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GROUP ACTIONS AND DIRECT SUM DECOMPOSITIONS OF L^p SPACES

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ABSTRACT. Let G be a locally compact group of measure preserving transformations on a σ-finite measure space $(X, \mathscr{B}, m)$, and let S be a subset of $M^1(G)$. Let $1 < p < \infty$, $I_p = \{f: f \in L^p(m)$ and ${}_g f = f$, for all $g \in G\}$, let $I_p(S) = \{f: f \in L^p(m)$ and $\mu * f = f$ for all $\mu \in S\}$, and let $K_p(S)$ be the closed subspace of $L^p(m)$ generated by functions of the form $\mu * f - f$, for $f \in L^p(m)$ and $\mu \in S$. Conditions are given on S which ensure that $I_p = I_p(S)$, and this is used to express $L^p(m)$ as a direct sum of I_p and $K_p(S)$.

1. INTRODUCTION

It is well known that the amenability of a semigroup S of linear transformations on a Banach space B may lead to the expression of B as a direct sum of the subspace of vectors left fixed by each transformation of S, and the closed subspace of B generated by vectors of the form $\{s(x) - x: s \in S$ and $x \in B\}$ (see [3, p. 85] and [6], for example). In fact, the classical mean ergodic theorem can be regarded as the special case of this situation which arises when B is a Hilbert space and S is the semigroup generated by a single contraction on B, for in this case S is abelian and hence amenable.

In a recent paper ([10]) J. Rosenblatt has considered decompositions of the above type in L^p spaces which arise from the action of a countable and discrete group G as a group of measure preserving transformations on a finite measure space $(X, \mathscr{B}, m)$. He gives conditions on a probability measure μ on G which ensure that $L^p(m)$ is the direct sum of the functions in $L^p(m)$ which are invariant under the action of G, and of the subspace of $L^p(m)$ generated by functions of the form $\{f - \mu * f: f \in L^p(m)\}$.

In the present paper, some results closely related to the above are obtained which apply for more general locally compact groups G. For example, a consequence of results in this paper is the following: if G is a compact connected

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group, if $1 < p < \infty$, and h is a positive function in $L^1(G)$ whose integral is one, then $L^p(G)$ is the direct sum of the subspace of constant functions and the subspace of functions which are limits in $L^p(G)$ of functions of the form $f - h * f$, for some $f \in L^p(G)$. Furthermore, if the regular representation of G upon the space of mean-zero functions in $L^2(G)$ does not weakly contain the trivial representation, then the subspace $\{f - h * f : f \in L^p(G)\}$ of $L^p(G)$ is closed and $L^p(G)$ is the direct sum of this subspace and the subspace of constant functions. Some notation and preliminaries now follow.

Let $(X, \mathscr{B}, m)$ be a σ-finite measure space, normalized if $(X, \mathscr{B}, m)$ is finite, and let G be a locally compact group. It will be assumed that there is a jointly measurable function $(s, x) \to sx$ from $G \times X$ to X (where the σ-algebra on G is the σ-algebra of Borel sets), under which G becomes a group of measure preserving transformations on $(X, \mathscr{B}, m)$. It will be further assumed that if $1 \leq p < \infty$ and $f \in L^p(m)$, then $s \to {}_sf$ is a continuous function from G to $L^p(m)$, where ${}_sf$ denotes the function $x \to f(s^{-1}x)$ in $L^p(m)$. It should be noted that should $(X, \mathscr{B}, m)$ be separable, this further assumption is a consequence of the other assumptions, as can be shown by an adaptation of a standard argument ([4, vol. I, p. 347]). A case of special interest is where $G = X$, m is the left Haar measure on G, and the action of G on X is simply given by the left group translations. If X is a topological space, $C(X)$ will denote all the bounded continuous scalar valued functions on X, and $C_0(X)$ the functions in $C(X)$ which vanish at infinity.

Let λ denote a given left invariant Haar measure on G, normalized if G is compact, and let $\mathscr{P}^1(G)$ denote those functions in $L^1(G)$ which are positive and such that $\int_G f\, d\lambda = 1$. Let $M(G)$ the Banach space of Borel measures of finite variation on G, and let $M^1(G)$ denote those elements of $M(G)$ which are positive and of norm one. If $s \in G$, let $\delta_S \in M^1(G)$ be the unit mass at s. $M^1(G)$ is a semigroup under convolution with an identity element δ_e, where e is the identity element of G. $\mathscr{P}^1(G)$ is a subsemigroup of $M^1(G)$. Let S denote a given nonempty subset of $M^1(G)$; S_{w^*} will denote S equipped with the weak* topology. It is proved in [7, Lemma 3.2] that in the weak* topology, $M(G)$ is a topological semigroup in the sense that convolution is separately continuous. The action of G upon each space $L^p(m)$, $1 \leq p \leq \infty$, can be extended to an action of $M(G)$ upon $L^p(m)$ by means of the formula

$$(\mu * f)(x) = \int_G f(s^{-1}x)\, d\mu(s), \quad \text{for } x \in X \text{ and } \mu \in M(G).$$

Accordingly, elements of $M(G)$ will sometimes be regarded as bounded operators on $L^p(m)$. If f is a function on G, f^* will denote the function on G given by $f^*(x) = \overline{f(x^{-1})}$. If $\mu \in M(G)$, define $\tilde{\mu}$ as the element of $M(G)$ given $\tilde{\mu}(A) = \overline{\mu(A^{-1})}$. Let the set $\{\tilde{\mu} : \mu \in S\}$ be denoted by $\tilde{S}$. S is said to be *left translation invariant* if $\delta_S * \mu \in S$ for all $s \in G$ and all $\mu \in S$. S is said to be *right translation invariant* if $\mu * \delta_S \in S$ for all $s \in G$ and $\mu \in S$. Note

that $(\delta_S * \mu)(A) = \mu(s^{-1}A)$, and $(\mu * \delta_S)(A) = \mu(As^{-1})$, for all Borel sets A. It is easy to check that both $\{\delta_S : s \in G\}$ and $\mathscr{P}^1(G)$ are left and right translation invariant subsemigroups of $M^1(G)$.

2. Expressing $L^p(m)$ as a direct sum

Let $K_p(S)$ denote the closed subspace of $L^p(m)$ generated by functions of the form $\mu * f - f$, for $f \in L^p(m)$ and $\mu \in S$, and let $I_p(S)$ denote those functions in $L^p(m)$ which are left fixed by the action of each element of S. That is, $I_p(S) = \{f : f \in L^p(m)$ and $\mu * f = f$ for all $\mu \in S\}$. The action of G upon $(X, \mathscr{B}, m)$ is said to be *disjoint at infinity on m-finite sets* if, for each set $K \in \mathscr{B}$ such that $m(K) < \infty$, there is a compact subset C of G such that $m(gK \cap K) = 0$, for all $g \notin C$. If G is compact the action of G is disjoint at infinity on m-finite sets, and the converse is true if $m(X) < \infty$.

Theorem 1. *Let $1 < p < \infty$, let the action of G upon $(X, \mathscr{B}, m)$ be disjoint at infinity on m-finite sets, and let S be a subsemigroup of $M^1(G)$ such that $C(S_{w^*})$ has an invariant mean. Then $L^p(m)$ is the direct sum of $K_p(S)$ and $I_p(S)$, and the associated projection P from $L^p(m)$ onto $I_p(S)$ has the property that $P\mu = \mu P = P$ for all $\mu \in S$.*

Proof. If $f \in L^p(m)$, the closure of $\{\mu * f : \mu \in S\}$ is weakly compact in $L^p(m)$. Also, if $h \in L^q(m)$, where $p^{-1} + q^{-1} = 1$, we have

$$\int_X (\mu * f) h \, dm = \int_G \left(\int_X ({}_s f) h \, dm \right) d\mu(s).$$

Because the action of G is disjoint at infinity on m-finite sets, a routine calculation (compare it with [4, vol. I, p. 295]) shows that the function on G given by $s \to \int_X ({}_s f) h \, dm$ belongs to $C_0(G)$. It follows that the action of S_{w^*} upon $L^p(m)$ is continuous in the weak operator topology. These observations show that a result of Kido and Takahashi [6, Theorem 1] may be applied. This gives the existence of a bounded projection P of $L^p(m)$ onto $I_p(S)$ which is such that $P\mu = \mu P = P$ for all $\mu \in S$.

Now each function of the form $P(f)$, for some f, is the weak limit of functions of the form $\sum_{i=1}^{i=n} c_i(\mu_i * f)$, where $c_i \geq 0$, and $\sum_{i=1}^{i=n} c_i = 1$. Hence, $f - P(f)$ is the weak limit of functions of the form $\sum_{i=1}^{i=n} c_i(f - \mu_i * f)$, and such functions belong to $K_p(S)$. As $K_p(S)$ is weakly closed, we deduce that $f - P(f) \in K_p(S)$. Conversely, we see that $P(f - \mu * f) = P(f) - \mu * P(f) = 0$, for $f \in L^p(m)$ and $\mu \in S$. Hence $K_p(S) = \{f : f \in L^p(m)$ and $P(f) = 0\}$. It follows that $L^p(m)$ is the direct sum of $I_p(S)$ and $K_p(S)$.

Definition 1. Let I_p denote the set $I_p(\{\delta_S : s \in G\})$. That is, $I_p = \{f : f \in L^p(m)$ and ${}_g f = f$, for all $g \in G\}$.

Definition 2. If A is a subset of G, $group(A)$ will denote the subgroup of G generated by A.

Definition 3. The subset S of $M^1(G)$ is said to be *adapted* if, whenever A is a Borel subset of G such that $\mu(A) = 1$ for each $\mu \in S$, then $group(A)$ is dense in G. Equivalently, S is adapted if and only if for each proper closed subgroup H of G, there is $\mu \in S$ so that $\mu(H) < 1$. The measure μ is said to be adapted if $\{\mu\}$ is adapted.

Lemma 1. *The following hold*:

(i) *Let* $1 \le p \le \infty$. *Then* $I_p(S) \supset I_p$.
(ii) *Let* $1 \le p \le \infty$ *and let* S *be right translation invariant. Then* $I_p(\tilde{S}) = I_p$.
(iii) *Let* $1 \le p \le \infty$ *and let* S *be left translation invariant. Then* $I_p(S) = I_p$.

Proof. Let $f \in I_p$, let $g \in L^q(m)$, where $p^{-1} + q^{-1} = 1$, and let $\mu \in S$. Then $\int_X g(\mu * f)\, dm = \int_G (\int_X f(s^{-1}x) g(x)\, dm(x))\, d\mu(s)$, by Fubini's theorem,

$$= \int_X gf\, dm.$$

As this holds for all $g \in L^q(m)$, it follows that $\mu * f = f$. Hence $f \in I_p(S)$, and $I_p(S) \supset I_p$. This proves (i).

Now let S be right invariant and let $f \in I_p(\tilde{S})$. Then for $s \in G$ we have ${}_sf = \delta_s * \tilde{\mu} * f = (\mu * \delta_{s^{-1}})^{\sim} * f = f$. Hence $f \in I_p$ and $I_p \supset I_p(\tilde{S})$. The converse follows from (i), and (ii) is proved. In a like manner, (iii) may also be proved.

The following result extends a result of J. Rosenblatt ([10, Lemma 2]) which was shown to apply to the case when G is countable and S consists of a single measure.

Proposition 1. *Let* $1 < p < \infty$ *and let* S *be an adapted subset of* $M^1(G)$. *Then* $I_p = I_p(S) = I_p(\tilde{S})$.

Proof. We have $I_p(S) \supset I_p$ by Lemma 1(i). Conversely, let $f \in I_p(S)$. Without loss of generality, we may assume that $\|f\|_p = 1$. By the uniform convexity of $L^p(m)$ ([5, p. 232]), there is g in $L^q(m)$, where $p^{-1} + q^{-1} = 1$, so that $\|g\|_q = 1$,

$$\text{(a)} \qquad \operatorname{Re}\left(\int_G h\overline{g}\, dm\right) < 1 \quad \text{for } \|h\|_p = 1 \text{ and } h \ne f, \quad \text{and} \quad \int_G f\overline{g}\, dm = 1.$$

Let Y_f be the weakly closed convex hull of $\{{}_gf : g \in G\}$ in $L^p(m)$. Then Y_f is weakly compact. Let $\alpha_f : G \to Y_f$ be defined by letting $\alpha_f(s) = {}_sf$. Then α_f is continuous in the weak topology on Y_f, and hence is Borel measurable. If $\mu \in S$ let μ_f denote the Borel measure $\mu \circ \alpha_f^{-1}$ on Y_f. If $k \in L^q(m)$, we

now have

$$\int_{Y_f}\left(\int_X h(t)\overline{k}(t)\,dm(t)\right)d\mu_f(h) = \int_G\left(\int_X f(s^{-1}t)\overline{k}(t)\,dm(t)\right)d\mu(s),$$
$$= \int_X (\mu * f)(t)\overline{k}(t)\,dm(t),$$
$$= \int_X f(t)\overline{k}(t)\,dm(t)), \text{ as } f \in I_p(S) \text{ and } \mu \in S.$$

Taking $k = g$ we now have

$$\int_{Y_f} \operatorname{Re}\left(\int_X h(t)\overline{g}(t)\,dm(t)\right)d\mu_f(h) = 1,$$

and using (a), we deduce that $\mu_f = \delta_{\{f\}}$. Hence, for all $\mu \in S$, $\mu(\{s: s \in G$ and ${}_sf = f\}) = 1$. Since S is adapted ${}_sf = f$, for all $s \in G$, so that $f \in I_p$. Thus, $I_p = I_p(S)$. Since S is adapted if and only if $\tilde{S}$ is adapted, it follows also that $I_p = I_p(\tilde{S})$.

Corollary 1. *Let $1 < p < \infty$, let G be a compact group, and let S be an adapted subset of $M^1(G)$. Then if $f \in L^p(G)$ and $\mu * f = f$ for all $\mu \in S$, f is constant.*

Corollary 2. *Let $1 < p < \infty$, let G be a noncompact, σ-compact group, and let S be an adapted subset of $M^1(G)$. Then if $f \in L^p(G)$ and $\mu * f = f$ for all $\mu \in S$, $f = 0$.*

Remark. The only nonadapted measures in $M^1(\mathbf{T})$, where $\mathbf{T}$ is the circle group, are those measures which are supported by a finite subgroup of $\mathbf{T}$.

The following result gives conditions which ensure that S is adapted.

Proposition 2. *Let $M^1(G) \supset S$. The following hold:*

(i) *if, for some $\mu \in S$, either $S \supset \{\mu * \delta_S : s \in G\}$ or $S \supset \{\delta_S * \mu : s \in G\}$, then S is adapted,*

(ii) *if $1 \le p < \infty$, then $\mathscr{P}^1(G) \cap L^p(G)$ is adapted, and*

(iii) *if G is connected and $1 \le p < \infty$, every nonempty subset of $\mathscr{P}^1(G) \cap L^p(G)$ is adapted.*

Proof. (i) Assume that $S \supset \{\mu * \delta_S : s \in G\}$, and let $\sigma(A) = 1$ for all $\sigma \in S$. Then $\mu(As^{-1}) = 1$ for all $s \in G$. Let H denote the closure of $group(A)$. Then $\mu(Hs^{-1}) = 1$ for all $s \in G$. If Hs^{-1} and H are disjoint, $\mu(Hs^{-1} \cup H) = 2$, a contradiction. Hence $Hs^{-1} = H$, for all $s \in G$. Thus $G = H$, which proves that S is adapted. On the other hand, it may be the case that $S \supset \{\delta_S * \mu : s \in G\}$. A similar argument suffices in this case. This proves (i).

As $\mathscr{P}^1(G) \cap L^p(G)$ is left invariant, (ii) is a consequence of (i).

For (iii), let $\mathscr{P}^1(G) \cap L^p(\lambda) \supset S$. Then, $\mu(A) = 1$ for all $\mu \in S$ implies $\lambda(A) > 0$. Hence AA^{-1} contains an open neighbourhood V of the identity

and, as G is connected, $G = \bigcup_{n=1}^{\infty} V^n$, so that $group(V) = G$ (here results in [4, vol. 1, pp. 296 and 62] have been used). It follows that $group(A) = G$, so S is adapted. This proves (iii).

Remark. Proposition 2(i) shows that if S is either left or right translation invariant, then S is adapted.

When G is compact, the following result relates the adaptedness of S to properties of the Fourier transforms of measures in S.

Proposition 3. *Let G be compact, let Σ be the dual object of G, and let S be a subset of $M^1(G)$. For each $\sigma \in \Sigma$, let $U_\sigma \in \sigma$, let H_σ denote the Hilbert space of U_σ, and let I_σ denote the identity operator on H_σ. Consider the following conditions:*

(a) *S is adapted,*
(b) *for each $\sigma \in \Sigma$ such that σ is not the class of the trivial representation, $\hat{\mu}(\sigma) \neq I_\sigma$ for some $\mu \in S$,*
(c) *if H is a proper closed normal subgroup of G, then $\mu(H) < 1$ for some $\mu \in S$,*
(d) *for any function $\alpha: S \to G$, $\{\mu * \delta_{\alpha(\mu)} : \mu \in S\}$ is adapted,*
(e) *for each $\sigma \in \Sigma$ such that σ is not the class of the trivial representation, $\hat{\mu}(\sigma)$ is not unitary for some $\mu \in S$, and*
(f) *if H is a proper closed normal subgroup of G and $\alpha: S \to G$ is any function, then $\mu(H\alpha(\mu)) < 1$ for some $\mu \in S$.*

Then (a) *implies* (b), (b) *implies* (c), (d) *implies* (e), *and* (e) *implies* (f). *When G is abelian and compact,* (a), (b), *and* (c) *are equivalent, and* (d), (e), *and* (f) *are also equivalent.*

Proof. Let (a) hold. Let $\sigma \in \Sigma$ be such that $\hat{\mu}(\sigma) = I_\sigma$ for all $\mu \in S$. Let χ_σ denote the trace of σ. Then a routine calculation shows that $\mu * \chi_\sigma = \chi_\sigma$ for all $\mu \in S$. By Proposition 1, χ_σ is constant on G. It now follows (see [4, vol. II, p. 14]), that σ is the class of the trivial representation. This proves that (a) implies (b).

If (c) fails, let $\mu(H) = 1$ for all $\mu \in S$, for some proper closed normal subgroup H of G. By [4, vol. II, p. 64], there is $\sigma \in \Sigma$ such that σ is not the class of the trivial representation and $U_\sigma(H) = \{I_\sigma\}$. Then it is easy to check that $\hat{\mu}(\sigma) = I_\sigma$, for all $\mu \in S$. Hence (b) implies (c).

Let (d) hold. Let $\sigma \in \Sigma$ be such that $\hat{\mu}(\sigma)$ is unitary for all $\mu \in S$. Then for each $\mu \in S$, $\hat{\mu}(\sigma)$ is an extreme point of the closed convex hull of $U_\sigma(G)$ in $B(H_\sigma)$. Since $U_\sigma(G)$ is compact, for each $\mu \in S$ there is $x_\mu^{-1} \in G$ such that $U_\sigma(x_\mu^{-1}) = \hat{\mu}(\sigma)$. Then for all $\xi, \eta \in H_\sigma$,

$$\int_G \langle U_\sigma(s)\xi, \eta\rangle d\mu(s) = \langle U_\sigma(x_\mu^{-1})\xi, \eta\rangle .$$

Thus, for all $\xi, \eta \in H_\sigma$,

$$\int_G \langle U_\sigma(sx_\mu)\xi, \eta\rangle \, d\mu(s) = \langle \xi, \eta\rangle .$$

Hence, $(\mu * \delta_{x_\mu})\hat{}(\sigma) = I_\sigma$ for all $\mu \in S$. By (a) implies (b) above, we deduce that (d) implies (e).

If (f) fails, there is a proper closed normal subgroup of G and a function $\alpha: S \to G$ so that $\mu(H\alpha(\mu)^{-1}) = 1$ for all $\mu \in S$. Then $(\mu * \delta_{x_\mu})(H) = 1$ for all $\mu \in S$. By (b) implies (c) above, for some $\sigma \in \Sigma$ such that σ is not the class of the trivial representation, $(\mu * \delta_{x_\mu})\hat{}(\sigma) = I_\sigma$ for all $\mu \in S$. Hence $\hat{\mu}(\sigma)U_\sigma(x_\mu) = I_\sigma$, so that $\hat{\mu}(\sigma) = U_\sigma(x_\mu^{-1})$ and $\hat{\mu}(\sigma)$ is unitary for all $\mu \in S$. Hence (e) implies (f).

In the abelian case, it is immediate that (c) implies (a) and (f) implies (d).

Corollary. *Let G be abelian and compact and let $\mu \in M^1(G)$. Then μ is adapted if and only if for all $\sigma \in \Sigma$ with $\sigma \neq 1$, $\hat{\mu}(\sigma) \neq 1$. Also, $\mu * \delta_x$ is adapted for all $x \in G$ if and only if for all $\sigma \in \Sigma$ with $\sigma \neq 1$, $|\hat{\mu}(\sigma)| < 1$.*

Theorem 2. *Let $1 < p < \infty$, let the action of G upon $(X, \mathscr{B}, m)$ be disjoint at infinity on m-finite sets, and let S be an adapted subsemigroup of $M^1(G)$ such that $C(S_{w^*})$ has an invariant mean. Then $L^p(m)$ is the direct sum of $K_p(S)$ and I_p, and the associated projection P from $L^p(m)$ onto I_p has the property that $P\mu = \mu P = P$ for all $\mu \in S$.*

Proof. This is immediate from Theorem 1 and Proposition 1.

3. The ergodic case

When the action of G upon $(X, \mathscr{B}, m)$ is ergodic a decomposition result for $L^p(m)$ (Theorem 3, below), which does not assume that S is a subsemigroup of $M^1(G)$, can be obtained.

Lemma 2. *Let $1 \leq p < \infty$ and $p^{-1} + q^{-1} = 1$. Then the following hold:*

(i) *if S is right translation invariant, the annihilator of $K_p(S)$ in $L^q(m)$ is I_q, so that $K_p(S)$ is the same for all right translation invariant subsets S of $M^1(G)$, and*

(ii) *if S is adapted and $1 < p < \infty$, the annihilator of $K_p(S)$ in $L^q(m)$ is I_q, so that $K_p(S)$ is the same for all adapted subsets S of $M^1(G)$ when $1 < p < \infty$.*

Proof. Let S be right translation invariant. If h is in the annihilator of $K_p(S)$ in $L^q(m)$, a simple calculation shows that $\tilde{\mu} * h = h$, so that $h \in I_q(\tilde{S})$. By Lemma 1(ii), $h \in I_q$. On the other hand, a calculation using Fubini's theorem shows that functions in I_q belong to the annihilator of $K_p(S)$ in $L^q(m)$. This proves that the annihilator of $K_p(S)$ in $L^q(m)$ is I_q. This shows that for all right translation invariant semigroups S, $K_p(S)$ has the same annihilator in $L^q(m)$, so that all such spaces $K_p(S)$ are identical. This proves (i). The proof of (ii) is analogous; simply use Proposition 1 in place of Lemma 1.

Definitions. When $p = 1$ or $m(X) < \infty$, let $L_0^p(m)$ denote those functions $f \in L^p(m)$ such that $\int_X f\,dm = 0$. The action of G upon $(X, \mathscr{B}, m)$ is said to be *ergodic* if $m(gA\Delta A) = 0$ for all $g \in G$ implies that $m(A) = 0$ or $m(X - A) = 0$.

A calculation using Fubini's theorem shows that if $p = 1$, or if $m(X) < \infty$ and $1 \le p < \infty$, then $L_0^p(m) \supset K_p(S)$.

Theorem 3. *Let the action of G on $(X, \mathscr{B}, m)$ be ergodic and let S be a nonempty subset of $M^1(G)$. Then the following hold*:

(i) *if S is right translation invariant, $K_1(S) = L_0^1(m)$,*
(ii) *if $m(X) < \infty$, $1 < p < \infty$ and S is adapted, $K_p(S) = L_0^p(m)$,*
(iii) *if $m(X) = \infty$, $1 < p < \infty$ and S is adapted, $K_p(S) = L^p(m)$, and*
(iv) *if $1 < p < \infty$ and S is adapted, $L^p(m)$ is the direct sum of $K_p(S)$ and the subspace of constant functions.*

Proof. Because the action of G is ergodic, I_q consists of constants only. Let $p = 1$ and let S be right translation invariant. Then by (i) of Lemma 2, $K_1(S)$ and $L_0^1(m)$ have the same annihilator in $L^\infty(m)$, namely I_∞. Hence $L_0^1(m) = K_1(S)$. This proves (i). If $m(X) < \infty$, $1 < p < \infty$, and S is adapted, the annihilator of $K_p(S)$ is the subspace of constants, by Lemma 2(ii). Hence $K_p(S)$ and $L_0^p(m)$ have the same annihilators and so are equal. This proves (ii). If $m(X) = \infty$ and $1 < p < \infty$, $I_q = \{0\}$. If S is also adapted, the argument just used, together with Lemma 2(ii), shows that $K_p(S) = L^p(m)$, so that (iii) holds. Finally, (iv) is immediate from (ii) and (iii).

Remark 1. Let the action of G be ergodic. Then if $m(X) < \infty$ and S is right translation invariant, $L^1(m)$ is the direct sum of $K_1(S)$ and I_1. If $m(X) = \infty$ and S is right translation invariant, (i) shows that $L^1(m)$ is not the direct sum of $K_1(S)$ and I_1. This shows that Theorem 1 generally fails when $p = 1$.

Remark 2. If the action of G is not ergodic, there may be nonzero functions in $L_0^p(m) \cap I_p$, so that $L^p(m)$ is not generally the direct sum of $K_p(S)$ and I_p.

Corollary. *Let G be a compact and connected, let $1 < p < \infty$, let $m(X) < \infty$, let the action of G be ergodic, and let S be a nonempty subset of $\mathscr{P}^1(G)$. Then $L^p(m)$ is the direct sum of the subspace of constant functions and $K_p(S)$.*

Proof. This is immediate from Proposition 2(iii) and Theorem 3(iv).

4. Strongly aperiodic measures and containment

This section contains some analogues for convolution of certain results of J. Rosenblatt which apply for group translations on countable, discrete groups ([9] and [10, Propositions 12 and 14]).

Definition 1. Let H be a discrete group and let π be a representation of H as a group of invertible operators on a Banach space B. Then π is said to

weakly contain the trivial representation if there is a net (ν_α) in B such that $\lim_\alpha \|\pi(x)(\nu_\alpha) - \nu_\alpha\| = 0$, for all $x \in H$.

Definition 2. A discrete group H is said to have *Kazhdan's property* if, for any unitary representation π of H which weakly contains the trivial representation, there is a nonzero vector ν in the Hilbert space of the representation so that $\pi(x)\nu = \nu$, for all $x \in H$.

Definition 3. If $\mu \in M^1(G)$, μ is called *strongly aperiodic* if $\tilde{\mu} * \mu(A) = 1$ implies that $group(A) = G$.

Remarks. When G is discrete, the concept of a strongly aperiodic measure was introduced by J. Rosenblatt ([10, p. 266]). He noted that in this case the strongly aperiodic measures $\mu \in M^1(G)$ are precisely those such that the set $\{x^{-1}y : x, y \in (\text{support of } \mu)\}$ generates G. In the general case, if $\mu \in M^1(G)$ is strongly aperiodic and H is a closed subgroup of G such that $\mu(H) = 1$, we have $\tilde{\mu} * \mu(H) = 1$ so that $H = G$. Hence, every strongly aperiodic measure is adapted.

Theorem 4. *Let $m(X) < \infty$, let $1 < p < \infty$, let μ be a strongly aperiodic measure in $M^1(G)$, and assume that the representation of G on $L_0^2(m)$ given by $s \to {}_sf$ does not weakly contain the trivial representation of G on $L_0^2(m)$. Then the following hold*:

(i) *$I - \mu$ is a bounded invertible operator on $L_0^p(m)$,*
(ii) *$L_0^p(m) = \{\mu * f - f : f \in L_0^p(m)\}$,*
(iii) *each function in $L^p(m)$ can be expressed uniquely in the form $\alpha + \mu * f - f$, where α is a constant function and $f \in L_0^p(m)$, and*
(iv) *any linear functional ϕ on $L^p(m)$ such that $\phi(\mu * f) = \phi(f)$ for all $f \in L^p(m)$ is automatically continuous and is a multiple of m.*

Proof. Consider the case $p = 2$. If $\|\mu\|_{L_0^2(m)} = 1$, there is a sequence (g_n) in $L_0^2(m)$ so that $\|g_n\|_2 = 1$ and $\lim_{n\to\infty} \|\mu * g_n\|_2 = 1$. Now,

$$\begin{aligned}\|\mu * g_n\|_2^2 &= \int_X \left(\int_G g_n(s^{-1}t)\, d\tilde{\mu} * \mu(s)\right) \overline{g}_n(t)\, dm(t),\\ &= \int_G \left(\int_X g_n(s^{-1}t)\overline{g}_n(t)\, dm(t)\right) d\tilde{\mu} * \mu(s).\end{aligned}$$

As $|\int_X g_n(s^{-1}t)\overline{g}_n(t)\, dm(t)| \le 1$, we see that the sequence of functions $(s \to \int_X g_n(s^{-1}t)\overline{g}_n(t)\, dm(t))$ converges in $\tilde{\mu} * \mu$-measure to 1 on G. Consequently, there is a subsequence (h_n) of (g_n) so that the sequence $(s \to \int_X h_n(s^{-1}t)\overline{h}_n(t)\, dm(t))$ converges $\tilde{\mu} * \mu$-almost everywhere to 1. Hence there is $A_\mu \in \mathscr{B}$ such that $\tilde{\mu} * \mu(A_\mu) = 1$ and, for all $s \in A_\mu$, $\lim_{n\to\infty} \int_X h_n(s^{-1}t)\overline{h}_n(t)\, dm(t) = 1$. It now follows, as in [1, p. 343], that for all $s \in A_\mu$, $\lim_{n\to\infty} \|{}_sh_n - h_n\|_2 = 0$. Now the set $\{s : s \in G \text{ and } \lim_{n\to\infty} \|{}_sh_n - h_n\|_2 = 0\}$ is easily seen to be a subgroup of G which is in fact equal to G, as it contains A_μ and μ is

strongly aperiodic. This contradicts the containment assumption, so it follows that $\|\mu\|_{L_0^2(m)} < 1$. Hence, $I - \mu$ is invertible on $L_0^2(m)$. An application of the complex interpolation method in Banach spaces now enables us to deduce that $I - \mu$ is a bounded invertible operator on $L_0^p(m)$ for $1 < p < \infty$ (compare with [10, Proposition 12]). This proves (i). Now (i) implies (ii), and (ii) easily implies (iii). Finally, if ϕ is a linear function on $L^p(m)$ such that $\phi(\mu * f) = \phi(f)$ for all $f \in L^p(m)$, it follows from (iii) that $\phi(f) = \phi(1) \int_X f\, dm$, for all $f \in L^p(m)$. This proves (iv).

Corollary. *Let G be a compact connected group such that the regular representation of G upon $L_0^2(G)$ does not weakly contain the trivial representation, and let $1 < p < \infty$. Let $h \in \mathscr{P}^1(G)$, and let ϕ be a linear functional on $L^p(G)$ such that $\phi(h * f) = \phi(f)$ for all $f \in L^p(G)$. Then ϕ is a multiple of Haar measure.*

Proof. An adaptation of the argument in (iii) of Proposition 2 shows that h is a strongly aperiodic measure in $M^1(G)$. Now the result follows from the above remarks and Theorem 4 applied to the action of G upon itself, which is given by left translations.

Remarks. The idea in the proof of Theorem 4 of showing that $\|\mu\|_{L_0^2(m)} < 1$ is to be found in [1], where it was applied in the case of an average of group translations. Theorem 4 should also be compared with [9] and [10, Propositions 12 and 14]. If G is compact, the assumption that the regular representation of G on $L_0^2(G)$ does not weakly contain the trivial representation is equivalent to the condition that the operator $f \to (\int_G f\, d\lambda)1$ on $L^2(G)$ belongs to the C^*-algebra generated by left translation operators on $L^2(G)$. This result is proved in [1], where compact groups with these equivalent properties are referred to as groups with property (A). If G has a dense subgroup with Kazhdan's property, it is proved in [1] that G has property (A). As noted in [1], it is a result of Margulis and also of Sullivan ([8] and [11]) that $SO(n)$ for $n \geq 5$ has a dense subgroup with Kazhdan's property and consequently has property (A). It is also noted in [9] that it is a consequence of the work of Drinfel′d in resolving the Banach-Ruziewicz problem for S^2 and S^3 ([2]), that $SO(3)$ and $SO(4)$ have property (A).

Added in proof. Since writing this paper, the author has been able to extend the result in the Corollary to Theorem 4 by showing that the conclusion of the corollary remains valid for any compact connected group.

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