \otimes S) \geq n \). But \( IS \subset (0 :_Sy) \), thus \( \text{grade}_S(M \otimes S, N \otimes S) \geq n \).

Assume that \( \text{grade}_S(M \otimes S, N \otimes S) \geq n \). Let \( y \in M \) and let \( I \subset (0 :_Sy \otimes 1) = (O :_Ry)S \) be a finitely generated ideal of \( S \) satisfying

\[ \text{grade}_S(S/I, N \otimes S) \geq n. \]

Let \( J \) be the finitely generated ideal contained in \( (0 :_Ry) \) satisfying \( I \subset JS \). Then \( \text{grade}_R(R/J, N) = \text{grade}_S(S/JS, N \otimes S) \geq n \), therefore \( \text{grade}_R(M, N) \geq n \).

Let \( (R, m) \) be a local ring with maximal ideal \( m \), and let \( M \) be an \( R \) module, the depth of \( M \) is defined as: \( \text{depth}_R M = \text{grade}_R(R/m, M) \).

Let \( R \) be a ring, the small finitistic projective dimension of \( R \), is defined as follows: \( \text{f. p. dim } R = \sup \{ \text{proj. dim } M \mid M \text{ is an } R \text{ module admitting a resolution consisting of finitely generated projective } R \text{ modules, and } \text{proj. dim } M < \infty \} \).

**Lemma 3.** Let \( R \) be a local coherent regular ring then \( \text{depth } R = \text{w. dim } R \).

**Proof.** By [3, Corollary 2.7] we have \( \text{depth } R = \text{f. p. dim } R \). Since \( R \) is a coherent ring any finitely presented \( R \) module \( M \) satisfies \( \text{w. dim } M = \text{proj. dim } M \).
[28, Lemma 1.2], and admits a resolution consisting of finitely generated free modules. Since \( R \) is a coherent regular ring any finitely generated ideal of \( R \) has finite projective dimension, hence any finitely presented cyclic \( R \) module has finite projective dimension. It follows by induction on the number of generators of a finitely presented \( R \) module \( M \), that \( \text{proj.dim} M < \infty \). We conclude that \( \text{f.p.dim} R = \text{w.dim} R \), and the claim follows.

**Lemma 4.** Let \( R \) be a ring for which \( R[x] \) is a coherent ring then

1. \( \text{w.dim} R \leq \text{w.dim} R(x) \leq \text{w.dim} R + 1 \).
2. \( \text{w.dim} R \leq \text{w.dim} R(x) \leq \text{w.dim} R + 1 \).

**Proof.** The left-hand side inequalities follow from the fact that \( R(x) \) and \( R(x) \) are faithfully flat \( R \) modules [27, Proposition 1.34]. The right-hand side inequalities follow from the fact that \( \text{w.dim} R[x] = \text{w.dim} R + 1 \) [34, Theorem 0.14].

**Theorem 2.** Let \( R \) be a stably coherent ring of \( \text{w.dim} R = n < \infty \), then

1. \( \text{w.dim} R(x) = \text{w.dim} R \).
2a. If for every prime ideal \( p \) of \( R \) which is not maximal we have \( \text{depth} R_p < n \), then \( \text{w.dim} R(x) = \text{w.dim} R \).
2b. Otherwise \( \text{w.dim} R(x) = \text{w.dim} R + 1 \).

**Proof.**

1. There is a 1:1 correspondence between maximal ideals of \( R \) and maximal ideals of \( R(x) \), given by \( m \mapsto mR(x) \), and satisfying \( R(x)_{mR(x)} = R_m(x) \). Consider the faithfully flat local homomorphism

\[
(R_m, mR_m) \rightarrow (R_m(x), mR_m(x)).
\]

By Lemma 2, \( \text{depth} R_m = \text{depth} R_m(x) \). By Lemma 3,

\[
\text{w.dim} R_m = \text{w.dim} R_m(x).
\]

Taking supremum over all the maximal ideals \( m \) of \( R \), on both sides we obtain \( \text{w.dim} R = \text{w.dim} R(x) \).

2. Note that for every prime ideal \( p \) of \( R \), depth \( R_p = \text{w.dim} R_p \leq \text{w.dim} R = n \).
2a. Assume that depth \( R_p < n \) for all nonmaximal ideals \( p \) of \( R \). By Lemma 4, our claim will be complete if we show that for every maximal ideal \( M \) of \( R(x) \) we have \( \text{w.dim} R(x)_M = n \).

Let \( M = PR(x) \), where \( P \) is a prime ideal of \( R[x] \) not containing a monic polynomial, then \( R(x)_M = (R[x]_U)_{PR[x]_U} = R[x]_P \).

Let \( p = P \cap R \).

If \( P = pR[x] \), then \( R[x]_P = R[x]_{pR[x]} = R_p[x]_{pR[x]} = R_p(x) \). By (1) we obtain \( \text{w.dim} R[x]_P = \text{w.dim} R_p(x) = \text{w.dim} R_p \leq n \).

If \( P \nsubseteq pR[x] \) we have two possibilities. If \( p \) is a maximal ideal of \( R \), then \( P \) contains a monic polynomial and need not be taken into account. If \( p \) is not
a maximal ideal of \( R \), then \( R_p[x] = R[x]_{(R-p)} \), thus \( R[x]_p = R_p[x]_{P_{R_p}[x]} \),
and \( \text{w.dim} \ R[x]_p = \text{w.dim} \ R_p[x]_{P_{R_p}[x]} \leq \text{w.dim} \ R_p[x] = \text{w.dim} \ R_p + 1 = \text{depth} \ R_p + 1 \leq n \).

(2b) Let \( p \) be a nonmaximal ideal of \( R \) satisfying \( \text{depth} \ R_p = n \). If \( n = 0 \), then \( R \) is a Von Neumann regular ring, hence every ideal of \( R \) is maximal, and this case falls in the category of (2a). Thus \( n \geq 1 \). By Lemma 4, it suffices to construct a prime ideal \( Q \) in \( R[x] \) satisfying \( Q \cap R = p \), \( Q \) contains no monic polynomial, \( pR[x] \not\subseteq Q \) and \( \text{depth} \ R[x]_Q \geq n + 1 \).

Let \( p \not\subseteq m \) for a maximal ideal \( m \) of \( R \) and let \( a \in m - p \). Set \( Q = pR[x] + (ax+1)R[x] \). It is clear that \( Q \) contains no monic polynomial.

To show that \( Q \) is a prime ideal we follow an argument given by Le Riche [24]. We note that it suffices to show that \( Q/pR[x] \) is a prime ideal of

\[ R[x]/pR[x] = R/p[x]. \]

Let \( F \) be the field of quotients of \( R/p \), then the image of \( Q/pR[x] \) in \( F[x] \) is generated by the irreducible polynomials \( (a+p)x + (1+p) \), and is therefore a prime ideal. Thus \( Q/pR[x] \) is a prime ideal.

To show that \( Q \cap R = p \), let \( r \in Q \cap R \). Write \( r = f(x) + g(x)(ax + 1) \), \( f(x) = b_kx^k + \cdots + b_0 \), \( g(x) = c_kx^k + \cdots + c_0 \), \( b_i \in p \), \( 0 \leq i \leq k \), \( c_j \in R \), \( 0 \leq j \leq k \). We substitute these expressions of \( f(x) \) and \( g(x) \) in the quality describing \( r \), and compare coefficients of powers of \( x \) on both sides. Since \( a \notin p \) but \( b_i \in p \), \( 0 \leq i \leq k \) we obtain that \( c_i \in p \), \( 0 \leq i \leq k \) and thus \( r = b_0 + c_0 \in p \).

We will now show that depth \( R[x]_Q \geq n + 1 \). Since \( n = \text{depth} \ R_p = \text{grade} \ R_p(R_p/pR_p, R_p) \) up to a polynomial extension of \( R_p \), we may assume that there are elements \( a_1, \ldots, a_n \in p \) such that \( a_1, \ldots, a_n \in pR_p \) form an \( R_p \) regular sequence. Now \( R[x]_Q = R_p[x]_{QR_p[x]} \), therefore it suffices to show that \( a_1, \ldots, a_n, ax+1 \in QR_p[x] \) form an \( R_p[x]_{QR_p[x]} \) regular sequence.

Since \( a_1, \ldots, a_n \) is an \( R_p \) regular sequence, it is an \( R_p[x] \) regular sequence, and as \( a_1, \ldots, a_n \in QR_p[x] \), it stays an \( R_p[x]_{QR_p[x]} \) regular sequence.

Now, let \( (f(x)/g(x))(ax + 1) = (f_1(x)/g(x))a_1 + \cdots + (f_n(x)/g(x))a_n \) with \( g(x), f(x), f_i(x) \in R_p[x] \) and \( g(x) \in R_p[x] - QR_p[x] \). Since \( R \) is a coherent ring of finite weak dimension, \( R_p \), and hence \( R_p[x] \) is a domain [34, Corollary 5.16], thus

\[ f(x)(ax + 1) = f_1(x)a_1 + \cdots + f_n(x)a_n. \]

Let \( f(x) = b_kx^k + \cdots + b_0 \), \( f_i(x) = c_i^jx^i + \cdots + c_0^i \) with \( b_j, c_j^i \in R_p \), \( 0 \leq j \leq k \), \( 1 \leq i \leq n \). Substituting these expressions of \( f(x) \) and \( f_i(x) \) in the above equality, and comparing coefficients of powers of \( x \) on both sides we obtain that \( b_j \in (a_1, \ldots, a_n)R_p \) for \( 0 \leq j \leq k \) and thus \( f(x)/g(x) \in (a_1, \ldots, a_n)R_p[x]_{QR_p[x]} \). We conclude that \( a_1, \ldots, a_n, ax + 1 \) form an \( R_p[x]_{QR_p[x]} \) regular sequence.

**Corollary 1.** Let \( R \) be a Noetherian regular ring, then \( \text{w.dim} \ R(x) = \text{w.dim} R \).
Proof. If \( w.\dim R = \infty \), by Lemma 4 we are done. Otherwise \( w.\dim R = n < \infty \). By [25, p. 156], for every prime ideal \( p \) of \( R \) we have \( \text{depth} R_p = w.\dim R_p = \text{gl.\dim} R_p = \text{Krull\ dim} R_p = htp \). Thus Krull \( \dim R = n \) and the only prime ideals \( p \) of height \( n \) are maximal ideals. By Theorem 2 (2a) the conclusion follows.

**Corollary 2.** Let \( R \) be a ring. The following conditions are equivalent.

1. \( R \) is a Von Neumann regular ring.
2. \( R(x) \) is a Von Neumann regular ring.
3. \( (x) \) is a Von Neumann regular ring.

Proof. A ring \( A \) is a Von Neumann regular ring if and only if \( w.\dim A = 0 \). To show (1) \( \rightarrow \) (2) use the fact that every ideal of \( R \) is maximal and Theorem 2 (2a). To show (3) \( \rightarrow \) (1) use Theorem 1 and Lemma 4.

Note that Corollary 2 also easily follows from the fact that \( R \) is a Von Neumann regular ring if and only if Krull \( \dim R = 0 \) and \( R \) is reduced. Also, since Krull \( \dim R = 0 \), here, \( R(x) = R(x) \).

Since a ring \( A \) is a semisimple ring if and only if \( A \) is a Von Neumann regular Noetherian ring we obtain that \( R \) is a semisimple ring if and only if \( R(x) \) is a semisimple ring if and only if \( R(x) \) is a semisimple ring.

**Corollary 3.** Let \( R \) be a ring, the following conditions are equivalent.

1. \( R \) is a semihereditary ring.
2. \( R(x) \) is a semihereditary ring.

Proof. A ring \( A \) is a semihereditary ring if and only if \( A \) is a coherent ring of \( w.\dim A \leq 1 \) [28, Proposition 2.2]. Now use Theorem 2 (1) and Lemma 4.

We now recapture Le Riche's results [24].

**Corollary 4.** Let \( R \) be a ring. The following conditions are equivalent.

1. \( R \) is a semihereditary ring of \( \text{Krull\ dim} R \leq 1 \).
2. \( R(x) \) is a semihereditary ring.

Proof. (1) \( \rightarrow \) (2). Let \( p \) be a nonmaximal ideal of \( R \); then \( p \) is minimal. Since \( R_p \) is a domain [34, Corollary 5.16], it is a field, thus \( \text{depth} R_p = 0 \). By Theorem 1 and Theorem 2 (2a), \( R(x) \) is semihereditary.

(2) \( \rightarrow \) (1). If \( R(x) \) is a semihereditary ring using Lemma 4, and the faithful flatness of \( R(x) \) over \( R \) we conclude that \( R \) is a semihereditary ring. If \( w.\dim R = 0 \) then \( \text{Krull\ dim} R = 0 \). If \( w.\dim R = 1 \) then \( w.\dim R = w.\dim R(x) \) and by Theorem 2 (2), \( \text{depth} R_p = 0 \) for every nonmaximal prime ideal \( p \) of \( R \). Since \( R_p \) is a domain, we conclude that \( R_p \) is a field, and thus \( p \) is minimal and \( \text{Krull\ dim} R \leq 1 \).

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References

31. L. Ratliff, $A(x)$ and GB-Noetherian rings, Rocky Mountain J. Math. 9 (1979), 337–353.

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