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EXISTENCE AND NONEXISTENCE OF RADIAL LIMITS OF MINIMAL SURFACES

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ABSTRACT. A bounded solution of the minimal surface equation is constructed which has no radial limits at a boundary point.

INTRODUCTION

The Dirichlet problem for the minimal surface equation consists of determining a function $f = f(x, y)$ satisfying the equation

$$(1 + f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1 + f_x^2)f_{yy} = 0$$

in a domain Ω and taking on assigned values on the boundary of Ω. The boundary behavior of solutions of the Dirichlet problem has been well studied (e.g. [5, 6, 7, 8, 9, 13, 15, 16, 19]); when Ω is convex and the assigned boundary values are continuous, the (unique) solution is in $C^0(\overline{\Omega})$ [16], but this is generally false when Ω is not convex (e.g. the intriguing example of Simon [20]). If Ω is locally convex at each point of $\partial\Omega$ except one, say N, and the assigned boundary values are continuous, then the (generalized) solution of the Dirichlet problem will (probably) not be continuous at N and yet will have radial limits at N from each direction in Ω [3, 10, 11]. If the assigned boundary values have a jump discontinuity at N, the (generalized) solution continues to have radial limits [11, 12].

Suppose f is a bounded solution of the minimal surface equation in a domain Ω, $N \in \partial\Omega$, and "$f|_{\partial\Omega}$" is not defined at N (e.g. [20]). How does f behave "at N"? In this note we will construct a bounded solution of the minimal surface equation which has no radial limits at a point of the boundary. We also have some comments on the existence of radial limits.

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1. Nonexistence of radial limits

Let Ω be a convex domain in the plane. Suppose $N = (0,0) \in \partial\Omega$ and, say, $\{(x,0): 0 < x < 1\} \subseteq \Omega$. We will use a "gliding hump" argument to construct a bounded solution f of the minimal surface equation in Ω such that $\lim_{x\to 0+} f(x,0)$ does not exist.

First we need a lemma similar to a "localization" lemma for harmonic functions.

Lemma. *Let $\varepsilon > 0$ and $h \in C^2(\Omega)$ be a solution of the minimal surface equation with $|h| \le 2$. Then for each $\delta \in (0,1)$, there exists $g \in C^2(\Omega) \cap C^0(\Omega \cup \{N\})$ such that g is a solution of the minimal surface equation in Ω, $g(0,0) = 2$, $|g| \le 2$, and $\sup\{|h(x,y) - g(x,y)|: (x,y) \in \Omega, \operatorname{dist}((x,y),\partial\Omega) \ge \delta\} < \varepsilon$.*

Proof. Let $g(x,y,t)$ be the solution of the Dirichlet problem with boundary values h^* on $\partial\Omega\backslash B(N,t)$ and 2 on $\partial\Omega \cap B(N,t)$, where $B(N,t)$ is the disc about N of radius t and h^* is the trace of h on $\partial\Omega$. If $0 < s < t$, then $h(x,y) \le g(x,y,s) \le g(x,y,t)$ for $(x,y) \in \Omega$ ($g(\cdot,s)$ and $g(\cdot,t)$ can be considered as lower Perron solutions, for example) and so $g(\cdot,t)$ converges uniformly on compacta in Ω to a solution H of the minimal surface equation as t decreases to zero (e.g. [9], 329). Now $h(x,y) \le H(x,y) \le g(x,y,t)$ for $(x,y) \in \Omega, t > 0$ and $g^*(x,y,t) = h^*(x,y)$ a.e. on $\partial\Omega\backslash B(N,t)$ so $H^* = h^*$ a.e. on $\partial\Omega$. Then the general maximum principle [14] implies $H = h$ on Ω and so $g(\cdot,t)$ converges uniformly on compacta to h. For t_0 close enough to zero, $g = g(\cdot,t_0)$ satisfies the conclusions of the lemma. Q.E.D.

We are in a position to prove the following

Theorem. *Let $\Omega \subseteq R^2$ be an open convex set, $P \in \partial\Omega$, and $Q \in \Omega$. There exists a bounded solution f of the minimal surface equation in Ω such that $f(x,y)$ has no limit as (x,y) approaches P along the line segment PQ.*

Proof. We may assume $P = N = (0,0)$ and $Q = (1,0)$, as before. Pick $f_1 \in C^2(\Omega) \cap C^0(\overline{\Omega})$ such that $|f_1| \le 2$ and $f_1(0,0) = -2$. Let $r_1 \in (0,1)$ such that $f_1(r_1,0) < -1$. Let $\varepsilon_1 = -(1 + f_1(r_1,0))$. From the Lemma, we see that there exists $f_2 \in C^2(\Omega) \cap C^0(\Omega \cup \{N\})$ such that $|f_2| \le 2, f_2(0,0) = 2$, and $\sup\{|f_2(x,y) - f_1(x,y)| \,|\, (x,y) \in \Omega, \operatorname{dist}((x,y),\partial\Omega) \ge \delta_1\} < \varepsilon_1$, where $\delta_1 = \operatorname{dist}((r_1,0),\partial\Omega)$. Then $f_2(r_1,0) < -1$. Now pick $r_2 \in (0,r_1)$ such that $r_2 < \frac{1}{2}$ and $f_2(r_2,0) > 1$.

In general, let $\varepsilon_n = \min_{1\le k\le n} |f_n(r_k,0) - (-1)^k|$ and $\delta_n = \operatorname{dist}((r_n,0),\partial\Omega)$. Use the Lemma to find $f_{n+1} \in C^2(\Omega)\cap C^0(\Omega\cup\{N\})$ with $f_{n+1}(0,0) = 2(-1)^{n+1}$, $|f_{n+1}| \le 2$, and $\sup\{|f_{n+1}(x,y) - f_n(x,y)|: (x,y) \in \Omega, \operatorname{dist}((x,y),\partial\Omega) \ge \delta_n\} < \varepsilon_n$. Then pick $r_{n+1} \in (0,r_n)$ with $r_{n+1} < 1/(n+1)$ such that $f_{n+1}(r_{n+1},0) > 1$ if $n+1$ is even and $f_{n+1}(r_{n+1},0) < -1$ if $n+1$ is odd. It follows from the definition of ε_n that $(-1)^k f_{n+1}(r_k,0) > 1$ for all $1 \le k \le n+1$. The sequence $\{f_n\}$ is uniformly bounded and so there is a subsequence, still denoted $\{f_n\}$, and a solution f of the minimal surface equation such that f_n converges to f uniformly on compacta in Ω ([9], p. 323). Notice that $r_n \to 0$ as $n \to \infty$, $f(r_n, 0) \geq 1$ if n is even, $f(r_n, 0) \leq -1$ if n is odd, and $|f| \leq 2$. Thus f is bounded and $\lim_{x \to 0} f(x, 0)$ does not exist. Q.E.D.

We may choose Ω to be symmetric about the non-negative x-axis and construct f so that $f(x, -y) = f(x, y)$ and $\lim_{x \to 0} f(x, 0)$ does not exist. It is easy to see then that f has no radial limits (e.g. [11]). A similar construction gives corresponding results for equations of bounded mean curvature and for nonparametric minimal hypersurfaces. In particular, this disproves the following conjecture. Set

$$Rf(\theta) = \lim_{t \to 0+} f(t\cos(\theta), t\sin(\theta))$$

when this limit exists. Conjecture [11]: Let f be a bounded solution of the minimal surface equation in a domain Ω. Then $Rf(\theta)$ exists for all θ for which $\{(r\cos(\theta), r\sin(\theta)): 0 < r < \varepsilon\} \subset \Omega$, for some $\varepsilon > 0$.

2. Remarks on the existence of radial limits

Suppose U is a connected, simply connected, bounded, open subset of $\mathfrak{R}^2$ with locally Lipschitz boundary such that $N = (0, 0) \in \partial U$ and

$$U = \{(r\cos(\theta), r\sin(\theta)): 0 < r < r(\theta), \alpha < \theta < \beta\}$$

for some $\alpha < \beta$. Let $U^+ = \{(r\cos(\theta), r\sin(\theta)) : \frac{1}{2}(\alpha + \beta) < \theta < \beta,\ 0 < r < r(\theta)\}$ and set $\partial^+ U = \partial U \cap \partial U^+$ and $\partial^- U = \partial U \backslash \partial^+ U$. Suppose $f \in C^0(\overline{U} \backslash \{N\})$ is a bounded solution of the minimal surface equation. The graph of f over U, S, then has finite area. Let us define

$$a^\pm = \liminf_{(x,y) \to (0,0)} \{f(x, y) : (x, y) \in \partial^\pm U\}$$

and

$$b^\pm = \limsup_{(x,y) \to (0,0)} \{f(x, y) : (x, y) \in \partial^\pm U\}.$$

Using a parametric representation of the graph of f similar to that in [10] and [12], we can prove the following.

Proposition. *Suppose $f \in C^0(\overline{U} \backslash \{N\})$ is a bounded solution of the minimal surface equation in U and*

$$a = \max\{a^-, a^+\} > b = \min\{b^-, b^+\}.$$

Then, for some $\alpha_0, \beta_0 \in [\alpha, \beta]$ with $\alpha_0 < \beta_0$, $Rf(\theta)$ is defined and continuous on (α_0, β_0). Further, either

(i) *$a = a^+(a^-)$, $Rf(\theta)$ is increasing (decreasing) on (α_0, β_0) and*

$$\lim_{\theta \to \alpha_0 +} Rf(\theta) = b^-(a^-) \quad \text{and} \quad \lim_{\theta \to \beta_0 -} Rf(\theta) = a^+(b^+)$$

or

(ii) *there exists* $\gamma \in (\alpha_0, \beta_0 - \pi)$ *such that* $Rf(\theta)$ *is increasing (decreasing) on* $(\alpha_0, \gamma]$, *constant on* $[\gamma, \gamma+\pi]$, *and decreasing (increasing) on* $[\gamma+\pi, \beta_0)$ *and*

$$\lim_{\theta\to\alpha_0+} Rf(\theta) = b^-(a^-) \quad \text{and} \quad \lim_{\theta\to\beta_0-} Rf(\theta) = b^+(a^+).$$

In particular, if $z \in (b, a)$, *then for some* $\theta_z \in (\alpha_0, \beta_0)$, $Rf(\theta_z) = z$.

A similar result can be obtained for nonparametric surfaces of (bounded) prescribed mean curvature (using a parametrization similar to that in [4]).

Proof. Since the area of S is finite, the Dirichlet integral of X, $D(X)$, is finite. The proposition follows using Courant's lemma ([1], Lemma 3.1) and the "jump" in the boundary values of f "at N". Q.E.D.

Examples.

1. Let $U = \{(x,y) : x^2+y^2 < 1, y > 0\}$ and let $f \in C^2(U) \cap C^0(\overline{U}\backslash\{N\})$ be the solution of the minimal surface equation in U with $f(x,0) = 0$ for $x \in (0,1)$, $f(x,0) = 9+\cos(2\pi/x)$ for $x \in (-1,0)$, and $f(\cos(\theta), \sin(\theta)) = (10/\pi)\theta$ for $0 \le \theta \le \pi$. Then $a^- = b^- = 0$, $a^+ = 8$, $b^+ = 10$, $\alpha_0 = 0$, $\beta_0 \ge 4\pi/5$, and conclusion (i) of the proposition holds. To see this, consider the function $g(r\cos(\theta), r\sin(\theta)) = (10/\pi)\theta$ whose graph is a helicoid. Since $f \le g$ on $\partial U\backslash\{N\}$, $f \le g$ in U. Since $Rg(\theta) = (10/\pi)\theta < 8$ iff $\theta < 4\pi/5$, we see that $Rf(\theta)$ exists and $Rf(\theta) \le Rg(\theta)$ if $0 \le \theta < 4\pi/5$. Thus $\beta_0 \in [4\pi/5, \pi]$.

2. Let $U = \{(x,y) : x^2+y^2 < 1, y > 0 \text{ or } x < 0\}$. Then $\alpha = 0$ and $\beta = 3\pi/2$. Let $g \in C^0(\overline{U}\backslash\{N\})$ be a solution of the minimal surface equation in U such that $g(x,0) = 9$ for $x \in (0,1)$, $g(0,y) = -1$ for $y \in (-1,0)$, and $Rg(\theta) = 20$ (such a function g can be obtained by making g large enough on $\partial U \cap \partial B$, where $B = \{(x,y) : x^2+y^2 < 1\}$). Let $f \in C^0(\overline{U}\backslash\{N\})$ be the solution in U of the minimal surface equation such that $f = g$ on $\partial U \cap \partial B$, $f(x,0) = 10+\cos(\pi/x)$ for $x \in (0,1)$, and $f(0,y) = \cos(\pi/y)$ for $y \in (-1,0)$. Then $g \le f$ in U. Now $a^- = 9$, $b^- = 11$, $a^+ = -1$, and $b^+ = 1$. Since $Rg(\pi) = 20$, conclusion (ii) of the proposition holds; in fact, $Rf(\theta)$ is increasing on $(\alpha_0, \gamma]$, constant (≥ 20) on $[\gamma, \gamma+\pi]$, and decreasing on $[\gamma+\pi, \beta_0)$, for some $\gamma \in (\alpha_0, \beta_0-\pi) \subseteq (0, \pi/2)$, $\lim_{\theta\to\alpha_0+} Rf(\theta) = 11$, and $\lim_{\theta\to\beta_0-} Rf(\theta) = 1$. Since $Rf(\theta) \ge Rg(\theta)$ for $\theta \in (\alpha_0, \beta_0)$, we see that $0 \le \alpha_0 \le \alpha_1$ and $\beta_1 \le \beta_0 \le 3\pi/2$, where $\alpha_1 \in (0, \pi/2)$ such that $Rg(\alpha_1) = 11$ and $Rg(\beta_1) = 1$.

In general, the example in section 1 shows that $Rf(\theta)$ need not be defined for even one value of θ. If we represent the graph of f parametrically as in [10] (and [12]), then the z-coordinate function $z(w)(w = u+iv)$ need not have a radial limit at a boundary point, say 1, of the parameter domain $D = \{w : |w| < 1\}$ which $x(w)$ and $y(w)$ both map to 0. How poorly behaved is $z(w)$ near 1? It can be shown that (a suitable restriction of) z is in $h^p(D)$ for all $p < 1$, $\tilde{z} \in C^0(\overline{D})$, $\tilde{z}_\theta \in h^1(D)$, and $z + i\tilde{z} \in H^\infty(D) \cap VMO$, where $\tilde{z}$

is the harmonic conjugate of z (here we use the notation of [2] and [17]). Of course, if z were a little nicer (e.g. $z \in h^1(D)$ or $\tilde{z}_\theta \in h^p(D)$ for some $p > 1$), then z would have a radial limit at 1.

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