PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 106, Number 4, August 1989

A DYNAMICAL SYSTEM ON R^3 WITH UNIFORMLY BOUNDED TRAJECTORIES AND NO COMPACT TRAJECTORIES

K. M. KUPERBERG AND COKE S. REED

(Communicated by Kenneth R. Meyer)

ABSTRACT. This paper contains an example of a rest point free dynamical system on R^3 with uniformly bounded trajectories, and with no circular trajectories. The construction is based on an example of a dynamical system described by P. A. Schweitzer, and on an example of a dynamical system on R^3 constructed previously by the authors.

INTRODUCTION

In order to answer S. Ulam's question posed in the Scottish Book (see Problem 110 in [5 or 3]) we constructed in [2] a rest point free dynamical system Φ on R^3 with all trajectories uniformly bounded by a given $\varepsilon > 0$. The function $f(x) = \Phi(1, x)$ provided a counterexample to Ulam's problem. The dynamical system Φ uses elements similar to Fuller's flow described in [1] (see also Wilson [6]), and hence Φ contains circular trajectories. The purpose of this paper is to eliminate all circular trajectories by modifying the example described in [2]. This is done with the help of Schweitzer's flow, constructed in [4] in order to solve the Seifert conjecture on closed integral curves of a nonvanishing vector field on S^3.

THE EXAMPLE

P. A. Schweitzer, see [4, pp. 393–394], based the construction of a "plug" with no circular trajectories on Denjoy's vector field on a surface of a torus. A C^1 Denjoy vector field is included in the Appendix of [4]. We will now define a Schweitzer flow in such a way that the plug "stops" an open set of trajectories.

Assume the following notation: R^n = Euclidean n-space, S^1 = one-dimensional sphere, $T = S^1 \times S^1$, d = distance function on T, $Z = C^1$ Denjoy's vector field on T, A = exceptional minimal set of Z, and $\Phi(t, x)$ = trajectory of a dynamical system Φ passing through x.

Let B be a compact proper subset of T, invariant under the dynamical system generated by Z, and such that $A \subset \operatorname{Int} B$. Let $p_0 \in T - B$ be a point.

Received by the editors June 19, 1988 and, in revised form, October 26, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 58F25; Secondary 54H20.

For $i = 0, 1, 2$, let $D_i \subset T$ be a disc centered at p_0 with radius r_i such that $r_0 < r_1 < r_2 \leq d(p_0, B)$. Let $f: T \to R^1$ be a C^∞ mapping such that $f|D_1 \equiv 0$, $f|T - D_2 \equiv 1$, and if $q_1, q_2 \in D_2 - D_1$ are such that $d(q_1, p_0) < d(q_2, p_0)$, then $f(q_1) < f(q_2)$. Let $g: R^1 \to R^1$ be an increasing C^∞ function such that $g|(-\infty, 0] \equiv 0$ and $g|[1/6, \infty) \equiv 1$. Define $\alpha, \beta: R^1 \to R^1$ by

$$\alpha(s) = \begin{cases} 0 & \text{if } s \leq \frac{1}{6}, \\ g(s - \frac{1}{6}) & \text{if } \frac{1}{6} \leq s \leq \frac{1}{3}, \\ g(\frac{1}{2} - s) & \text{if } \frac{1}{3} \leq s \leq \frac{1}{2}, \\ -\alpha(1 - s) & \text{if } s \geq \frac{1}{2}, \end{cases}$$

$$\beta(s) = \begin{cases} \alpha(s) & \text{if } s \leq \frac{1}{2}, \\ -\alpha(s) & \text{if } s \geq \frac{1}{2}. \end{cases}$$

Set $M = T - D_0$. Let V be a vector field defined on the manifold $M \times R^1$ by setting

$$V((p, s)) = (f(p)Z(p)\alpha(s), 1 - f(p)\beta(s)).$$

Notice that if $V((p, s)) = (v_1, v_2)$, then $V((p, 1 - s)) = (-v_1, v_2)$; and $V((p, s)) = (0, 1)$ if either $p \in D_1 - D_0$, $s \leq 1/6$, or $s \geq 5/6$. The dynamical system Φ generated by V has the following properties:

(i) for every point $p \in B$ the trajectory containing the point $(p, 1)$ has its α-limit points in the interior of the set $M \times [0, 1]$, and the trajectory containing the point $(p, 0)$ has its ω-limit points in the interior of the set $M \times [0, 1]$,

(ii) for any number $t > 0$ and any point $(p, s) \in [M \times (R^1 - \{1/3, 2/3\})] \cup \operatorname{Int}[(D_2 - D_0) \times R^1]$ if $\Phi(t, (p, s)) = (q, r)$, then $r > s$,

(iii) if $(p, 0)$ and $(q, 1)$ belong to the same trajectory, then $p = q$,

(iv) Φ contains no circular trajectories.

There exists a C^∞ embedding $\mu: M \times [0, 1] \to R^3$ such that for any $p \in M$, the set $\{p\} \times [0, 1]$ is contained in a line parallel to the z-axis, and for any $(p, s) \in M \times [0, 1]$, there exists a neighborhood N of p in M such that the projection onto the xy-plane restricted to the set $\mu(N \times \{s\})$ is an embedding, see [2]. We may also assume that $\mu(M \times [0, 1])$ is a subset of the cube $\{(x, y, z) \in R^3: 0 \leq x, y, z \leq 1\}$.

There exists a C^1 vector field U defined on R^3 such that U and the dynamical system ψ generated by U satisfy the following conditions:

(i) $(x, y, z) \notin \mu(M \times [0, 1])$, then $U((x, y, z)) = (0, 0, 1)$,

(ii) for any trajectory $\psi(t, (x, y, z))$, the set $\psi(t, (x, y, z)) \cap \mu(M \times [0, 1])$ is contained in the set $\mu(\Phi(t, (p, s))$ for some $(p, s) \in M \times [0, 1]$,

(iii) for any trajectory $\psi(t, (x, y, z))$, the set $\psi(t, (x, y, z)) \cap (R^3 - \mu(M \times [0, 1])$ is contained in a line parallel to the z-axis.

Remark. No trajectory of ψ is compact, and if a trajectory of ψ passes through the points $(x_1, y_1, 0)$ and $(x_2, y_2, 1)$, then $(x_1, y_1) = (x_2, y_2)$.

There exists a nonempty subset of the plane $Q = \{(x,y) \in R^2 : a \leq x \leq a+\delta,\ b \leq y \leq b+\delta\}$ such that if $(x,y) \in Q$, then the α-limit points of $\psi(t,(x,y,1))$ and the ω-limit points of $\psi(t,(x,y,0))$ are in the interior of the cube $\{(x,y,z) \in R^3 : 0 \leq x,y,z \leq 1\}$.

Let $[x]$ denote the greatest integer less than or equal to x. Let N_0 be an integer greater than $1/\delta$. We define a vector field W on R^3 in the following way: if $iN_0+j \leq z < iN_0+j+1$, where i and j are integers, and $0 \leq j < N_0$, then put

$$W((x,y,z)) = U((x - i/N_0 - [x - i/N_0], y - j/N_0 - [y - j/N_0], z - [z])).$$

Clearly, W is a C^1 vector field defined on R^3. Similarly as in [1], the trajectories of the dynamical system generated by W are uniformly bounded; every trajectory is contained in a box of dimensions 3, 3, and N_0^2, and hence it is bounded by $\sqrt{18 + N_0^4}$.

Remark 1. By rescaling the example, the uniform bound for the diameter of the trajectories can be changed to an arbitrarily pre-assigned $\varepsilon > 0$.

Remark 2. The vector field constructed in this paper is of class C^1. The following question remains unanswered: Does there exist a C^∞ dynamical system on R^3 with uniformly bounded trajectories and no compact trajectories?

References

1. F. B. Fuller, *An index of fixed point type for periodic orbits*, Amer. J. Math. **89** (1967), 133–148.
2. K. Kuperberg and C. Reed, *A rest point free dynamical system on R^3 with uniformly bounded trajectories*, Fund. Math. **114** (1981), 229–234.
3. R. D. Mauldin (ed.), *The Scottish book: Mathematics from the Scottish Café*, Birkhauser, Boston, 1981.
4. P. A. Schweitzer, *Counterexamples to the Seifert conjecture and opening closed leaves of foliations*, Ann. of Math. (2) **100** (1974), 386–400.
5. S. M. Ulam, *The Scottish book*, Los Alamos Scientific Monograph LA-6832.
6. F. W. Wilson, Jr., *On the minimal sets of non-singular vector fields*, Ann. of Math. (2) **84** (1966), 529–536.

Department of Mathematics, Auburn University, Auburn, Alabama 36849

Supercomputing Research Center, Lanham, Maryland 20706