STONE-CECH REMAINDERS
WHICH MAKE CONTINUOUS IMAGES NORMAL

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Abstract. If $f$ is a continuous surjection from a normal space $X$ onto a regular space $Y$, then there are a space $Z$ and a perfect map $bf : Z \to Y$ extending $f$ such that $X \subseteq Z \subseteq \beta X$. If $f$ is a continuous surjection from normal $X$ onto Tychonov $Y$ and $\beta X\setminus X$ is sequential, then $Y$ is normal. More generally, if $f$ is a continuous surjection from normal $X$ onto regular $Y$ and $\beta X\setminus X$ has the property that countably compact subsets are closed (this property is called C-closed), then $Y$ is normal. There is an example of a normal space $X$ such that $\beta X\setminus X$ is C-closed but not sequential. If $X$ is normal and $\beta X\setminus X$ is first countable, then $\beta X\setminus X$ is locally compact.

We began the study of the class of spaces ACRIN (all continuous regular images normal) in [FL], where we showed that if $X$ is a normal space such that $\beta X\setminus X$ is finite, then $X$ is ACRIN. Here, we give more general conditions on $\beta X\setminus X$ which imply that a normal space is ACRIN.

For a Tychonov space $X$, we denote the Stone-Cech compactification of $X$ by $\beta X$. We call a space $Z$ such that $X \subseteq Z \subseteq \beta X$ an intermediate space. If $X$ and $Y$ are Tychonov spaces and $f : X \to Y$ is a continuous map, we denote by $\beta f : \beta X \to \beta Y$ the Stone extension of $f$.

Let us recall some well-known properties of maps.

1. Lemma. Let $f : X \to Y$ be a continuous surjection. (a) If $X$ is normal and $f$ is a closed mapping, then $Y$ is normal.
   (b) If $X$ is countably compact, then $Y$ is countably compact.
   (c) If $f$ is perfect and $Y$ is countably compact, then $X$ is countably compact.
   (d) If $\beta f^{-1}(\beta X\setminus X) = \beta Y\setminus Y$, then $f$ is perfect.

Let us fix a continuous surjection $f : X \to Y$ from a normal space to a regular space. Let $bf : P \to Y$ be a perfect map extending $f$, where $P$ is an intermediate space. If $Y$ is Tychonov, we will set $P = \beta f^{-1}(Y)$ and
We defer the discussion of how to define $P$ and $\beta f$ in the case when $Y$ is regular but not Tychonov. From Lemma 1, we see that if $P$ is normal, then $Y$ is normal. Thus, we seek conditions which imply that $P$ is normal.

One such condition is to assert that every intermediate space $Z$ is normal. Barr and Hajek introduced this notion in [BH] and called it normality inducing. Further, they showed that if $X$ is normality inducing, then $X$ is countably compact, and they showed that $X$ is normality inducing if and only if every compact subset of $\beta X \setminus X$ is finite.

A condition which implies that an intermediate space $Z$ is normal is that $Z \setminus X$ is closed in $\beta X \setminus X$. Then $Z$ is normal because it is the union of the normal space $X$ and the compact space $\text{Cl}_{\beta X}(Z \setminus X)$. (See [FL, Lemma 1.1(c)].) Thus, if $X$ is normal and $\beta X \setminus X$ is finite, or more generally, discrete, then $X$ is normality inducing, and hence ACRIN.

Let us assume that $X$ is countably compact. Again from Lemma 1, we see that $P$ is countably compact. Thus, it is not necessary that every intermediate space $Z$ be normal; it is enough to require that countably compact intermediate spaces be normal.

2. Proposition. If $\beta X \setminus X$ is sequential and $Y$ is Tychonov, then $P \setminus X$ is closed in $\beta X \setminus X$. Hence, $Y$ is normal.

Proof. Suppose that $q \in \beta X \setminus X$ and there is a sequence $(p_n)_{n \in \omega}$ in $P \setminus X$ converging to $q$. By the definition of $P$, each $\beta f(p_n)$ is an element of $Y$. Since $Y$ is countably compact, $(\beta f(p_n))_{n \in \omega}$ has a cluster point $y$ in $Y$. By continuity, $(\beta f(p_n))_{n \in \omega}$ converges to $y$ and $y = \beta f(q)$. Therefore, $q \in P$, and $P \setminus X$ is sequentially closed in $\beta X \setminus X$. Because $\beta X \setminus X$ is sequential, $P \setminus X$ is closed in $\beta X \setminus X$. □

A space is called $C$-closed if every countably compact subset is closed. (See [IN].) For example, sequential spaces are $C$-closed. Countable spaces, which need not be sequential, and $P$-spaces, which are sequential if and only if they are discrete, are $C$-closed. If $X$ is normal and not countably compact, then $X$ contains a closed copy of $\omega$. Thus, $\beta X \setminus X$ contains a closed copy of $\beta \omega \setminus \omega$. Therefore, $\beta X \setminus X$ is not $C$-closed—it contains the nonclosed countably compact subset $(\beta \omega \setminus \omega) \setminus \{p\}$ where $p \in \beta \omega \setminus \omega$.

3. Theorem. If $\beta X \setminus X$ is $C$-closed, then every countably compact intermediate space $Z$ is normal. Hence $X$ is ACRIN.

Proof. Towards a contradiction, assume that $H$ and $K$ are disjoint closed subsets of a countably compact intermediate space $Z$ and that $q \in (\text{Cl}_{\beta X}(H) \cap \text{Cl}_{\beta X}(K)) \setminus Z$. If $q$ were an element of $\text{Cl}_{\beta X}(H \cap X) \cap \text{Cl}_{\beta X}(K \cap X)$, then $X$ would not be normal. Assume without loss of generality that $q \notin \text{Cl}_{\beta X}(H \cap X)$. By regularity, there is an open subset $U$ of $\beta X$ with $\text{Cl}_{\beta X}(H \cap X) \subseteq U \subseteq \text{Cl}_{\beta X}(U) \subseteq \beta X \setminus \{q\}$. Then $H \setminus U$ is a countably compact subset of $\beta X \setminus X$...
(since it is a closed subset of \( Z \)), and since \( \beta X \setminus X \) is assumed to be \( C \)-closed, 
\[ q \in \text{Cl}_{\beta X \setminus X}(H \setminus X) = H \setminus U \subseteq Z, \]
contradicting \( q \notin Z \). \( \square \)

If \( Y \) is a locally compact space and \( \alpha Y = Y \cup \{\infty\} \) is its one-point compactification, then for large enough cardinal \( \gamma \), the subspace \( (\alpha Y \times \gamma) \cup (\infty, \gamma) \) of \( \alpha Y \times (\gamma + 1) \) is a normal space whose Stone-Cech remainder is \( Y \). Thus, every locally compact space is a remainder of a normal space. The following result limits the applicability of Theorem 3 to exactly the locally compact spaces if \( \beta X \setminus X \) is first countable.

**4. Proposition.** If \( X \) is normal and \( \beta X \setminus X \) is first countable, then \( \beta X \setminus X \) is locally compact.

**Proof.** Towards a contradiction, suppose that \( p \in \beta X \setminus X \) has a countable nested base in \( \beta X \setminus X \) such that for all \( n \), \( \text{Cl}_{\beta X \setminus X}(B_n) \) is not compact. Since \( \text{Cl}_{\beta X}(B_n) \) is compact, we may choose distinct \( x_n \in \text{Cl}_{\beta X}(B_n) \cap X \). Then \( \{x_{2n} : n \in \omega\} \) and \( \{x_{2n+1} : n \in \omega\} \) are disjoint closed subsets of \( X \), both of whose closures in \( \beta X \) contain \( p \). This contradicts the normality of \( X \). \( \square \)

There are normal spaces whose Stone-Cech compactifications are \( C \)-closed, but not locally compact. For example, it is not hard to show that if \( E \) is an \( \eta_1 \)-set with the order topology and \( X \) is the set of non-\( P \)-points of the Dedekind compactification of \( E \), then \( \beta X \setminus X \) is a \( P \)-space without isolated points, and hence is \( C \)-closed but not locally compact. Since \( X \), a linearly ordered space is normal, it follows from Theorem 3 that every continuous image of \( X \) is normal.

We now give alternative definitions of \( bf \) and \( P \) which require only that \( Y \) be regular. The general situation is this: \( f \) is a continuous surjection from the normal space \( X \) to a regular space \( Y \), and \( Z \) is a Hausdorff extension of \( X \), that is, a Hausdorff space which contains \( X \) as a dense subspace. For \( p \in Z \), define \( M_p = \{N \cap X : N \text{ is a neighborhood of } p \text{ in } Z\} \). Let \( G \) be \( \text{Cl}_{Z \times Y}(\text{graph } f) \).

**5. Lemma.** Let \( f \), \( X \), \( Y \), \( Z \), and \( G \) be as above. (a) For \( p \in Z \setminus X \), \( f \cup \{(p, y)\} \) is continuous if and only if \( f[M_p] \) converges to \( y \).

(b) If for all \( p \in Z \), \( f \cup \{(p, y_p)\} \) is a continuous function, then \( f \cup \{(p, y_p) : p \in Z\} \) is a continuous function.

(c) \( (p, y) \in G \) if and only if \( f[M_p] \) adheres to \( y \).

Further assume that \( X \) is normal and \( Z = \beta X \).

(d) If \( f[M_p] \) adheres to \( y \), then \( f[M_p] \) converges to \( y \).

(e) Hence, \( G \) is the graph of a function \( bf \) and \( bf \) is perfect.

**Proof.** (a) and (b) are [PW, 4.1(1) and 4.1(n)]; (c) is routine. Proof of (d): Because \( X \) is normal and \( Z = \beta X \), we may consider points of \( Z \) to be ultrafilters of closed subsets of \( X \) and basic open sets have the form \( N(F) = \{q \in \beta X : F \notin q\} \). We prove the contrapositive. Suppose that \( f[M_p] \) does not converge to \( y \). There is a neighborhood \( V \) of \( y \) which does not contain \( f^{-1}(N) \)
for any $N \in \mathcal{N}_p$. Because $Y$ is regular, there is an open $W$ such that $y \in W$ and $\text{Cl}_Y W \subseteq \bar{V}$. Consider $H = f^{-1}(\text{Cl}_Y W) = \{ x \in X : f(x) \in \text{Cl}_Y W \}$. Then $N(H) \in \mathcal{N}_p$ and $f^{-1}(N(H)) \cap W = \emptyset$. Thus, $f[\mathcal{N}_p]$ does not adhere to $y$.

We have arranged things so that the proof of (e) is easy. By (c), (d), and the fact that points of convergence are unique in Hausdorff spaces, $bf$ is a function. Continuity follows from the previous parts. Because $Z$ is compact, the projection onto $Y$ is a closed map; $bf$ is the restriction of projection to the closed set $G$, so it is also closed. Finally, $bf^{-1}\{y\}$ is $G \cap (Z \times \{y\})$, the intersection of a closed set and a compact set. □

We wish to thank Jack Porter for discussions which transformed our original ad hoc, dot and circle construction into Lemma 5.

We close with some questions.

**Question 1.** If $X$ is normal and $\beta X \setminus X$ is countable, is $\beta X \setminus X$ sequential? If $X$ is normal and $\beta X \setminus X$ is sequential, is $\beta X \setminus X$ locally compact?

**Question 2.** If $X$ is normal and $\beta X \setminus X$ has countable tightness, is $X$ ACRIN? Is there, without extra axioms of set theory, a regular space of countable tightness which is not $C$-closed? (Balogh proved, assuming PFA, that locally compact spaces of countable tightness are $C$-closed. Fedorchuk constructed, assuming $\Diamond$, a Tychonov space of countable tightness which is not $C$-closed.)

**References**


