A CONSTRUCTION FOR PSEUDOCOMPLEMENTED SEMILATTICES AND TWO APPLICATIONS

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Abstract. A method is given by which pseudocomplemented semilattices can be constructed from graphs. Two consequences of the method are obtained, namely: there exist continuum-many quasivarieties of pseudocomplemented semilattices; for any non-zero cardinal \( \kappa \), there exist \( \kappa \) pairwise non-isomorphic pseudocomplemented semilattices with isomorphic endomorphism monoids.

1. Introduction

A pseudocomplemented semilattice is an algebra \((S; \wedge, *, 0, 1)\) of type \((2, 1, 0, 0)\) consisting of a semilattice \((S; \wedge)\) with a least element 0, a greatest element 1, and a pseudocomplementation operation \(*\) such that, for \(x, y \in S\), \(x \wedge y = 0\) if and only if \(y \leq x^*\).

We shall exhibit a method by which pseudocomplemented semilattices can be constructed from graphs. By applying this construction, the following results will be established.

Theorem 1. There exist \(2^{\aleph_0}\) quasivarieties of pseudocomplemented semilattices.

Theorem 2. Given a non-zero cardinal \( \kappa \), there exists a family of pseudocomplemented semilattices \((S_i : i \in I)\) satisfying

\(\begin{align*}
(i) & \quad S_i \neq S_j \text{ for distinct } i, j \in I; \\
(ii) & \quad \text{End}(S_i) \cong \text{End}(S_j) \text{ for all } i, j \in I; \\
(iii) & \quad \text{if } \kappa \text{ is infinite, then } |I| = 2^\kappa \text{ and } |S_i| = \kappa \text{ for all } i \in I; \\
(iv) & \quad \text{if } \kappa \text{ is finite, then } |I| = \kappa \text{ and each } S_i \text{ is finite.}
\end{align*}\)

In Theorem 2, and throughout this paper, the notation \(\text{End}(S)\) denotes the monoid (semigroup with identity) of all endomorphisms of \(S\) with composition as multiplication.

It should be noted that Theorem 1 stands in striking contrast to the fact that there are only two non-trivial varieties of pseudocomplemented semilattices.
proved by Jones [7] (cf. Sankappanavar [10]). That not every quasivariety of pseudocomplemented semilattices is necessarily a variety had already been observed by Sankappanavar (unpublished) and, independently, Schmid [12].

Both theorems have precise analogues for pseudocomplemented distributive lattices, i.e., algebras \((L; \lor, \land, *, 0, 1)\) such that \((L; \lor, \land)\) is a distributive lattice and \((L; \land, *, 0, 1)\) is a pseudocomplemented semilattice. The analogue of Theorem 1 is given in [1] and, independently, Wróński [13]; the analogue of Theorem 2 was proved in [2]. For Boolean algebras, both theorems fail spectacularly: there is only one non-trivial quasivariety, and Boolean algebras having isomorphic endomorphism monoids are isomorphic (Magill [8], Maxson [9], and Schein [11]).

The reader in need of background material concerning pseudocomplemented semilattices and related topics is directed to Grätzer [3]. We shall use the following notations. If \(S\) is a pseudocomplemented semilattice, \(S^*\) denotes the skeleton of \(S\), that is, \(S^* = \{x^* : x \in S\}\). The Glivenko congruence on \(S\), denoted \(\Gamma_S\), is defined by \(\Gamma_S = \{(x, y) \in S \times S : x^* = y^*\}\). The Glivenko endomorphism of \(S\), denoted \(\gamma_S\), is defined by \(\gamma_S(x) = x^{**}\) for all \(x \in S\). Because \(x^* = x^{***}\) for all \(x \in S\), it follows that \(\gamma_S\) is the identity on \(S^*\) and that \(\Gamma_S\) is the congruence induced by \(\gamma_S\), i.e., \(\Gamma_S = \{(x, y) \in S \times S : \gamma_S(x) = \gamma_S(y)\}\).

Since \(S^*\) is a Boolean lattice where \(x^* \lor y^* = (x^{**} \land y^{**})^*\) for any \(x, y \in S\), it follows that if \(\varphi : S \to T\) is a homomorphism to a pseudocomplemented semilattice \(T\), then \(\varphi \upharpoonright S^* : S^* \to T^*\) is a Boolean homomorphism.

2. The construction

The immediate goal of this section is to define a pseudocomplemented semilattice \(S_G\) for every graph \(G\).

For a graph \(G = (V, E)\) (i.e., a set \(V\) together with a set \(E\) of two-element subsets of \(V\)), let \(B_G\) denote the Boolean lattice of finite/co-finite subsets of \(V\) ordered by inclusion. Further, let

\[
S_G = (B_G \times 2) \setminus \{(\emptyset, 1), (V, 0)\} \cup \{(a, 0) : |a| = 1 \text{ or } a \in E\}
\]

where \(2\) denotes the two-element chain \(\{0, 1\}\). Let \(\leq\) denote the usual ordering on \(B_G \times 2\) and define a relation \(\leq\) on \(S_G\) by \((a, i) \leq (b, j)\) iff either \((a, i) \leq (b, j)\) in \(B_G \times 2\),

\[
\text{or } i = 1, j = 0, a \leq b, \text{ and } |a| = 1, \\
\text{or } i = 1, j = 0, a \leq b, \text{ and } a \in E.
\]

It must be shown that \((S_G, \leq)\) is indeed a pseudocomplemented semilattice.

**Lemma 1.** \(\leq\) is an order relation.

**Proof.** Since \(\leq\) is reflexive, so too is \(\leq\).

To see that \(\leq\) is anti-symmetric, first note that the only new pairs added to \(\leq\) are of the form \((a, 1) \leq (b, 0)\), whence we may suppose that

\[
(a, 1) \leq (b, 0) \quad \text{and} \quad (b, 0) \leq (a, 1).
\]
It follows that $a = b$. Further, by the first inequality, either $|a| = 1$ or $a \in E$ which is absurd, since in neither case is $(a, 0)$ an element of $S_G$.

Finally, to see that $\leq$ is transitive, we need only consider cases in which one inequality is of the form $(a, 1) \leq (b, 0)$. Thus, either

\[(c, i) \leq (a, 1) \quad \text{and} \quad (a, 1) \leq (b, 0) \quad \text{or} \quad (a, 1) \leq (b, 0) \quad \text{and} \quad (b, 0) \leq (c, i)\]

for $i = 0, 1$. In either case, $a \leq b$ and $|a| = 1$ or $a \in E$. If, in the former case, $c = \emptyset$, then $i = 0$ and $(c, 0) \leq (b, 0)$. Otherwise, since $c \leq a \leq b$, $|c| = 1$ or $c = a \in E$. No matter which, $i = 1$ and $(c, 1) \leq (b, 0)$. In the latter case $a \leq c$ and, hence, $(a, 1) \leq (c, i)$. □

**Lemma 2.** $S_G$ is a semilattice where, for $(a, i), (b, j) \in S_G$,

\[ (a, i) \wedge (b, j) = \begin{cases} (\emptyset, 0) & \text{if } a \wedge b = \emptyset, \\ (a \wedge b, 1) & \text{if } |a \wedge b| = 1 \text{ or } a \wedge b \in E, \\ (a \wedge b, i \wedge j) & \text{otherwise}. \end{cases} \]

**Proof.** Suppose $(a, i), (b, j) \in S_G$ and $(c, k)$ is a common lower bound. In particular, $c \leq a \wedge b$. We consider the various possibilities.

If $a \wedge b = \emptyset$, then $(\emptyset, 0)$ is a common lower bound. Since $c = \emptyset$ in this case, $(\emptyset, 0)$ is the only lower bound.

If $|a \wedge b| = 1$ or $a \wedge b \in E$, then $(a \wedge b, 1)$ is a common lower bound. Moreover, it is the greatest since $c \leq a \wedge b$ implies that $(c, k) \leq (a \wedge b, 1)$ for any $k$.

Finally, suppose $|a \wedge b| \geq 2$ and $a \wedge b \notin E$. If $(a \wedge b, i \wedge j) = (V, 0)$, then either $(a, i)$ or $(b, j) = (V, 0)$ which is absurd. Thus, $(a \wedge b, i \wedge j) \in S_G$ by hypothesis. Clearly, $(a \wedge b, i \wedge j)$ is a common lower bound; it is to be seen that it is the greatest. If $k = 0$, then $(c, 0) \leq (a \wedge b, i \wedge j)$ automatically. Suppose, on the other hand, that $k = 1$. For $i = j = 1$, $(c, 1) \leq (a \wedge b, 1)$. Otherwise, say, $i = 0$. Thus $(c, 1) \leq (a, 0)$ and so $|c| = 1$ or $c \in E$. Either way, $(c, 1) \leq (a \wedge b, i \wedge j)$. □

**Lemma 3.** $S_G$ is a pseudocomplemented semilattice where $(V, 1)^* = (\emptyset, 0)$ and, for $(V, 1) \neq (a, i) \in S_G$, $(a, i)^* = (a^*, 1)$.

**Proof.** Obviously, $(V, 1)^* = (\emptyset, 0)$. For $(V, 1) \neq (a, i) \in S_G$, $a \neq V$. Thus, $a^* \neq \emptyset$ and, hence, $(a^*, 1) \in S_G$. By Lemma 2, $(a, i) \wedge (a^*, 1) = (\emptyset, 0)$. Furthermore, if $(a, i) \wedge (b, j) = (\emptyset, 0)$, then $a \wedge b = \emptyset$. In particular, $b \leq a^*$ and $(b, j) \leq (a^*, 1)$. □

The remainder of this section establishes the properties of pseudocomplemented semilattices of the form $S_G$ that will be required in the proofs of Theorems 1 and 2.

Let $G = (V, E)$ and $H = (W, F)$ be two graphs such that $|V| \geq 5$ and, for every $x \in V$, $x \in e$ for some $e \in E$. Further, let $\varphi: S_G \to S_H$ be a homomorphism and let $\theta$ denote the congruence on $S_G$ induced by $\varphi$. 

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Lemma 4. If, for every co-atom \( a \) of \( B_G \), \((a,0) \equiv (a,1)(\theta)\), then \( \theta \supseteq \Gamma_{S_G} \).

**Proof.** For any \((b,0), (b,1) \in S_G\), there exists a co-atom \( a \in B_G \) such that \( a \geq b \). Since \((b,0) \in S_G, |b| \neq 1 \) and \( b \notin E \) and, since \((b,1) \in S_G, b \neq \emptyset \). Thus, by Lemma 2, \((a,0) \land (b,1) = (b,0) \) and \((a,1) \land (b,1) = (b,1) \). By hypothesis, it follows that \((b,0) \equiv (b,1)(\theta)\). \( \square \)

Lemma 5. If \( \theta \nsubseteq \Gamma_{S_G} \), then \( \phi \) is one-to-one on \( S_G^* \).

**Proof.** Suppose \( \phi \) is not one-to-one on \( S_G^* \). Then, since \( \phi \upharpoonright S_G^* : S_G^* \to S_H^* \) is a Boolean homomorphism, \( \phi(a,1) = \phi(V,1) = (W,1) \) for some co-atom \( a \in B_G \). Thus, by Lemma 3, \((\phi(a,0))^* = \phi((a,0)^*) = \phi(a^*,1) = \phi((a,1)^*) = (\phi(a,1))^* = (W,1)^* = (S,0)\). It follows that \( \phi(a,0) = (W,1) \) and, hence, \((a,0) \equiv (S,1)(\theta)\). Let \( b \) be any other co-atom of \( B_G \). By Lemma 4, it is sufficient to show \((b,0) \equiv (b,1)(\theta)\). Since \(|V| \geq 5, |a \land b| \geq 3 \). Thus, by Lemma 2, \((a \land b,0) = (a,0) \land (b,1) \equiv (V,1) \land (b,1) = (b,1) \). It follows that \((b,0) \equiv (b,1)(\theta)\) since \((a \land b,0) \leq (b,0) \leq (b,1)\). \( \square \)

Lemma 6. If \( \theta \nsubseteq \Gamma_{S_G} \), then, for \( \emptyset \neq a \in B_G, \phi(a,1) = (r,1) \) for some \( r \in B_H \). Furthermore,

(i) if \(|a| = 1, then \(|r| = 1\);  
(ii) if \( a \in E \), then \( r \in F \); and

(iii) if \((a,0) \equiv (a,1)(\theta)\), then \(|a| = 2 \) and \( r \in F \).

**Proof.** By Lemma 5, \( \phi \) is one-to-one on \( S_G^* \). Further, since \( \phi \upharpoonright S_G^* : S_G^* \to S_H^* \), it follows that, for \( \emptyset \neq a \in B_G \), \( \phi(a,1) = (r,1) \) for some \( r \in B_H \).

With (iii) in mind, suppose \((a,0) \equiv (a,1)(\theta)\) for some \(|a| \geq 3 \). Then there exists a co-atom \( b \in B_G \) such that \( b \geq a \). Since \( \phi \) is order-preserving, \( s \geq r \) where \( \phi(b,1) = (s,1) \). By hypothesis, \((a,1) \land (b,0) = (a,0) \) and, hence, \((r,1) \lor \phi(b,0) = (r,1) \) in particular, \( \phi(b,0) \geq (r,1) \). However, since \( \phi \) preserves \( \land \), \( \phi(b,0) \in \{(s,0), (s,1)\} \). But, because \( \phi \) is one-to-one on \( S_G^* \), \(|r| \geq 3 \) and, hence, \((s,0) \notin (r,1) \). It follows that \( \phi(b,0) = (s,1) \) and, consequently, that \((b,0) \equiv (b,1)(\theta)\). We claim that, for any co-atom \( c \in B_G \), \((c,0) \equiv (c,1)(\theta)\). To see this observe that, since \(|b \land c| \geq 3, (b \land c,0) = (b,0) \land (b \land c,1) \equiv (b,1) \land (b \land c,1) = (b \land c,1) \). Hence, \( \phi(b \land c,1) = \phi(b \land c,0) = \phi((c,0) \land (b \land c,1)) = \phi(c,0) \land \phi(b \land c,1) \); that is, \( \phi(c,0) \geq (b \land c,1) \). If \( \phi(c,1) = (t,1) \), then, since \( \phi \) preserves \( \land \), \( c,0 \in \{t,0\} \). But, if \( \phi(b \land c,1) = (u,1) \), then \(|u| \geq 3 \) since \( \phi \) is one-to-one on \( S_G^* \). It follows that \((t,0) \notin (u,1) \) and, hence, \( \phi(c,0) = (t,1) \). Thus, \((c,0) \equiv (c,1)(\theta)\) for every co-atom \( c \in B_G \). By Lemma 4, this is absurd. Thus, we conclude that, for any \( a \in B_G \), if \(|a| \geq 3 \) then \((a,0) \nsubseteq (a,1)(\theta)\).

Suppose now that \(|a| = 2 \). Set \( b = a \cup \{x\} \) for some \( x \in V \setminus a \). Since \( \phi \) is one-to-one on \( S_G^* \), \(|s| > |r| \geq 2 \) where \( \phi(a,1) = (r,1) \) and \( \phi(b,1) = (s,1) \). Thus, as shown above, \((b,0) \equiv (b,1)(\theta)\) and, in particular, \( \phi(b,0) = (s,0) \). To prove (ii), suppose \( a \in E \). Then \((a,1) \leq (b,0) \) and, hence, \((r,1) \leq (s,0) \) which, since \(|r| \geq 2 \), implies \( r \in F \). Thus, (ii) is seen to hold. To prove...
(iii), suppose \( a \notin E \). For \( r \notin F \), since \((a,0) = (a,1) \land (b,0)\), \( \varphi(a,0) = (r,1) \land (s,0) = (r,0) \) and, hence, \((a,0) \neq (a,1)(\theta) \) which verifies (iii).

Since \( \varphi \) is one-to-one on \( S^*_G \) and \( V \subseteq \cup E \), (i) is an immediate consequence of (ii). \( \square \)

**Proposition 1.** If \( \theta \notin \Gamma_{S_G} \), then the following hold:

(i) for \( x \in V \), \( \varphi(\{x\},1) = (\{\psi(x)\},1) \) defines a one-to-one compatible mapping \( \psi: G \to H \) (a mapping is compatible if \( \{\psi(x), \psi(y)\} \in F \) whenever \( \{x, y\} \in E \) which is also onto whenever \( G \) is finite;

(ii) if \( G = H \) and \( \varphi \upharpoonright S^*_G \) is the identity, then \( \varphi \) is the identity.

**Proof.** By Lemma 5, \( \varphi \) is one-to-one on \( S^*_G \). Thus, by Lemma 6 (i) and (ii), \( \psi: G \to H \) as given above is a well-defined one-to-one compatible mapping.

To complete the proof of (i), suppose \( G \) is finite. Then, since \( f(\{x\},1) = (0,0) \), it follows that

\[
\bigwedge((\{\psi(x)\}^*,1): x \in V) = \bigwedge((\{\psi(x)\},1)^*: x \in V) = \bigwedge((\varphi(\{x\},1))^*: x \in V)
\]

\[
= \bigwedge(\varphi(\{x\},1)^*: x \in V) = \bigwedge(\varphi(\{x\},1)^*: x \in V)
\]

\[
= \varphi(\bigwedge((\{x\}^*,1): x \in V)) = (\varnothing,0).
\]

But as a meet of co-atoms of \( S^*_H \), \( \bigwedge((\{\psi(x)\}^*,1): x \in V) \) can be \((\varnothing,0)\) only if every co-atom is present, that is, \( \psi(V) \supseteq W \).

Finally, if \( G = H \) and \( \varphi \upharpoonright S^*_G \) is the identity, then, by Lemma 6 (iii), \( \varphi \) is the identity and (ii) holds. \( \square \)

3. **Proof of Theorem 1**

For \( i < \aleph_0 \), let \( G(i) = (V(i), E(i)) \) be the complete graph on \( 5 + i \) elements, and let \( S^*_G(i) \) be the associated pseudocomplemented semilattice. Clearly, for \( i < \aleph_0 \), \( |V(i)| \geq 5 \) and \( V(i) \subseteq \bigcup E(i) \).

Let \( U_a : \alpha < 2^{\aleph_0} \) be a family of \( 2^{\aleph_0} \) distinct subsets of \( \aleph_0 \), and for each \( \alpha < 2^{\aleph_0} \) let \( Q_\alpha \) denote the quasivariety generated by the set \( \{S^*_G(i) : i \in U_\alpha\} \).

Fix \( \alpha, \beta < 2^{\aleph_0} \) with \( \alpha \neq \beta \). Without loss of generality, we may choose \( m \in U_\alpha \setminus U_\beta \). If \( S^*_G(m) \in Q_\alpha \), then \( S^*_G(m) \in \text{SPP}_u(S^*_G(i) : i \in U_\beta) \) (see Grätzer and Lakser [4]), and it follows (see Grätzer, Lakser, and Quackenbush [5]) from the fact that pseudocomplemented semilattices are locally finite (see Jones [7] and also Sankappanavar [10]) that \( S^*_G(m) \in \text{SP}(S^*_G(i) : i \in U_\beta) \). Given any \( x \in S^*_G(m) \) with \( x \neq x^{**} \), it follows that there exists \( i \in U_\beta \) and a homomorphism \( \varphi: S^*_G(m) \to S^*_G(i) \) such that \( \varphi(x) \neq \varphi(x^{**}) \). Hence the congruence induced by \( \varphi \) fails to contain the Glivenko congruence. By Proposition 1 (i) it follows that \( |G(m)| = |G(i)| \), which is absurd, and so the quasivarieties \( (Q_\alpha : \alpha < 2^{\aleph_0}) \) are distinct.
4. Proof of Theorem 2

To establish Theorem 2, we again choose suitable families of graphs, but this time with a little more care. By Hedrlín and Sichler [6], given a non-zero cardinal $\kappa$, there exists a family of graphs $(G(i) = (V(i), E(i)) : i \in I)$ with the following properties: (i) each $G(i)$ is rigid, i.e., the only compatible mapping from $G(i)$ to itself is the identity; (ii) for $i \neq j$, there is no compatible mapping from $G(i)$ to $G(j)$; (iii) if $\kappa$ is infinite, then $|I| = 2^\kappa$ and $|V(i)| = \kappa$ for $i \in I$; (iv) if $\kappa$ is finite, then $|I| = \kappa$ and $5 \leq |V(i)| = |V(j)| < \kappa_0$ for $i, j \in I$.

Consider $(SG, \gamma_i : i \in I)$. By Proposition 1 (i), the absence of a compatible map from $G(i)$ to $G(j)$ implies that $S_{G(i)} \neq S_{G(j)}$ for $i \neq j$.

For $i \in I$, let $\gamma_i$ be the Glivenko endomorphism of $S_{G(i)}$.

We show that for $i \in I$, $\text{End}(S_{G(i)})$ is isomorphic to the monoid $M \cup \{i\}$ where $M = \text{End}(B_{G(i)})$ and $i$ is an adjoined identity element, that is, $i \notin M$, $i^2 = i$, and $i \psi = \psi i = \psi$ for all $\psi \in M$. Since $B_{G(i)} \cong B_{G(j)}$ for all $i, j \in I$, it will follow that $\text{End}(S_{G(i)}) \cong \text{End}(S_{G(j)})$.

As $B_{G(i)} \cong S_{G(i)}^{\ast}$ it suffices to establish an embedding $T : \text{End}(S_{G(i)}^{\ast}) \rightarrow \text{End}(S_{G(i)})$ such that the image of $T$ consists precisely of all non-identity endomorphisms of $S_{G(i)}$. Define $T$ by $T(\alpha) = \alpha \gamma_i$ for all $\alpha \in \text{End}(S_{G(i)}^{\ast})$.

Since $\gamma_i$ is the identity on $S_{G(i)}^{\ast}$, $T$ is one-to-one and a homomorphism. Moreover, all $T(\alpha)$ are non-identity because $\gamma_i$ is not one-to-one.

To show that every non-identity $\varphi \in \text{End}(S_{G(i)})$ belongs to the image of $T$, it will suffice to show that each such $\varphi$ is equal to $\varphi' \gamma_i$, where $\varphi'$ denotes the restriction of $\varphi$ to $S_{G(i)}^{\ast}$. Equivalently, we need to show that the congruence induced by $\varphi$ contains the Glivenko congruence on $S_{G(i)}$. If such is not the case, then, by Proposition 1 (i), the rigidity of $G(i)$ implies that $\varphi'$ is the identity. Whence, by Proposition 1 (ii), $\varphi$ is the identity in violation of the hypothesis.

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