EXISTENCE AND NONEXISTENCE OF RADIAL LIMITS OF MINIMAL SURFACES

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Abstract. A bounded solution of the minimal surface equation is constructed which has no radial limits at a boundary point.

Introduction

The Dirichlet problem for the minimal surface equation consists of determining a function $f = f(x, y)$ satisfying the equation

$$(1 + f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1 + f_x^2)f_{yy} = 0$$

in a domain $\Omega$ and taking on assigned values on the boundary of $\Omega$. The boundary behavior of solutions of the Dirichlet problem has been well studied (e.g. [5, 6, 7, 8, 9, 13, 15, 16, 19]); when $\Omega$ is convex and the assigned boundary values are continuous, the (unique) solution is in $C^0(\bar{\Omega})$ [16], but this is generally false when $\Omega$ is not convex (e.g. the intriguing example of Simon [20]). If $\Omega$ is locally convex at each point of $\partial\Omega$ except one, say $N$, and the assigned boundary values are continuous, then the (generalized) solution of the Dirichlet problem will (probably) not be continuous at $N$ and yet will have radial limits at $N$ from each direction in $\Omega$ [3, 10, 11]. If the assigned boundary values have a jump discontinuity at $N$, the (generalized) solution continues to have radial limits [11, 12].

Suppose $f$ is a bounded solution of the minimal surface equation in a domain $\Omega$, $N \in \partial\Omega$, and $f|_{\partial\Omega}$ is not defined at $N$ (e.g. [20]). How does $f$ behave at $N$? In this note we will construct a bounded solution of the minimal surface equation which has no radial limits at a point of the boundary. We also have some comments on the existence of radial limits.

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1. Nonexistence of radial limits

Let $\Omega$ be a convex domain in the plane. Suppose $N = (0,0) \in \partial \Omega$ and, say, $\{(x,0): 0 < x < 1\} \subseteq \Omega$. We will use a “gliding hump” argument to construct a bounded solution $f$ of the minimal surface equation in $\Omega$ such that $\lim_{x \to 0^+} f(x,0)$ does not exist.

First we need a lemma similar to a “localization” lemma for harmonic functions.

**Lemma.** Let $\varepsilon > 0$ and $h \in C^2(\Omega)$ be a solution of the minimal surface equation with $|h| \leq 2$. Then for each $\delta \in (0,1)$, there exists $g \in C^2(\Omega) \cap C^0(\Omega \cup \{N\})$ such that $g$ is a solution of the minimal surface equation in $\Omega$, $g(0,0) = 2$, $|g| \leq 2$, and $\sup\{|h(x,y) - g(x,y)|: (x,y) \in \Omega, \text{dist}((x,y),\partial \Omega) \geq \delta\} < \varepsilon$.

**Proof.** Let $g(x,y,t)$ be the solution of the Dirichlet problem with boundary values $h^*$ on $\partial \Omega \setminus B(N,t)$ and $2$ on $\partial \Omega \cap B(N,t)$, where $B(N,t)$ is the disc about $N$ of radius $t$ and $h^*$ is the trace of $h$ on $\partial \Omega$. If $0 < s < t$, then $h(x,y) < g(x,y,s) < g(x,y,t)$ for $(x,y) \in \Omega$ (see [9], 329). Now $h(x,y) < H(x,y) < g(x,y,t)$ for $(x,y) \in \Omega$, $t > 0$ and $g^*(x,y,t) = h^*(x,y)$ a.e. on $\partial \Omega \setminus B(N,t)$ so $H^* = h^*$ a.e. on $\partial \Omega$. Then the general maximum principle [14] implies $H = h$ on $\Omega$ and so $g(-,t)$ converges uniformly on compacta in $\Omega$ to a solution $H$ of the minimal surface equation as $t$ decreases to zero (e.g. [9], 329).

We are in a position to prove the following

**Theorem.** Let $\Omega \subseteq \mathbb{R}^2$ be an open convex set, $P \in \partial \Omega$, and $Q \in \Omega$. There exists a bounded solution $f$ of the minimal surface equation in $\Omega$ such that $f(x,y)$ has no limit as $(x,y)$ approaches $P$ along the line segment $PQ$.

**Proof.** We may assume $P = N = (0,0)$ and $Q = (1,0)$, as before. Pick $f_1 \in C^2(\Omega) \cap C^0(\overline{\Omega})$ such that $|f_1| \leq 2$ and $f_1(0,0) = -2$. Let $r_1 \in (0,1)$ such that $f_1(r_1,0) < -1$. Let $\varepsilon_1 = -(1 + f_1(r_1,0))$. From the Lemma, we see that there exists $f_2 \in C^2(\Omega) \cap C^0(\Omega \cup \{N\})$ such that $|f_2| \leq 2$, $f_2(0,0) = 2$, and $\sup\{|f_2(x,y) - f_1(x,y)|: (x,y) \in \Omega, \text{dist}((x,y),\partial \Omega) \geq \delta_1\} < \varepsilon_1$, where $\delta_1 = \text{dist}(r_1,0,\partial \Omega)$. Then $f_2(r_1,0) < -1$. Now pick $r_2 \in (0,r_1)$ such that $r_2 < \frac{1}{2}$ and $f_2(r_2,0) > 1$.

In general, let $\varepsilon_n = \min\{1 \leq k \leq n: |f_n(r_k,0) - (-1)^k| \}$ and $\delta_n = \text{dist}(r_n,0,\partial \Omega)$. Use the Lemma to find $f_{n+1} \in C^2(\Omega) \cap C^0(\Omega \cup \{N\})$ with $f_{n+1}(0,0) = 2(-1)^n$, $|f_{n+1}| \leq 2$, and $\sup\{|f_{n+1}(x,y) - f_n(x,y)|: (x,y) \in \Omega, \text{dist}((x,y),\partial \Omega) \geq \delta_n\} < \varepsilon_n$. Then pick $r_{n+1} \in (0,r_n)$ with $r_{n+1} < 1/(n+1)$ such that $f_{n+1}(r_{n+1},0) > 1$ if $n+1$ is even and $f_{n+1}(r_{n+1},0) < -1$ if $n+1$ is odd. It follows from the definition of $\varepsilon_n$ that $(-1)^k f_{n+1}(r_k,0) > 1$ for all $1 \leq k \leq n + 1$. The sequence $\{f_n\}$ is uniformly bounded and so there is a subsequence, still denoted $\{f_n\}$,
and a solution $f$ of the minimal surface equation such that $f_n$ converges to $f$ uniformly on compacta in $\Omega$ ([9], p. 323). Notice that $r_n \to 0$ as $n \to \infty$, $f(r_n, 0) \geq 1$ if $n$ is even, $f(r_n, 0) \leq -1$ if $n$ is odd, and $|f| \leq 2$. Thus $f$ is bounded and $\lim_{x \to 0} f(x, 0)$ does not exist. Q.E.D.

We may choose $\Omega$ to be symmetric about the non-negative $x$-axis and construct $f$ so that $f(x, -y) = f(x, y)$ and $\lim_{x \to 0} f(x, 0)$ does not exist. It is easy to see then that $f$ has no radial limits (e.g. [11]). A similar construction gives corresponding results for equations of bounded mean curvature and for nonparametric minimal hypersurfaces. In particular, this disproves the following conjecture. Set

$$Rf(\theta) = \lim_{t \to 0^+} f(t \cos(\theta), t \sin(\theta))$$

when this limit exists. Conjecture [11]: Let $f$ be a bounded solution of the minimal surface equation in a domain $\Omega$. Then $Rf(\theta)$ exists for all $\theta$ for which $\{(r \cos(\theta), r \sin(\theta)): 0 < r < \varepsilon\} \subset \Omega$, for some $\varepsilon > 0$.

2. Remarks on the existence of radial limits

Suppose $U$ is a connected, simply connected, bounded, open subset of $\mathbb{R}^2$ with locally Lipschitz boundary such that $N = (0, 0) \in \partial U$ and

$$U = \{(r \cos(\theta), r \sin(\theta)): 0 < r < r(\theta), \alpha < \theta < \beta\}$$

for some $\alpha < \beta$. Let $U^+ = \{(r \cos(\theta), r \sin(\theta)) : \frac{1}{2}(\alpha + \beta) < \theta < \beta, 0 < r < r(\theta)\}$ and set $\partial^+ U = \partial U \cap \partial U^+$ and $\partial^- U = \partial U \setminus \partial^+ U$. Suppose $f \in C^0(\bar{U}\setminus\{N\})$ is a bounded solution of the minimal surface equation. The graph of $f$ over $U$, $S$, then has finite area. Let us define

$$a^\pm = \liminf_{(x, y) \to (0, 0)} \{f(x, y) : (x, y) \in \partial^\pm U\}$$

and

$$b^\pm = \limsup_{(x, y) \to (0, 0)} \{f(x, y) : (x, y) \in \partial^\pm U\}.$$ 

Using a parametric representation of the graph of $f$ similar to that in [10] and [12], we can prove the following.

**Proposition.** Suppose $f \in C^0(\bar{U}\setminus\{N\})$ is a bounded solution of the minimal surface equation in $U$ and

$$a = \max\{a^-, a^+\} > b = \min\{b^-, b^+\}.$$ 

Then, for some $\alpha_0, \beta_0 \in [\alpha, \beta]$ with $\alpha_0 < \beta_0$, $Rf(\theta)$ is defined and continuous on $(\alpha_0, \beta_0)$. Further, either

(i) $a = a^+(a^-)$, $Rf(\theta)$ is increasing (decreasing) on $(\alpha_0, \beta_0)$ and

$$\lim_{\theta \to \alpha_0^+} Rf(\theta) = b^-(a^-) \quad \text{and} \quad \lim_{\theta \to \beta_0^-} Rf(\theta) = a^+(b^+)$$

or
(ii) there exists $\gamma \in (\alpha_0, \beta_0 - \pi)$ such that $Rf(\theta)$ is increasing (decreasing) on $(\alpha_0, \gamma)$, constant on $[\gamma, \gamma + \pi]$, and decreasing (increasing) on $[\gamma + \pi, \beta_0)$ and

$$\lim_{\theta \to \alpha_0^+} Rf(\theta) = b^-(a^-) \quad \text{and} \quad \lim_{\theta \to \beta_0^-} Rf(\theta) = b^+(a^+).$$

In particular, if $z \in (b, a)$, then for some $\theta_z \in (\alpha_0, \beta_0)$, $Rf(\theta_z) = z$.

A similar result can be obtained for nonparametric surfaces of (bounded) prescribed mean curvature (using a parametrization similar to that in [4]).

**Proof.** Since the area of $S$ is finite, the Dirichlet integral of $X$, $D(X)$, is finite. The proposition follows using Courant’s lemma ([1], Lemma 3.1) and the “jump” in the boundary values of $f$ “at $N$”. Q.E.D.

**Examples.**

1. Let $U = \{(x, y) : x^2 + y^2 < 1, y > 0\}$ and let $f \in C^2(U) \cap C^0(\overline{U} \setminus \{N\})$ be the solution of the minimal surface equation in $U$ with $f(x,0) = 0$ for $x \in (0,1)$, $f(x,0) = 9 + \cos(2\pi/x)$ for $x \in (-1,0)$, and $f(\cos(\theta), \sin(\theta)) = (10/\pi)^0$ for $0 \leq \theta \leq \pi$. Then $a^- = b^- = 0$, $a^+ = 8$, $b^+ = 10$, $\alpha_0 = 0$, $\beta_0 \geq 4\pi/5$, and conclusion (i) of the proposition holds. To see this, consider the function $g(x \cos(\theta), x \sin(\theta)) = (10/\pi)^0$ whose graph is a helicoid. Since $f \leq g$ on $\partial U \setminus \{N\}$, $f \leq g$ in $U$. Since $Rg(\theta) = (10/\pi)^0 < 8$ iff $\theta < 4\pi/5$, we see that $Rf(\theta)$ exists and $Rf(\theta) \leq Rg(\theta)$ if $0 \leq \theta < 4\pi/5$. Thus $\beta_0 \in [4\pi/5, \pi]$.

2. Let $U = \{(x, y) : x^2 + y^2 < 1, y > 0 \text{ or } x < 0\}$. Then $\alpha = 0$ and $\beta = 3\pi/2$. Let $g \in C^0(\overline{U} \setminus \{N\})$ be a solution of the minimal surface equation in $U$ such that $g(x,0) = 9$ for $x \in (0,1)$, $g(0,y) = -1$ for $y \in (-1,0)$, and $Rg(\theta) = 20$ (such a function $g$ can be obtained by making $g$ large enough on $\partial U \cap \partial B$, where $B = \{(x, y) : x^2 + y^2 < 1\}$). Let $f \in C^0(\overline{U} \setminus \{N\})$ be the solution in $U$ of the minimal surface equation such that $f = g$ on $\partial U \cap \partial B$, $f(x,0) = 10 + \cos(\pi/x)$ for $x \in (0,1)$, and $f(0,y) = \cos(\pi/y)$ for $y \in (-1,0)$. Then $g \leq f$ in $U$. Now $a^- = 9$, $b^- = 11$, $a^+ = -1$, and $b^+ = 1$. Since $Rg(\pi) = 20$, conclusion (ii) of the proposition holds; in fact, $Rf(\theta)$ is increasing on $(\alpha_0, \pi)$, constant $(\geq 20)$ on $[\gamma, \gamma + \pi]$, and decreasing on $[\gamma + \pi, \beta_0)$, for some $\gamma \in (\alpha_0, \beta_0 - \pi) \subseteq (0, \pi/2)$, $\lim_{\theta \to \alpha_0^+} Rf(\theta) = 11$, and $\lim_{\theta \to \beta_0^-} Rf(\theta) = 1$. Since $Rf(\theta) \geq Rg(\theta)$ for $\theta \in (\alpha_0, \beta_0)$, we see that $0 \leq \alpha_0 \leq \alpha_1$ and $\beta_1 \leq \beta_0 \leq 3\pi/2$, where $\alpha_1 \in (0, \pi/2)$ such that $Rg(\alpha_1) = 11$ and $Rg(\beta_1) = 1$.

In general, the example in section 1 shows that $Rf(\theta)$ need not be defined for even one value of $\theta$. If we represent the graph of $f$ parametrically as in [10] (and [12]), then the $z$-coordinate function $z(w)(w = u + iv)$ need not have a radial limit at a boundary point, say 1, of the parameter domain $D = \{w : |w| < 1\}$ which $x(w)$ and $y(w)$ both map to 0. How poorly behaved is $z(w)$ near 1? It can be shown that (a suitable restriction of) $z$ is in $h^p(D)$ for all $p < 1$, $\tilde{z} \in C^0(\overline{D})$, $\tilde{z}_0 \in h^1(D)$, and $z + i\tilde{z} \in H^\infty(D) \cap VM^0$, where $\tilde{z}$.
is the harmonic conjugate of $z$ (here we use the notation of [2] and [17]). Of course, if $z$ were a little nicer (e.g. $z \in h^1(D)$ or $\bar{z}_\theta \in h^p(D)$ for some $p > 1$), then $z$ would have a radial limit at 1.

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