Abstract. If \( f \) is a continuous surjection from a normal space \( X \) onto a regular space \( Y \), then there are a space \( Z \) and a perfect map \( bf : Z \to Y \) extending \( f \) such that \( X \subseteq Z \subseteq \beta X \). If \( f \) is a continuous surjection from normal \( X \) onto Tychonov \( Y \) and \( \beta X \setminus X \) is sequential, then \( Y \) is normal. More generally, if \( f \) is a continuous surjection from normal \( X \) onto regular \( Y \) and \( \beta X \setminus X \) has the property that countably compact subsets are closed (this property is called \( C \)-closed), then \( Y \) is normal. There is an example of a normal space \( X \) such that \( \beta X \setminus X \) is \( C \)-closed but not sequential. If \( X \) is normal and \( \beta X \setminus X \) is first countable, then \( \beta X \setminus X \) is locally compact.

We began the study of the class of spaces ACRIN (all continuous regular images normal) in [FL], where we showed that if \( X \) is a normal space such that \( \beta X \setminus X \) is finite, then \( X \) is ACRIN. Here, we give more general conditions on \( \beta X \setminus X \) which imply that a normal space is ACRIN.

For a Tychonov space \( X \), we denote the Stone-Cech compactification of \( X \) by \( \beta X \). We call a space \( Z \) such that \( X \subseteq Z \subseteq \beta X \) an intermediate space. If \( X \) and \( Y \) are Tychonov spaces and \( f : X \to Y \) is a continuous map, we denote by \( \beta f : \beta X \to \beta Y \) the Stone extension of \( f \).

Let us recall some well-known properties of maps.

1. Lemma. Let \( f : X \to Y \) be a continuous surjection. (a) If \( X \) is normal and \( f \) is a closed mapping, then \( Y \) is normal.
(b) If \( X \) is countably compact, then \( Y \) is countably compact.
(c) If \( f \) is perfect and \( Y \) is countably compact, then \( X \) is countably compact.
(d) If \( \beta f^{-1}(\beta X \setminus X) = \beta Y \setminus Y \), then \( f \) is perfect.

Let us fix a continuous surjection \( f : X \to Y \) from a normal space to a regular space. Let \( bf : P \to Y \) be a perfect map extending \( f \), where \( P \) is an intermediate space. If \( Y \) is Tychonov, we will set \( P = \beta f^{-1}(Y) \) and...
$bf = \beta f \upharpoonright P$. We defer the discussion of how to define $P$ and $bf$ in the case when $Y$ is regular but not Tychonov. From Lemma 1, we see that if $P$ is normal, then $Y$ is normal. Thus, we seek conditions which imply that $P$ is normal.

One such condition is to assert that every intermediate space $Z$ is normal. Barr and Hajek introduced this notion in [BH] and called it normality inducing. Further, they showed that if $X$ is normality inducing, then $X$ is countably compact, and they showed that $X$ is normality inducing if and only if every compact subset of $\beta X \setminus X$ is finite.

A condition which implies that an intermediate space $Z$ is normal is that $Z \setminus X$ is closed in $\beta X \setminus X$. Then $Z$ is normal because it is the union of the normal space $X$ and the compact space $\text{Cl}_{\beta X}(Z \setminus X)$. (See [FL, Lemma 1.1(c)].) Thus, if $X$ is normal and $\beta X \setminus X$ is finite, or more generally, discrete, then $X$ is normality inducing, and hence ACRIN.

Let us assume that $X$ is countably compact. Again from Lemma 1, we see that $P$ is countably compact. Thus, it is not necessary that every intermediate space $Z$ be normal; it is enough to require that countably compact intermediate spaces be normal.

2. Proposition. If $\beta X \setminus X$ is sequential and $Y$ is Tychonov, then $P \setminus X$ is closed in $\beta X \setminus X$. Hence, $Y$ is normal.

Proof. Suppose that $q \in \beta X \setminus X$ and there is a sequence $(p_n)_{n \in \omega}$ in $P \setminus X$ converging to $q$. By the definition of $P$, each $\beta f(p_n)$ is an element of $Y$. Since $Y$ is countably compact, $(\beta f(p_n))_{n \in \omega}$ has a cluster point $y$ in $Y$. By continuity, $(\beta f(p_n))_{n \in \omega}$ converges to $y$ and $y = \beta f(q)$. Therefore, $q \in P$, and $P \setminus X$ is sequentially closed in $\beta X \setminus X$. Because $\beta X \setminus X$ is sequential, $P \setminus X$ is closed in $\beta X \setminus X$. 

A space is called $C$-closed if every countably compact subset is closed. (See [IN].) For example, sequential spaces are $C$-closed. Countable spaces, which need not be sequential, and $P$-spaces, which are sequential if and only if they are discrete, are $C$-closed. If $X$ is normal and not countably compact, then $X$ contains a closed copy of $\omega$. Thus, $\beta X \setminus X$ contains a closed copy of $\beta \omega \setminus \omega$. Therefore, $\beta X \setminus X$ is not $C$-closed—it contains the nonclosed countably compact subset $(\beta \omega \setminus \omega \setminus \{p\})$ where $p \in \beta \omega \setminus \omega$.

3. Theorem. If $\beta X \setminus X$ is $C$-closed, then every countably compact intermediate space $Z$ is normal. Hence $X$ is ACRIN.

Proof. Towards a contradiction, assume that $H$ and $K$ are disjoint closed subsets of a countably compact intermediate space $Z$ and that $q \in [\text{Cl}_{\beta X}(H) \cap \text{Cl}_{\beta X}(K)] \setminus Z$. If $q$ were an element of $\text{Cl}_{\beta X}(H \cap X) \cap \text{Cl}_{\beta X}(K \cap X)$, then $X$ would not be normal. Assume without loss of generality that $q \not\in \text{Cl}_{\beta X}(H \cap X)$.

By regularity, there is an open subset $U$ of $\beta X$ with $\text{Cl}_{\beta X}(H \cap X) \subseteq U \subseteq \text{Cl}_{\beta X}(U) \subseteq \beta X \setminus \{q\}$. Then $H \setminus U$ is a countably compact subset of $\beta X \setminus X$. Hence $Y$ is normal.
(since it is a closed subset of $Z$), and since $\beta X \setminus X$ is assumed to be $C$-closed, 
$q \in \text{Cl}_{\beta X \setminus X}(H \setminus X) = H \setminus U \subseteq Z$, contradicting $q \notin Z$. □

If $Y$ is a locally compact space and $\alpha Y = Y \cup \{\infty\}$ is its one-point compactification, then for large enough cardinal $\gamma$, the subspace $(\alpha Y \times \gamma) \cup (\infty, \gamma)$ of $\alpha Y \times (\gamma + 1)$ is a normal space whose Stone-Cech remainder is $Y$. Thus, every locally compact space is a remainder of a normal space. The following result limits the applicability of Theorem 3 to exactly the locally compact spaces if $\beta X \setminus X$ is first countable.

4. Proposition. If $X$ is normal and $\beta X \setminus X$ is first countable, then $\beta X \setminus X$ is locally compact.

Proof. Towards a contradiction, suppose that $p \in \beta X \setminus X$ has a countable nested base in $\beta X \setminus X$ such that for all $n$, $\text{Cl}_{\beta X \setminus X}(B_n)$ is not compact. Since $\text{Cl}_{\beta X \setminus X}(B_n)$ is compact, we may choose distinct $x_n \in \text{Cl}_{\beta X \setminus X}(B_n) \cap X$. Then 
\{\{x_{2n} : n \in \omega\} \text{ and } \{x_{2n+1} : n \in \omega\}\} are disjoint closed subsets of $X$, both of whose closures in $\beta X$ contain $p$. This contradicts the normality of $X$. □

There are normal spaces whose Stone-Cech compactifications are $C$-closed, but not locally compact. For example, it is not hard to show that if $E$ is an $\aleph_1$-set with the order topology and $X$ is the set of non-$P$-points of the Dedekind compactification of $E$, then $\beta X \setminus X$ is a $P$-space without isolated points, and hence is $C$-closed but not locally compact. Since $X$, a linearly ordered space is normal, it follows from Theorem 3 that every continuous image of $X$ is normal.

We now give alternative definitions of $bf$ and $P$ which require only that $Y$ be regular. The general situation is this: $f$ is a continuous surjection from the normal space $X$ to a regular space $Y$, and $Z$ is a Hausdorff extension of $X$, that is, a Hausdorff space which contains $X$ as a dense subspace. For $p \in Z$, define $\mathcal{M}_p = \{N \cap X : N \text{ is a neighborhood of } p \text{ in } Z\}$. Let $G$ be $\text{Cl}_{Z \setminus Y}(\text{graph } f)$.

5. Lemma. Let $f$, $X$, $Y$, $Z$, and $G$ be as above. (a) For $p \in Z \setminus X$, $f \cup \{(p, y)\}$ is continuous if and only if $f[\mathcal{M}_p]$ converges to $y$.

(b) If for all $p \in Z$, $f \cup \{(p, y_p)\}$ is a continuous function, then $f \cup \{(p, y_p) : p \in Z\}$ is a continuous function.

(c) $(p, y) \in G$ if and only if $f[\mathcal{M}_p]$ adheres to $y$.

Further assume that $X$ is normal and $Z = \beta X$.

(d) If $f[\mathcal{M}_p]$ adheres to $y$, then $f[\mathcal{M}_p]$ converges to $y$.

(e) Hence, $G$ is the graph of a function $bf$ and $bf$ is perfect.

Proof. (a) and (b) are [PW, 4.1(1) and 4.1(n)]; (c) is routine. Proof of (d): Because $X$ is normal and $Z = \beta X$, we may consider points of $Z$ to be ultrafilters of closed subsets of $X$ and basic open sets have the form $N(F) = \{q \in \beta X : F \notin q\}$. We prove the contrapositive. Suppose that $f[\mathcal{M}_p]$ does not converge to $y$. There is a neighborhood $V$ of $y$ which does not contain $f^{-1}(N)$
for any $N \in \mathcal{N}_p$. Because $Y$ is regular, there is an open $W$ such that $y \in W$ and $\text{Cl}_Y W \subseteq V$. Consider $H = f^{-1}(\text{Cl}_Y W) = \{x \in X : f(x) \in \text{Cl}_Y W\}$. Then $N(H) \in \mathcal{N}_p$ and $f^{-1}(N(H)) \cap W = \emptyset$. Thus, $f[\mathcal{N}_p]$ does not adhere to $y$.

We have arranged things so that the proof of (e) is easy. By (c), (d), and the fact that points of convergence are unique in Hausdorff spaces, $b\beta f$ is a function. Continuity follows from the previous parts. Because $Z$ is compact, the projection onto $Y$ is a closed map; $b\beta f$ is the restriction of projection to the closed set $G$, so it is also closed. Finally, $b\beta f^{-1}\{y\}$ is $G \cap (Z \times \{y\})$, the intersection of a closed set and a compact set. □

We wish to thank Jack Porter for discussions which transformed our original ad hoc, dot and circle construction into Lemma 5.

We close with some questions.

**Question 1.** If $X$ is normal and $\beta X \setminus X$ is countable, is $\beta X \setminus X$ sequential? If $X$ is normal and $\beta X \setminus X$ is sequential, is $\beta X \setminus X$ locally compact?

**Question 2.** If $X$ is normal and $\beta X \setminus X$ has countable tightness, is $X$ ACRIN? Is there, without extra axioms of set theory, a regular space of countable tightness which is not $C$-closed? (Balogh proved, assuming PFA, that locally compact spaces of countable tightness are $C$-closed. Fedorchuk constructed, assuming $\diamondsuit$, a Tychonov space of countable tightness which is not $C$-closed.)

**References**


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