AUTOMORPHISMS OF EXTENDED CURRENT ALGEBRAS

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ABSTRACT. We construct a (noncentral) extension of current algebras and study the adjoint action induced by the current group.

1. INTRODUCTION

Let us consider the affine Kač-Moody algebra [5] associated to a finite-dimensional linear reductive Lie algebra $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{g} \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

where $\tilde{\mathfrak{g}} = \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} \mathfrak{g}$. $\tilde{\mathfrak{g}}$ is a central extension of $\mathfrak{g}$ and the bracket is defined as follows ($u, v \in \mathfrak{g}; \lambda, \mu, \lambda_1, \mu_1 \in \mathbb{C}$):

$$[t^k \otimes u \oplus \lambda c \oplus \mu d , t^{k_1} \otimes v \oplus \lambda_1 c \oplus \mu_1 d]^\wedge = (t^{k+k_1} \otimes [u, v] + \mu k_1 t^{k} \otimes v - \mu_1 k t^{k} \otimes u) \oplus k \delta_{k, -k_1} (u|v)c.$$

There is a nondegenerate invariant symmetric bilinear form $(\cdot | \cdot)^\wedge$ on $\tilde{\mathfrak{g}}$ defined by:

$$(t^k \otimes u | t^{k_1} \otimes v)^\wedge = \delta_{k, -k_1} (u|v)$$

$$\langle c | d \rangle^\wedge = \langle d | c \rangle^\wedge = 1$$

for every $u, v \in \mathfrak{g}$ and $k, k_1 \in \mathbb{Z}$. The group $\tilde{G} = \mathbb{C}[t, t^{-1}] \otimes G$ acts on $\tilde{\mathfrak{g}}$ by automorphisms [6, 2] as follows:

$$(\widehat{\text{Ad}}g)(u(t)) = gug^{-1}(t) + \text{Res}_0 \text{tr}(ug^{-1}g')c$$

$$(\widehat{\text{Ad}}g)c = c$$

for every $g \in \tilde{G}$ and $u \in \tilde{\mathfrak{g}}$; this action preserves the bilinear form $(\cdot | \cdot)^\wedge$. Here $\tilde{\mathfrak{g}}$ is the algebra of finite Laurent polynomial maps. By a change of variable...
$t = e^{i\theta}$, each Laurent polynomial map can be realized as a smooth map from $S^1$ to $\mathfrak{g}$. This means that we have an injection from the algebra of Laurent polynomial maps into the loop algebra, the algebra of all smooth maps from $S^1$ to $\mathfrak{g}$. Since our main interest is the latter, we will keep the notation $\mathfrak{g}$ for the loop algebra unless otherwise specified. More generally the Lie algebra $\mathfrak{g}(M)$ of all smooth maps from an arbitrary manifold $M$ to $\mathfrak{g}$ is known as the current algebra [3, 1].

In this paper, we consider a (noncentral) extension of the current algebra and study the automorphisms on the extended algebra induced by the current group. The paper is organized as follows. In §2, inspired by [4], we define an extension $\mathfrak{g}(M)$ of the current algebra. In §3, we introduce a bilinear form on $\mathfrak{g}(M)$ which is shown to be invariant and nondegenerate. In §4, we prove a useful lemma. In §5, we study the adjoint action of the current group on $\mathfrak{g}(M)$ by automorphisms preserving the bilinear form. We find out that this can be done only when $\dim M \leq 2$. Finally, in §6, we show the previous results generalize the loop algebra case (formulas (2) to (4)) when we take $M = S^1$.

2. Extension of current algebras

Let $G$ be a reductive Lie group and $\mathfrak{g}$ its Lie algebra. There is a nondegenerate symmetric bilinear invariant form $\langle \cdot , \cdot \rangle$ on $\mathfrak{g}$. Suppose $M$ is a compact, closed, orientable manifold with a normalized volume form $\Omega$. Let $\mathfrak{g}$ be the set of all $C^\infty$ maps $x : M \rightarrow \mathfrak{g}$. $\mathfrak{g}$ is the Lie algebra of the infinite-dimensional Lie group $\tilde{G}$ of all $C^\infty$ maps $g : G \rightarrow M$. It is called the current algebra. We now construct a (noncentral) extension of $\mathfrak{g}$:

\begin{equation}
\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{c} \oplus \mathfrak{d},
\end{equation}

where

\begin{equation}
\mathfrak{c} = \Lambda^1(M)/d\Lambda^0(M),
\end{equation}

and

\begin{equation}
\mathfrak{d} = \{ X \in \text{Vect}(M) | L_X \Omega = 0 \},
\end{equation}

where $L_X = d i_X + i_X d$ is the Lie derivative operator.

The commutation relations are given by

\begin{align}
[x, y] &= [x, y] + (dx|y) = [x, y] - (x|dy), \\
[\alpha, x] &= 0, \\
[X, x] &= L_X x, \\
[\alpha, \beta] &= 0, \\
[X, \alpha] &= L_X \alpha, \\
[X, Y] &= [X, Y], \text{ the usual bracket of vector fields},
\end{align}

for any $x, y \in \mathfrak{g}$, $\alpha, \beta \in \mathfrak{c}$ and $X, Y \in \mathfrak{d}$. The anticommutativity and the Jacobi identities follow easily from the properties of usual brackets and Lie derivatives.
A similar definition was given by Kac in [4], where $\hat{g}$ is a central extension of $\hat{g}$. In our case, $[X, \alpha]^\wedge$ is not zero in order to define automorphisms of the current algebra. See remark of §6 for details.

3. The nondegenerate invariant form on $\hat{g}$

We define a symmetric bilinear form on $\hat{g}$ by the following formulas

$$ (x|y)^\wedge = \int_M (x|y)\Omega, $$

$$ (X|\alpha)^\wedge = \int_M (i_x \alpha)\Omega, $$

and the others are zero. It is well defined because for each $f \in C^\infty(M)$,

$$ (X|df)^\wedge = \int_M (i_x df)\Omega = \int_M (L_x f)\Omega = \int_M d_i_X (f\Omega) = 0. $$

The nondegeneracy and invariance properties are given by the following two propositions.

**Proposition 3.1.** The form $(\cdot|\cdot)^\wedge$ is nondegenerate.

**Proof.** Notice that if $x \neq 0$, then there exists a $y \in \hat{g}$ such that $(x|y)^\wedge \neq 0$; also if $X \neq 0$, then there exists an $\alpha \in \mathfrak{c}$ such that $(X|\alpha)^\wedge \neq 0$. So it suffices to check that $\alpha \in \Lambda^1(M)$ is exact if it satisfies $(X|\alpha)^\wedge = 0$ for all $X \in \mathfrak{d}$. Since

$$ (X|\alpha)^\wedge = \int_M (i_x \alpha)\Omega = \int_M \alpha \wedge i_x \Omega, $$

we only have to show that the map $\Theta: \mathfrak{d} \to \mathbb{Z}^{n-1}(M)$ defined by $\Theta(X) = i_x \Omega$ is onto (by Poincaré duality). It is easy to see that for each $\beta \in \mathbb{Z}^{n-1}(M)$, there is an $X \in \text{Vect}(M)$ such that $i_x \Omega = \beta$; $d\beta = 0$ implies that $L_x \Omega = d_i_X \Omega = 0$, which means $X \in \mathfrak{d}$. □

**Proposition 3.2.** The form $(\cdot|\cdot)^\wedge$ is invariant.

**Proof.** By definition of invariance, we have to check that the left-hand sides of the following two formulas are equal:

$$ ([x + \alpha + X, y + \beta + Y]^\wedge |z + \gamma + Z)^\wedge $$

$$ = \int_M \{([x, y]|z) + (L_x y - L_y x|z) + i_z (dx|y) $$

$$ + i_z (L_x \beta - L_y \alpha) + i_{[x, y]\gamma}\Omega, $$

$$ (x + \alpha + X|[y + \beta + Y, z + \gamma + Z]^\wedge)^\wedge $$

$$ = \int_M \{(x|[y, z]) + (x|L_Y z - L_Z y) + i_x (dy|z) $$

$$ + i_x (L_Y \gamma - L_Z \beta) + i_{[Y, Z]\alpha}\Omega. $$

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By using the well-known formula $i_{[X,Y]} = [i_X, L_Y] = [L_X, i_Y]$, the difference of the two integrands on the right-hand sides is a linear combination of Lie derivatives of functions on $M$

\begin{equation}
-L_Y(x|z) + L_Z(x|y) - L_Y(i_X Y) + L_Y(i_Z \alpha),
\end{equation}

hence the integration over $M$ is zero (see equation (16)).

4. A LEMMA

We establish here a lemma which will be frequently used in §5.

**Lemma.** If $g: M \rightarrow G$ and $x: M \rightarrow g$ are $C^\infty$-maps, $X \in \text{Vect}(M)$, then

\begin{align}
(21) \quad & (a) \quad L_X(\text{Ad} \, g)_* x = (\text{Ad} \, g)_*(L_X x + [i_X g^* \omega, x]), \\
(22) \quad & (b) \quad d(\text{Ad} \, g)_* x = (\text{Ad} \, g)_* (dx + [d g^* \omega, x]),
\end{align}

where $(\text{Ad} \, g)_*$ is the adjoint action of $g \in G$ on $g$ and $\omega$ is the Maurer-Cartan form of the Lie group $G$.

**Proof.** Let $\phi_t$ be the flow on $M$ generated by $X$. Set $g_t = g \circ \phi_t: M \rightarrow G, x_t = x \circ \phi_t; M \rightarrow g$. Then

\begin{align}
(23) \quad L_X(\text{Ad} \, g)_* x &= \lim_{t \to 0} \frac{(\text{Ad} \, g_t)_* x_t - (\text{Ad} \, g)_* x}{t} \\
(24) \quad &= (\text{Ad} \, g)_* \left( L_X x + \lim_{t \to 0} \frac{(\text{Ad} \, g^{-1} g_t)_* - 1}{t} x \right).
\end{align}

Notice that the tangent vector of the flow $g^{-1} g_t$ at the unit element of $G$ is $i_X g^* \omega$, an element in the Lie algebra $g$. Hence we have

\begin{equation}
\lim_{t \to 0} \frac{(\text{Ad} \, g^{-1} g_t)_* - 1}{t} x = [i_X g^* \omega, x].
\end{equation}

This proves part (a). Part (b) follows easily.

5. AUTOMORPHISMS OF $\hat{\mathfrak{g}}$

We now investigate how to define the adjoint action of $\tilde{G}$ on the extended current algebra $\hat{\mathfrak{g}}$ so that the bilinear form $(\cdot, \cdot)^\wedge$ is preserved. That is to say, for each $g \in \tilde{G}, x \in \hat{\mathfrak{g}}$ and $X \in \mathfrak{d}$ we have to find $a(g, X) \in \mathfrak{g}$ and $\xi(g, x), \eta(g, X) \in \Lambda^1(M)$ such that the action $\hat{\text{Ad}} g$ on $\hat{\mathfrak{g}}$ defined by

\begin{align}
(\hat{\text{Ad}} g)_* x &= (\text{Ad} \, g)_* x + \xi(g, x) \\
(\hat{\text{Ad}} g)_* \alpha &= \alpha \\
(\hat{\text{Ad}} g)_* X &= a(g, X) + \eta(g, X) + X
\end{align}

is an automorphism which preserves $(\cdot, \cdot)^\wedge$; here $(\text{Ad} \, g)_*$ is the pointwise adjoint action of $\tilde{G}$ on $\hat{\mathfrak{g}}$. 

Since we require
\[(\text{Ad}g)X, (\text{Ad}g)x]^\wedge = (\text{Ad}g)[X, x]^\wedge,\]
we must have, by comparing the \(g\)-components,
\[(a, (\text{Ad} g)_* x) + L_X(\text{Ad} g)_* x = (\text{Ad} g)_* L_X x.\]
We can choose
\[a(g, X) = -(\text{Ad} g)_* i_X g^* \omega\]
by the lemma, where \(\omega\) is the Maurer–Cartan form of \(G\). Next, invariance implies
\[(\text{Ad} g)_* x, (\text{Ad} g)_* X]^\wedge = (x, X)^\wedge = 0.\]
By the choice of \(a(g, X)\), we have
\[(\xi|x)^\wedge = ((\text{Ad} g)_* x|(\text{Ad} g)_* i_X g^* \omega)^\wedge,\]
i.e.
\[\int_M (i_X \xi) \Omega = \int_M ((\text{Ad} g)_* x|(\text{Ad} g)_* i_X g^* \omega) \Omega = \int_M (i_X (x|g^* \omega)) \Omega.\]
This means we can choose
\[\xi(g, X) = (x|g^* \omega).\]
Finally,
\[(\text{Ad} g)_* X|((\text{Ad} g)_* X)^\wedge = (X|X)^\wedge = 0\]
implies
\[(\eta|x)^\wedge = -\frac{1}{2}((\text{Ad} g)_* i_X g^* \omega|(\text{Ad} g)_* i_X g^* \omega)^\wedge.\]
Proceeding as in (32), we can choose
\[\eta(g, X) = -\frac{1}{2}(i_X g^* \omega|g^* \omega).\]
We summarize in the following.

**Theorem.** For each \(g \in \tilde{G}\), the action \(\text{Ad}g\) given by
\[(\text{Ad}g)_* x = (\text{Ad} g)_* x + (x|g^* \omega)\]
\[(\text{Ad}g)_* \alpha = \alpha\]
\[(\text{Ad}g)_* X = - (\text{Ad} g)_* i_X g^* \omega - \frac{1}{2}(i_X g^* \omega|g^* \omega) + X\]
preserves the bilinear form \((\cdot|\cdot)^\wedge\). Furthermore, it is an automorphism of \(\tilde{g}\) if \(\dim M \leq 2\).

In order to keep the proof to a reasonable size, routine calculations will be omitted.
Proof. From the construction of $\hat{\text{Ad}g}$, it is easy to see that it preserves the bilinear form $(\cdot | \cdot)$. That $\hat{\text{Ad}g}$ preserves the bracket of $x$ and $\alpha$, $X$ and $\alpha$, $\alpha$ and $\beta$ is also straightforward. Next, we calculate

$$[(\text{Ad}g)x , (\text{Ad}g)y]^A = [(\text{Ad}g)_x, (\text{Ad}g)_y] + (d(\text{Ad}g)_x)(\text{Ad}g)_y$$

(38) $$= (\text{Ad}g)_x[x, y] + ((\text{Ad}g)_x(dx + [g^*\omega, x])(\text{Ad}g)_y)$$

(by the lemma)

$$= (\text{Ad}g)_x[x, y] + (g^*\omega)[x, y] + (dx)y$$

$$= (\hat{\text{Ad}g})[x, y]^A$$

and

$$[(\hat{\text{Ad}g})X, (\hat{\text{Ad}g})X]^A - (\hat{\text{Ad}g})[X, X]^A$$

(39) $$= (ixg^*\omega|x + [g^*\omega, x]) + (x|L_xg^*\omega)$$

(by the lemma)

$$= d(ixg^*\omega|x) + (ixd(g^*\omega|x)) + (L_xg^*\omega)|X, x\rangle$$

which is an exact form. The last step above follows from the Maurer-Cartan equation $d\omega + \frac{1}{2}[\omega, \omega] = 0$. We finally examine the difference of $[(\hat{\text{Ad}g})X, (\hat{\text{Ad}g})Y]^A$ and $(\hat{\text{Ad}g})[X, Y]^A$. Their $d$-components are both $[X, Y]$. The difference of the $g$-components is

$$[(\text{Ad}g)_x[i_xg^*\omega, i_yg^*\omega] + L_Y(\text{Ad}g)_x i_xg^*\omega - L_X(\text{Ad}g)_x i_yg^*\omega$$

$$+ (\text{Ad}g)_x i_{xy}g^*\omega$$

$$= (\text{Ad}g)_x (L_y i_x - i_y L_x)g^*\omega - [i_xg^*\omega, i_yg^*\omega]$$

$$= (\text{Ad}g)_x (L_y i_x - i_y L_x)g^*\omega - [i_xg^*\omega, i_yg^*\omega]$$

(40) $$= 0.$$}

The difference of the $c$-components is

$$\frac{1}{2}(L_Y i_x g^*\omega|L_Y g^*\omega) + \frac{1}{2}(i_x g^*\omega|L_Y g^*\omega) - \frac{1}{2}(L_X i_y g^*\omega|L_Y g^*\omega) - \frac{1}{2}(i_y g^*\omega|L_X g^*\omega)$$

$$+ (di_x g^*\omega + [g^*\omega, i_x g^*\omega])i_y g^*\omega + \frac{1}{2}(di_y i_x g^*\omega|L_Y g^*\omega)$$

$$= \frac{1}{2}((L_Y i_X - L_X i_Y) + [L_X, i_Y])g^*\omega + g^*\omega) - (i_x g^*\omega, i_y g^*\omega)$$

$$+ \frac{1}{2}(i_x g^*\omega)(di_y + i_y d)g^*\omega + \frac{1}{2}((di_x - i_x d)g^*\omega)$$

$$= \frac{1}{2}d(i_x g^*\omega|L_Y g^*\omega) + \frac{1}{2}((di_y i_X - i_y L_x)g^*\omega|L_Y g^*\omega)$$

$$+ (L_x g^*\omega, i_y g^*\omega)\]g^*\omega)$$

$$= \frac{1}{2}d(i_x g^*\omega|L_Y g^*\omega) + \frac{1}{2}([i_x g^*\omega, i_y g^*\omega]g^*\omega).$$

The first term above is an exact form, but the second one is only when $\dim M \leq 2$. To see this, we introduce local coordinates $(\sigma, \tau)$ when $\dim M = 2$ and
choose $X = \partial/\partial \sigma$ and $Y = \partial/\partial \tau$. Then

\begin{equation}
(i_X g^\ast \omega, i_Y g^\ast \omega) = \int g^\ast \omega \ d\sigma \ + \ (i_{\partial/\partial \sigma} g^\ast \omega, i_{\partial/\partial \tau} g^\ast \omega) d\tau = 0,
\end{equation}

since the bilinear form is invariant. 

**Remark.** If $\dim M \geq 3$, the second term in (41) is not necessarily exact. Take $M = T^3$ parametrized by $(\theta, \varphi, \psi)$ and $G = SU(2)$. Choose $X = \partial/\partial \theta$, $Y = \partial/\partial \varphi$ and

\begin{equation}
g = \begin{pmatrix} e^{i\theta} \cos \varphi & e^{i\varphi} \sin \varphi \\
-e^{i\varphi} \sin \varphi & e^{-i\theta} \cos \varphi \end{pmatrix}.
\end{equation}

Then

\begin{equation}
\frac{1}{2} (i_X g^\ast \omega, i_Y g^\ast \omega) g^\ast \omega) = -2 \sin 2\varphi \ d\psi,
\end{equation}

which is not even closed.

### 6. Comparisons with Kac-Moody Algebras

We now take $M = S^1$ parametrized by $e^{i\theta}$. Its normalized volume form is $\Omega = \Omega/2\pi$. In this case, $\mathfrak{g}$ is the loop algebra, $c = H^1(S^1, \mathbb{R})$ is a one-dimensional vector space generated by the element $c = \alpha \cos \theta$; every one form $\alpha$ is represented by $((1/2\pi) \int_{S^1} \alpha) c$. $d$ is also one dimensional, generated by $d = (1/2\pi) \int_{S^1} d\theta$. Then the commutation relations (8) to (13) become

\begin{equation}
[x, y] = [x, y] + \frac{1}{2\pi} \int_{S^1} x'(\theta) y(\theta) d\theta c
\end{equation}

\begin{equation}
[x, c] = 0.
\end{equation}

The invariant form on $\mathfrak{g}$ is given by

\begin{equation}
(x|y) = \frac{1}{2\pi} \int_{S^1} (x(\theta)|y(\theta)) d\theta
\end{equation}

\begin{equation}
(c|d) = (d|c) = 1
\end{equation}

others are zero.

Finally, by the Theorem the action of $\tilde{G}$ on $\mathfrak{g}$ is

\begin{equation}
(\tilde{Ad}g)x = gxg^{-1} + \frac{1}{2\pi} \int_{S^1} \text{tr}(xg^{-1} g) = c
\end{equation}

\begin{equation}
(\tilde{Ad}g)d = ig'g^{-1} - \frac{1}{4\pi} \int_{S^1} \text{tr}((g^{-1})' d g) c + d.
\end{equation}

All these formulas agree with those given in the introduction if we make a change of variable $t = e^{i\theta}$ there.
Remark. If $M$ is one dimensional, then $c$ is in the center of $\hat{\mathfrak{g}}$ because $[X, \alpha]^\wedge = L_X\alpha$ is exact. This is not the case in general. If we had defined $[X, \alpha]^\wedge = 0$ instead of $L_X\alpha$, then the $c$-component of $[(\hat{\text{Ad}}g)X, (\hat{\text{Ad}}g)x]^\wedge - (\hat{\text{Ad}}g)[X, x]^\wedge$ would be $-L_X(x|g^*\omega)$; this may not be exact.

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