EACH HYPERINVARIANT SUBSPACE FOR MULTIPLICATION OPERATOR IS SPECTRAL

HUANG SENZHONG

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Abstract. We consider multiplication operators on general separable complex $L^p$-spaces, for $1 \leq p < +\infty$, and obtain the result announced in the title. Moreover, a result of Douglas and Pearcy on normal operators is given an alternate proof.

In this paper, our notation is standard. We refer the reader to [1] and [4]. We consider only complex spaces $L^p(X,\mathcal{A},\mu,\mathbb{C})$ for $1 \leq p < +\infty$. We also simply denote $L^p(X,\mathcal{A},\mu,\mathbb{C})$ by $L^p(X,\mathcal{A},\mu)$ or $L^p(X,\mu)$ in the circumstances where there is no confusion.

Let $\phi \in L^\infty(X,\mu)$. The multiplication operator corresponding to $\phi$ is the bounded operator $M_\phi$ on $L^p(X,\mu)$ defined by $(M_\phi f)(x) = \phi(x)f(x)$ for all $f \in L^p(X,\mu)$. For a Borel set $S$ of $\mathbb{C}$, define $E^c(S) = M_{1_{S}}$, where by $1_{S}$ we always denote the characteristic function corresponding to a set $S$. Following Dunford [3], $E^c(\cdot)$ is a spectral measure which makes $M_\phi$ a spectral operators. For a spectral operator, there is the following remarkable theorem of Fuglede and Dunford (its proof can be found in [5]).

F-D Theorem. If $A$ is a spectral operator with spectral measure $E(\cdot)$ and if $AB = BA$, then $BE(S) = E(S)B$ for all Borel sets $S$.

As a result of this theorem, we have

Corollary. Let $E(\cdot)$ be any spectral measure for $M_\phi$. If $\mathcal{M}$ is the range of $E(S)$ for some Borel set $S$, then $\mathcal{M}$ is a hyperinvariant subspace for $M_\phi$. Thus each nonscalar multiplication operator has a nontrivial hyperinvariant subspace.

Proof. Trivial.

From this corollary we see that the range of $E^c(S)$ for each Borel set $S$ is a hyperinvariant subspace for $M_\phi$. We will next see that each hyperinvariant subspace for $M_\phi$ can be written in this form whenever $M_\phi$ is defined on a separable space. Because of this, by $E^c(\cdot)$ we always mean this special spectral...
measure defined as at the beginning. Moreover, by a subspace we always mean a linear subspace and we call a closed subspace $\mathcal{M}$ of $L^p(X, \mu)$ a spectral subspace if $\mathcal{M} = E^v(S) L^p(X, \mu)$ for some Borel set of $\mathbb{C}$. We also denote the algebra of all multiplication operators on $L^p(X, \mu)$ by $\mathcal{L}^\infty$, i.e.

$$\mathcal{L}^\infty = \{ M_\phi : \phi \in L^\infty(X, \mu) \}.$$  

Let us start with a converse of the F-D Theorem for multiplication operators.

**Lemma 1.** If $B$ commutes with all $E^v(S)$, then $B$ commutes with $M_\phi$.

**Proof.** This is trivial by the definition of the spectral integral.

Now we pass to the structure of hyperinvariant subspaces for multiplication operators.

**Lemma 2.** Let $\mathcal{M}$ be a closed separable subspace of $L^p(X, \mathcal{A}, \mu)$ for $1 \leq p < +\infty$. If $\mathcal{M}$ is invariant for $\mathcal{L}^\infty$, i.e. $M_\phi \mathcal{M} \subseteq \mathcal{M}$ for all $M_\phi \in \mathcal{L}^\infty$, then there exists some $A \in \mathcal{A}$ such that $\mathcal{M} = 1_A \cdot L^p(X, \mathcal{A}, \mu)$.

**Proof.** Since $\mathcal{M}$ is separable, a theorem of Ando (see [4], page 152, Lemma 1) implies that there exists an $f \in \mathcal{M}$ with maximal support $A$ of all functions in $\mathcal{M}$. It follows that $A \in \mathcal{A}$ and $\mathcal{M} \subseteq 1_A \cdot L^p(X, \mu)$. We must show that $\mathcal{M} = 1_A \cdot L^p(X, \mu)$. To this end, assume $A \supseteq B \in \mathcal{A}$ with $\mu(B) < +\infty$. Let

$$A_n = \{ x \in X : \frac{1}{n} \leq |f(x)| \leq n \} \quad \text{for all } n \geq 1.$$  

Put $\phi_n = f^{-1} \cdot 1_A \cdot 1_B$. Then $\phi_n \in L^\infty(X, \mu)$; and hence $1_{B \cap A_n} = \phi_n \cdot f \in \mathcal{M}$. We have that $\mu(B \setminus B \cap A_n) \to 0$ as $n \to \infty$. Since $\mathcal{M}$ is closed, this implies that $1_B \in \mathcal{M}$. And thus $1_B \cdot L^\infty(X, \mu) \subseteq \mathcal{M}$ for all $A \supseteq B \in \mathcal{A}$ with $\mu(B) < +\infty$. It is now routine to check that $\mathcal{M} = 1_A \cdot L^p(X, \mu)$ since $\mathcal{M}$ is closed and since the span of subspaces $1_B \cdot L^\infty(X, \mu)$ for $A \supseteq B \in \mathcal{A}$ with $\mu(B) < +\infty$ is dense in $1_A \cdot L^p(X, \mu)$. This completes the proof.

To continue, we need one more notion. Let $(X, \mathcal{A}, \mu)$ be a finite measure space and let $\phi \in L^\infty(X, \mathcal{A}, \mu)$. Let $\mathcal{A}_0 = \{ \phi^{-1}(S) : S \text{ a Borel set of } \mathbb{C} \}$. It is easily seen that $\mathcal{A}_0$ is also a sub-sigma-algebra of $\mathcal{A}$. We regard $(X, \mathcal{A}_0, \mu)$ as a finite measure space. Let $A \in \mathcal{A}$ be given. By the well-known Radon-Nikodym theorem we may define the conditional expectation operator $E_A$, for the measure $\mu$ relative to $\mathcal{A}_0$. $E_A$ is uniquely determined by the equation

$$\int_B h \cdot 1_A d\mu = \int_B (E_A h) d\mu \quad (B \in \mathcal{A}_0)$$

for $h \in L^1(X, \mathcal{A}, \mu)$ and by the condition that $E_A h$ is $\mathcal{A}_0$-measurable. This class of operators $\{ E_A : A \in \mathcal{A} \}$ has the following interesting properties.

**Lemma 3.** (1) Each $E_A$ is a positive linear operator on $L^p(X, \mu)$, $1 \leq p \leq \infty$ with $\|E_A\| \leq 1$. In particular, $0 \leq E_A(h) \leq \|h\|_{\infty}$ for all $h \in L^\infty(X, \mu)$ with $0 \leq h$.

(2) Each $E_A$ commutes with $M_\phi$, i.e. $E_A M_\phi = M_\phi E_A$. 
Proof. (1) We may write $E_A = E_X \circ M_{1_A}$, where $E_X$ is the usual conditional expectation operator determined by the sub-sigma-algebra $\mathcal{A}_0$ (cf. [1]). Since both $E_X$ and $M_{1_A}$ are positive contractions on each $L^p(X, \mu)$, $1 \leq p \leq \infty$, as is well known, (1) follows.

(2) By Lemma 1, it suffices to show that $E_A$ commutes with all $E^c(S)$ for Borel sets $S$ of $\mathbb{C}$. Fix such an $S$. Then $E^c(S) = M_{1_{\phi^{-1}(S)}}$ and $\phi^{-1}(S) \in \mathcal{A}_0$. By the definition of $E_A$, for all $B \in \mathcal{A}_0$,

$$\int_B E^c(S)(E_A h) \, d\mu = \int_{B \cap \phi^{-1}(S)} (E_A h) \, d\mu = \int_{B \cap \phi^{-1}(S)} h \cdot 1_A \, d\mu$$

$$= \int_B (E^c(S)h) \cdot 1_A \, d\mu = \int_B E_A(E^c(S)h) \, d\mu,$$

for all $h \in L^1(X, \mathcal{A}, \mu)$. Since both $E^c(S)(E_A h)$ and $E_A(E^c(S)h)$ are $\mathcal{A}_0$-measurable, we conclude from the above that $E_A E^c(S) = E^c(S) E_A$. This finishes the proof.

**Lemma 4.** Let $(X, \mathcal{A}, \mu)$ be a finite measure space and $1 \leq p \leq +\infty$. Let $\phi \in L^\infty(X, \mu)$ and $\mathcal{M} = 1_A \cdot L^p(X, \mu)$ for some $A \in \mathcal{A}$. Then $\mathcal{M}$ is hyperinvariant for $M_\phi$ iff $A = \phi^{-1}(S)$ for some Borel set $S$, i.e. $\mathcal{M}$ is a spectral subspace.

**Proof.** Assume that $\mathcal{M} = 1_A \cdot L^p(X, \mu)$ is hyperinvariant for $M_\phi$. By Lemma 3 again, $E_A$ is a bounded operator which commutes with $M_\phi$. So $E_A(\mathcal{M}) \subseteq \mathcal{M}$. In particular, there exists $f_0 \in L^p(X, \mu)$ such that $E_A(1_A) = 1_A \cdot f_0$. Let $f_1 = E_A(1_A)$. By Lemma 3, again $0 \leq f_1 \leq 1$ a.e. Note that

$$\int_X (1 - f_1) \cdot 1_A \, d\mu = \int_X 1_A \, d\mu - \int_X (E_A 1_A) \, d\mu = 0,$$

and so $(1 - f_1) \cdot 1_A = 0$ a.e. Hence $1_A = 1_A \cdot f_1 = E_A(1_A)$ is $\mathcal{A}_0$-measurable. This implies that $A = \phi^{-1}(S)$ for some Borel set $S$. The converse is the corollary to the F-D Theorem.

**Lemma 5.** Let $(X, \mu)$ be a finite measure space and $1 \leq p < +\infty$. Let $\phi \in L^\infty(X, \mu)$ and let $M_\phi$ be the multiplication operator on $L^p(X, \mu)$. Then a closed separable subspace $\mathcal{M}$ of $L^p(X, \mu)$ is hyperinvariant for $M_\phi$ iff $\mathcal{M}$ is spectral, i.e. $\mathcal{M} = E^c(S)L^p(X, \mu)$ for some Borel set $S$.

**Proof.** Use Lemma 2 and Lemma 4 while observing that each element of $L^\infty$ commutes with $M_\phi$.

For a Banach space $Z$ and a bounded linear operator $T$ on $Z$, let $\text{com}t(T)$ be the commutant of $T$, i.e. the set of all bounded linear operators on $Z$ which commute with $T$. If $V: Z \rightarrow Y$ is an onto-isomorphism, then $\text{com}t(VTV^{-1}) = V \cdot \text{com}t(T) \cdot V^{-1}$.

Next, we state our main theorem.
Theorem 1. Let $1 < p < +\infty$ and $(X, \mu)$ be a measure space such that the space $L^p(X, \mu)$ is separable. Let $\phi \in L^\infty(X, \mu)$ and $M_\phi$ be the multiplication operator on $L^p(X, \mu)$. Then a closed subspace $M$ of $L^p(X, \mu)$ is hyperinvariant for $M_\phi$ iff $M$ is spectral, i.e. $M = E^c(S)L^p(X, \mu)$ for some Borel set $S$.

Proof. Since $L^p(X, \mu)$ is separable, by the same theorem of Ando used in the proof of Lemma 2 above, one can easily build a finite measure $\nu$ on $X$ such that $\nu$ is equivalent to $\mu$. Define $V: L^p(X, \mu) \to L^p(X, \nu)$ by

$$Vf = f \cdot \left( \frac{d\mu}{d\nu} \right)^{1/p} \text{ for all } f \in L^p(X, \mu).$$

Then $V$ is an onto-isometric isomorphism such that $VM_\phi V^{-1}$ is the multiplication operator $M_\phi$ on $L^p(X, \nu)$. From the previous remark, we immediately obtain that $M$ is hyperinvariant for $M_\phi$ on $L^p(X, \mu)$ iff $VM$ is also hyperinvariant for $M_\phi$ on $L^p(X, \nu)$. Use Lemma 5 while observing the definition of $V$, the latter assertion is equivalent to

$$M = E^c(S)L^p(X, \mu) \text{ for some Borel set } S.$$

This completes the proof.

We give two applications of Theorem 1.

Corollary 1. Suppose $1 < p \neq 2 < +\infty$. Let $L^p(X, \mu)$ be separable and let $\phi$ be $L^\infty(X, \mu)$. Let $E^c(\cdot)$ be the special spectral measure corresponding to $M_\phi$ on the space $L^p(X, \mu)$. Then $E^c(\cdot)$ is maximal in the sense that if $E(\cdot)$ is another spectral measure for $M_\phi$ in the sense of Dunford [3] with contractive projections, then the range of $E(\cdot)$ is contained in that of $E^c(\cdot)$.

Proof. Fix a Borel set $S$. Let $J = E(S)L^p(X, \mu)$. By the corollary to the F-D Theorem, $M$ is hyperinvariant for $M_\phi$. Now Theorem 1 implies that $M = E^c(S_1)L^p(X, \mu)$ for some Borel set $S_1$. It follows that $E(S)E^c(C \setminus S_1) = 0$ since $E(S)$ commutes with $E^c(C \setminus S_1)$ by the F-D Theorem. We consider the space $L^p(X, \mu)$ as a complex Banach lattice, and we refer the reader to [4] for a general theory of Banach lattices. We have, then, that $M = M_\phi$, and $E^c(S_1)$ is the unique band projection on $M$. Since $E(S)$ is a contractive projection, a classical theorem (see [4], page 160, Theorem 2) implies that $E(S)E^c(S_1) = E^c(S_1)$. Combining these we finally obtain $E(S) = E^c(S_1)$. This finishes the proof.

Corollary 2 (Douglas-Pearcy [2]). If $A$ is a normal operator on the separable Hilbert space $H$ with spectral measure $\{E_\cdot\}$, then $M$ is hyperinvariant for $A$ iff $M = E(S)H$ for some Borel set $S$ of $\mathbb{C}$.

Proof. By the spectral theorem, we may assume that $A = M_\phi$ on $L^2(X, \mu)$ for some measurable space $(X, \mu)$. Also, since $H$ is separable by hypothesis, so is $L^2(X, \mu)$. The result now follows from Theorem 1.
Note. This result was proved by Douglas and Pearcy [2] using facts from the theory of von Neumann algebras. Our proof seems to be more elementary and comes almost directly from the spectral theorem.

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REFERENCES


Department of Mathematics, Nankai University, Tianjin City, People’s Republic of China