A WEAK-TYPE ESTIMATE
FOR FOURIER-BESSEL MULTIPLIERS

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Abstract. We apply Hörmander's technique to prove a weak-type $(1,1)$ estimate for multiplier operators with respect to the Fourier-Bessel transform. This improves a result in [4, 5].

1. Introduction

Consider the measure space $(\mathbb{R}^+, d\mu(x))$ where $d\mu(x) = x^r dx$, $r$ being a fixed positive real number. By $L^p(\mu)$, $1 \leq p \leq \infty$, we denote the corresponding Lebesgue spaces equipped with the norms

$$||f||_p = \left( \int_0^\infty |f|^p d\mu \right)^{1/p}.$$

Let $\hat{f}(\lambda)$, $\lambda \geq 0$, denote the Fourier-Bessel transform of the function $f \in L^1(\mu)$. Explicitly this means

$$\hat{f}(\lambda) = \int_0^\infty f(x) \phi_\lambda(x) d\mu(x),$$

where $\phi_\lambda(x) = a(r)(\lambda x)^{-\frac{r-1}{2}} J_{\frac{r-1}{2}}(\lambda x)$, $x \geq 0$, $a(r) = 2^{\frac{r-1}{2}} \Gamma((r+1)/2)$, and $J_\alpha$ denotes the Bessel function of the first kind of order $\alpha$. The functions $\phi_\lambda$, $\lambda \geq 0$, are eigenfunctions of the second order Bessel differential operator

$$L = - \left( \frac{d}{dx^2} + \frac{r}{x} \frac{d}{dx} \right).$$

More precisely, we have

$$(1.1) \quad L \phi_\lambda = \lambda^2 \phi_\lambda.$$ 

If $r = n - 1$, $n \geq 2$ being an integer, then

$$\hat{f}(||\vec{\lambda}||) = \frac{1}{2\pi} \left( \frac{n}{2} \right)^{\frac{n-2}{2}} \left( \frac{n}{2} \right) \int_{\mathbb{R}^n} f(||\vec{x}||) \exp(-i\langle \vec{\lambda}, \vec{x} \rangle) d\vec{x}$$

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for \( \vec{\lambda} \in \mathbb{R}^n \). Thus \( \hat{f} \) gives the radial part of the Fourier transform of the radial function \( f'(|\vec{x}|) \) on \( \mathbb{R}^n \) (observe that \( f \in L^1(\mu) \) if and only if \( f'(|\vec{x}|) \in L^1(\mathbb{R}^n, d\vec{x}) \) where \( \| \cdot \| \) denotes the Euclidean norm and \( d\vec{x} \) is Lebesgue measure on \( \mathbb{R}^n \)). In the case \( r = n - 1 \), one can easily see that for the Fourier-Bessel transform, both the inversion theorem and Plancherel's formula hold. In particular we have

\[
\hat{f}(\lambda) = a(r)^{-1} \int_0^\infty \hat{f}(\lambda) \phi_\lambda(x) d\mu(\lambda)
\]

almost everywhere on \( \mathbb{R}^+ \) providing \( f \in L^1(\mu) \) and \( \hat{f} \in L^1(\mu) \). For \( f \in L^2(\mu) \) we have

\[
\int_0^\infty |f(x)|^2 d\mu(x) = a(r)^{-2} \int_0^\infty |\hat{f}(\lambda)|^2 d\mu(\lambda).
\]

The above results for general \( r > 0 \) can be established via a summability argument similar to that in the case of the ordinary Fourier transform. The key idea is that the Gaussian is invariant under the Fourier-Bessel transform when appropriately normalized. We also note that in the case \( r = n - 1 \), the operator \( -L \) is the radial part of the ordinary Laplace operator on \( \mathbb{R}^n \). Finally, we mention that \( C \) will be used to denote a constant which may vary from line to line.

For any bounded function \( m \) on \( (0, \infty) \) we define the multiplier operator \( T_m \) by \( (T_m f)^\sim = m \hat{f} \). The aim of this note is to prove the following theorem which is a generalization of the celebrated Hörmander-Mihlin multiplier theorem for the Fourier transform (see [3]).

**Theorem 1.1.** Let \( k \) denote the least even integer > \( (r + 1)/2 \) and let \( m \in C^k(0, \infty) \) be a bounded function which satisfies

\[
(1.2) \quad \left( \int_{R/2}^R |m^{(s)}(\lambda)|^2 d\mu(\lambda) \right)^{1/2} \leq BR^{(r+1)/2-s},
\]

where \( B \) is a constant independent of \( R > 0 \) and \( s = 0, 1, \ldots, k \). Then the operator \( T_m \) is of weak-type \( (1, 1) \), i.e.

\[
(1.3) \quad \mu(\{x \in (0, \infty) : |T_m f(x)| > \alpha\}) \leq C \alpha^{-1} \|f\|_1
\]

where \( C \) is independent of \( \alpha > 0 \) and \( f \in L^1(\mu) \).

Using the interpolation theorem of Marcinkiewicz and duality we also obtain the following

**Corollary 1.2.** Let \( m \) satisfy the assumptions of the Theorem 1.1. Then for \( 1 < p < \infty \), \( T_m \) is of strong type \( (p, p) \), i.e. \( \|T_m f\|_p \leq C_p \|f\|_p \).

**Remark.** The result of the above corollary was obtained previously by the second author by means of a Littlewood-Paley theory for the Fourier-Bessel transform (see [4, 5]). The proof of the above theorem is more direct and is based upon a generalized convolution structure which is natural for the Fourier-Bessel
transform. The novel ingredient of our proof is a Bernstein-type inequality for this convolution structure. We establish this inequality in the following section.

2. A Bernstein-type inequality for generalized translations

Since Hirschman and Bochner considered the Banach algebra of radial functions over thirty years ago, it has been known that $L^1(\mu)$ admits a commutative Banach algebra structure for which the Fourier-Bessel transform becomes the Gelfand transform (see [4 and 5] for details and for other references). In order to describe this structure we define the generalized translation operator $T^y$, $y \geq 0$, for suitable functions $f$, by

\begin{equation}
T^y f(x) = \int_0^\infty f((x, y) \theta) d\nu(\theta)
\end{equation}

where $(x, y) \theta = (x^2 + y^2 - 2xy \cos \theta)^{1/2}$, $0 \leq \theta \leq \pi$, $x, y \geq 0$ and $d\nu(\theta)$ denotes the probability measure $b(r)(\sin \theta)^{-1} d\theta$ on $[0, \pi]$. Here

$$b(r) = \pi^{-1/2} \Gamma((r + 1)/2) \Gamma(r/2)^{-1}.$$

The operator $T^y$ can also be described by

\begin{equation}
T^y f(x) = \int_{|x-y|}^{x+y} f(z) dW_{x,y}(z)
\end{equation}

where the probability measure $dW_{x,y}(z)$ is supported on $[|x - y|, x + y]$ and given by

$$dW_{x,y}(z) = c(r) \Delta(x, y, z)^{r-2} (xyz)^{-1} d\mu(z),$$

where $c(r) = 2r^{-2} \Gamma((r + 1)/2) \Gamma(r/2)^{-1} \pi^{-1/2}$. In the above formula $x, y, z \geq 0$, $|x - y| \leq z \leq x + y$, and $\Delta(x, y, z)$ denotes the area of a triangle with sides $x, y, z$. It is quite straightforward to go from (2.2) to (2.1) by a change of variables. Next, it is easy to check that the measure $dW_{x,y}(z) d\mu(y)$ is symmetric with respect to $y$ and $z$, i.e. $dW_{x,y}(z) d\mu(y) = dW_{x,z}(y) d\mu(z)$. From this it follows that the operators $T^y$, $y \geq 0$, are selfadjoint on $L^2(\mu)$ and moreover that

\begin{equation}
\int_0^\infty f(x) T^y g(x) d\mu(x) = \int_0^\infty T^y f(x) g(x) d\mu(x),
\end{equation}

for any reasonable pair of functions $f$ and $g$, e.g. if $f \in L^\infty(\mu)$ and $g \in L^1(\mu)$. It is also easy to see that generalized translations are submarkovian contractions, i.e. $0 \leq f \leq 1$ implies $0 \leq T^y f \leq 1$ and $\|T^y f\|_p \leq \|f\|_p$, for $1 \leq p \leq \infty$. We now define the generalized convolution $f * g$ for $f, g \in L^1(\mu)$ by

\begin{equation}
f * g(x) = \int_0^\infty f(y) T^y g(x) d\mu(y).
\end{equation}
It is well known that the following identity involving the functions $\phi_\lambda$ and the generalized translations $T_\gamma$ is valid

\begin{equation}
T_\gamma \phi_\lambda(x) = \phi_\lambda(x)\phi_\gamma(y)
\end{equation}

(see [4, 5] for references). Using (2.3) and (2.5) one can easily verify that for any $f, g \in L^1(\mu)$ we have $(f \ast g)\gamma = \hat{f} \hat{g}$ and $(T_\gamma f)\gamma = \phi_\gamma(y) \hat{f}(\lambda)$. Therefore

\begin{equation}
T_\gamma g \ast f = g \ast T_\gamma f.
\end{equation}

It is also easy to check that with the pseudodistance $\rho(x, y) = |x - y|$, the space $(\mathbb{R}^+, \mu, \rho)$ becomes a space of homogeneous type in the sense of [1]. Finally, from (2.2) it is immediate that $T_\gamma f(x) = 0$ if $f$ vanishes on the interval $[x - y, x + y]$ and that $|T_\gamma f(x)| \leq T_\gamma(|f|)(x)$ for $x, y \geq 0$.

In the proof of the Hörmander-Mihlin theorem (cf. [3]), Bernstein's inequality plays an important role. We now prove a Bernstein-type inequality for the generalized translations $T_\gamma$, $\gamma \geq 0$.

**Theorem 2.1.** Suppose $h \in L^1(\mu)$ is a differentiable function on $\mathbb{R}^+$ with $h'(x) \in L^1(\mu)$. Then

\begin{equation}
\|T_\gamma h - T_{\gamma'} h\|_1 \leq \|h'\|_1 |y_1 - y_2|.
\end{equation}

**Proof.** Using (2.1) we write

\begin{equation}
\|T_\gamma h - T_{\gamma'} h\|_1 = \int_0^\infty \int_0^\pi \left| h((x,y_1)\theta) - h((x,y_2)\theta) \right| d\nu(\theta) d\mu(x)
\end{equation}

For fixed $x, y_1, y_2 \geq 0$ and $\theta \in (0, \pi)$, let

\begin{equation}
\Phi(s) = \Phi_{x, y_1, y_2}(s) = (x, y_2 + s(y_1 - y_2)) \theta, \quad s \in [0, 1].
\end{equation}

Then

\begin{equation}
\left| \frac{d}{ds} \Phi(s) \right| \leq |y_1 - y_2|
\end{equation}

independently of $x, y_1, y_2, \theta$. To verify (2.9) consider the vectors $X = (x, 0)$, $Y_1 = (y, \cos \theta, y, \sin \theta)$, $i = 1, 2$, in $\mathbb{R}^2$. Letting $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^2$ it is easy to see that

\begin{equation}
(x, y_2 + s(y_1 - y_2))\theta = \|Y_2 + s(Y_1 - Y_2) - X\| \equiv \|\xi_z\|.
\end{equation}

Since $\nabla(\|\cdot\|)(\xi) = \xi / \|\xi\|$ for $\xi \in \mathbb{R}^2$, Schwarz' inequality implies

\begin{equation}
\left| \frac{d}{ds} \Phi(s) \right| \leq \|Y_1 - Y_2\| = |y_1 - y_2|.
\end{equation}
Here \( \langle \cdot, \cdot \rangle \) denotes the ordinary Euclidean inner product on \( \mathbb{R}^2 \). This gives (2.9). Now, using (2.9) and the contraction property of generalized translations, we have from (2.8)

\[
\|T^{y_1}h - T^{y_2}h\|_1 \leq |y_1 - y_2| \int_0^1 \int_0^\infty \int_0^\pi |h'((x, y_2 + s(y_1 - y_2))_0)| d\nu(\theta) d\mu(x) ds
\]

\[
\leq |y_1 - y_2| \int_0^1 \|T^{y_1+\delta(y_1-y_2)}(|h'|)\|_1 ds
\]

\[
\leq |y_1 - y_2||h'|_1.
\]

This completes the proof of Theorem 2.1.

As in the case of the Fourier transform (cf. [2]) Theorem 2.1 implies the following corollary. We include its proof for the sake of completeness.

**Corollary 2.2.** There exists a constant \( C > 0 \) such that for any \( \lambda > 0 \) and any function \( f \in L^1(\mu) \) with \( \text{supp} \hat{f} \subset (0, \lambda) \)

\[
(2.10) \|T^{y_1}f - T^{y_2}f\|_1 \leq C\lambda |y_1 - y_2||f||_1.
\]

**Proof.** Choose \( \chi \in C^\infty(\mathbb{R}^+) \) such that \( \chi(\lambda) = 1 \) for \( 0 < \lambda \leq 1 \) and \( \chi(\lambda) = 0 \) for \( \lambda \geq 2. \) Then by the inversion theorem we have \( \chi = \hat{h} \) for some \( C^\infty \) function \( h \in L^1(\mu) \) with \( h' \in L^1(\mu) \). Let \( h_\varepsilon(x) = \varepsilon h(\varepsilon x) \) for \( \varepsilon > 0 \). Then \( h_\varepsilon(\lambda) = \chi(\lambda/\varepsilon) = 1 \) for \( 0 < \lambda \leq \varepsilon \). Thus, for \( \varepsilon = \lambda \) we have

\[
T^{y_1}f - T^{y_2}f = h_\varepsilon \ast (T^{y_1}f - T^{y_2}f) = (T^{y_1}h_\varepsilon - T^{y_2}h_\varepsilon) \ast f
\]

and therefore by (2.7) we have

\[
\|T^{y_1}f - T^{y_2}f\|_1 \leq \|T^{y_1}h_\varepsilon - T^{y_2}h_\varepsilon\|_1 \|f\|_1 = \|T^{y_1}h - T^{y_2}h\|_1 \|f\|_1
\]

\[
\leq \lambda |y_1 - y_2||h'|_1 \|f\|_1.
\]

This establishes (2.10) with \( C = ||h'||_1 \) and completes the proof of the corollary.

### 3. Proof of Theorem 1.1

We closely follow the original proof of Hörmander (see [3, pp. 121–123]). Choose a function \( \psi \in C_0^\infty(\mathbb{R}^+) \) with support in \((1/2, 2)\) such that \( \sum_{-\infty}^\infty \psi(2^{-j}\lambda) = 1, \lambda > 0, \) and let

\[
m_j(\lambda) = m(\lambda) \psi(2^{-j}\lambda) \equiv m(\lambda) \psi_j(\lambda)
\]

and

\[
k_j(x) = m_j(x) = a(r)^{-1} \int_0^\infty m_j(\lambda) \phi_\lambda(x) \lambda' d\lambda.
\]

Then \( T_m = \sum_{-\infty}^{\infty} T_{m_j} \) where \( T_{m_j} f = k_j \ast f \) and \( k_j \in L^1(\mu) \). In order to prove the weak-type \((1, 1)\) inequality (1.3), it suffices to establish (see [1, p. 75])

\[
(3.3) \quad \sum_{j=-\infty}^{\infty} \int_{|x-y_0|>2|y-y_0|} |T^{y}k_j(x) - T^{y_0}k_j(x)| d\mu(x) \leq C,
\]
where $C$ is independent of $y, y_0 \geq 0$. This will be implied in a standard way by the following estimates:

$$(3.4) \quad \int_{|x-y_0| \geq 2|y-y_0|} |T^y k_j(x) - T^{y_0} k_j(x)| \, d\mu(x) \leq C (2^j |y - y_0|)^{(r+1)/2 - k},$$

which is good when $2^j |y - y_0| \geq 1$, and

$$(3.5) \quad \int_{|x-y_0| \geq 2|y-y_0|} |T^y k_j(x) - T^{y_0} k_j(x)| \, d\mu(x) \leq C 2^j |y - y_0|,$$

which is good when $2^j |y - y_0| < 1$.

To establish (3.4) we first note that conditions (1.2) imply

$$(1.2) \quad \sum_{l=0}^{k/2} (\int_0^\infty |L^l m_j(\lambda)|^2 \, d\mu(\lambda))^{1/2} \leq CB(2^j)^{(r+1)/2 - 2l}, \quad l = 0, 1, \ldots, k/2.$$  

In fact, by induction, one can easily verify the following Leibniz rule for $L$:

$$L^l(fg) = \sum_{0<\alpha+\beta \leq 2l} C^l_{\alpha,\beta} f^{(\alpha)}(\lambda)^{\alpha+\beta-2l}. $$

Thus, taking $R = 2^j$ in (1.2), we can estimate the left-hand side of (3.6) by

$$C \sum_{0<\alpha+\beta \leq 2l} \left( \int_0^\infty |m_j^{(\alpha)}(\lambda)|^2 \, d\mu(\lambda) \right)^{1/2} \leq C \sum_{0<\alpha+\beta \leq 2l} 2^{-\beta j} \left( \int_{2^{-j-1}}^{2^{2j+1}} |m_j^{(\alpha)}(\lambda)|^2 \, d\mu(\lambda) \right)^{1/2} \leq C \sum_{0<\alpha+\beta \leq 2l} (2^j)^{\alpha-2l} \left( \int_{2^{-j-1}}^{2^{2j+1}} |m_j^{(\alpha)}(\lambda)|^2 \, d\mu(\lambda) \right)^{1/2} \leq C B(2^j)^{(r+1)/2 - 2l}. $$

This gives (3.6). Next, using the Plancherel theorem and (3.6) we obtain

$$\|(1 + (2^j x)^2)^{k/2} k_j\|_2 = \|[(1 + 2^j L)^{k/2} m_j]^\vee\|_2 = a(r)^{-2} \|(1 + 2^j L)^{k/2} m_j\|_2 \leq C \sum_{l=0}^{k/2} (2^j)^l \|L^l m_j\|_2 \leq CB(2^j)^{(r+1)/2}. $$

Thus by Schwarz' inequality and the estimate $\|(1 + (2^j x)^2)^{-k/2}\|_2 \leq C(2^j)^{-(r+1)/2}$ we obtain

$$\| k_j \|_1 \leq \|(1 + (2^j x)^2)^{k/2} k_j\|_2 \|(1 + (2^j x)^2)^{-k/2}\|_2 \leq CB. $$
Moreover, since \( \|(2^j x)^k k_j\|_2 \leq \|(1 + (2^j x)^2)^{k/2} k_j\|_2 \) we obtain in the same way
\[
\int_{x \geq t} |k_j(x)| d\mu(x) \leq \|(2^j x)^k k_j\|_2 \left( \int_{x \geq t} (2^j x)^{-2k} d\mu(x) \right)^{1/2} \leq CB(2^j t)^{(r+1)/2-k}.
\] (3.8)

If \( \chi_A \) is the characteristic function of a set \( A \subset \mathbb{R}^+ \), then \( 0 \leq T^y \chi_A(x) \leq 1 \) for \( x, y \in \mathbb{R}^+ \). We claim that
\[
T^{y_0} \chi \{ z : |z-y_0| > 2|y-y_0| \} \leq \chi \{ z : z > 2|y-y_0| \}
\] (3.9)
and
\[
T^y \chi \{ z : z-y_0 > 2|y-y_0| \} \leq \chi \{ z : z > |y-y_0| \}.
\] (3.10)

We only prove (3.9) and note that the proof of (3.10) is similar. Let \( E \) denote the set \( \{ z : |z-y_0| > 2|y-y_0| \} \). To verify (3.9) we need only check that \( 0 \leq x \leq 2|y-y_0| \) implies \( T^{y_0} \chi_E(x) = 0 \). Since \( 0 \leq x \leq 2|y-y_0| \) implies \( (|y_0-x|, y_0+x) \cap E = \emptyset \), it follows that \( T^{y_0} \chi_E(x) = 0 \) (see remark after (2.6)). This proves (3.9).

Using (3.7) through (3.10) and the selfadjointness of generalized translations, cf. (2.3), we find that the left-hand side of (3.4) is majorized by
\[
\int_0^\infty \chi_E(x)|T^y k_j(x)| d\mu(x) + \int_0^\infty \chi_E(x)|T^{y_0} k_j(x)| d\mu(x) \\
\leq \int_{2|y-y_0|}^\infty |k_j(x)| d\mu(x) + \int_{2|y-y_0|}^\infty |k_j(x)| d\mu(x) \\
\leq CB(2^j |y-y_0|)^{(r+1)/2-k}.
\]
Thus (3.4) is established.

Finally, to establish (3.5) we use Bernstein's inequality. Since \( \tilde{k}_j = m_j \) has support in \( (0, 2^{j+1}) \) it follows from Corollary (2.2) and (3.7) that
\[
\int_{|x-y_0| > 2|y-y_0|} |T^y k_j - T^{y_0} k_j| d\mu \leq \|T^y k_j - T^{y_0} k_j\|_1 \leq 2^{j+1}|y-y_0||k_j||_1 \leq CB2^j |y-y_0|.
\]
Thus (3.5) is established. Combining (3.4) and (3.5) we obtain (3.3) which completes the proof of the theorem.

4. A FINAL REMARK

In connection with the Bernstein-type inequality (2.7) the following question arises. Consider the “mean value” operators \( T^y \), \( y \geq 0 \), defined for functions on \( \mathbb{R}^n \), \( n \geq 2 \), by
\[
T^y f(x) = \int_{S^{n-1}} f(x + y\tilde{t}) d\sigma(\tilde{t}),
\]
where \( d\sigma \) is normalized surface measure on the sphere \( S^{n-1} \). In other words \( T^y \) is the convolution operator with the uniformly distributed probability measure on \( \{ x \in \mathbb{R}^n : \|x\| = y \} \). Is it true that for any \( h \in C^1(\mathbb{R}^n) \) with, say, \( \|\nabla h\| \in L^1(\mathbb{R}^n) \), the following estimate holds
\[
\|T^{y_1} h - T^{y_2} h\|_{L^1(\mathbb{R}^n)} \leq C|y_1 - y_2|,
\]
where \( C \) depends on \( h \) but not on \( y_1 \) and \( y_2 \)? We note that Theorem 2.1 gives a positive answer to this problem only in the case when \( h \) is radial.

But one can rather easily remark that the above problem may be answered affirmatively by writing out \( \|T^{y_1} h - T^{y_2} h\|_{L^1(\mathbb{R}^n)} \), interchanging the order of integration and then applying the ordinary Bernstein-type inequality on \( \mathbb{R}^n \). This yields an easy proof of Theorem 2.1 in the case of integral values of \( r \).

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