LINEAR OPERATORS COMMUTING WITH TRANSLATIONS ON $\mathcal{D}(\mathbb{R})$ ARE CONTINUOUS

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Dedicated to the Memory of Dr. Lilian Louise Colombe Asam (née Graue)
Born 18 July 1953; died 12 September 1987

Lilian was an early, enthusiastic and dedicated TILFer. (A TILFer is one who has investigated Translation Invariant Linear Functionals.) Her joyfulness, good-nature, intelligence and energy made a unique, important and permanent impact on the lives of all who knew her. Lilian was tragically drowned in a boating accident on the Starnbergersee near Munich on September 12, 1987. We cherish her memory and we miss her presence among us on Earth.

Abstract. Let $\mathcal{D}(\mathbb{R})$ denote the Schwartz space of all $C^\infty$-functions $f: \mathbb{R} \to \mathbb{C}$ with compact supports in the real line $\mathbb{R}$. An earlier result of the author on the automatic continuity of translation-invariant linear functionals on $\mathcal{D}(\mathbb{R})$ is combined with a general version of the Closed-Graph Theorem due to A. P. Robertson and W. J. Robertson in order to prove that every linear mapping $S$ of $\mathcal{D}(\mathbb{R})$ into itself, which commutes with translations, is automatically continuous.

1. Introduction

As usual, $\mathcal{D}(\mathbb{R})$ denotes the Schwartz space of all $C^\infty$ complex-valued test functions with compact supports on the real line $\mathbb{R}$. There have been a number of papers written on the automatic continuity of translation-invariant linear functionals (TILFs) on $\mathcal{D}(\mathbb{R})$ and other spaces of functions and Schwartz distributions. These include [11, 12] by the author and, more recently, [1, 3, 8, 9, 16, 18, 20] by several other TILFers. See the author's Math Review of Willis [20] for a brief survey of results and some further references.

The purpose of this note is to prove a new result of this general type, except for operators rather than functionals, which was stated without proof in [13, §6, p. 442].

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Barry Johnson [6, 7] was one of the first to study the automatic continuity of linear operators commuting with translations or other continuous linear operators. See Sinclair [19], Dales [4], and Bachar [2] for surveys of the entire area of automatic continuity and many other references. See also Loy [10].

It is known that every continuous linear operator \( S : \mathcal{D}(\mathbb{R}) \to \mathcal{D}(\mathbb{R}) \) which commutes with translations

\[
S \tau_a = \tau_a S
\]

can be represented as convolution with a unique distribution \( T \) of compact support:

\[
Sf = T \ast f, \quad \text{for all } f \text{ in } \mathcal{D}(\mathbb{R}).
\]

See, for example, Donoghue [5, pp. 121–122] or Rudin [17, Theorem 6.33].

It is the purpose of this note to show that the hypothesis of continuity in this statement is superfluous. Specifically we prove the following

**Theorem.** A linear mapping \( S : \mathcal{D}(\mathbb{R}) \to \mathcal{D}(\mathbb{R}) \) satisfying (1) for all \( a \) in \( \mathbb{R} \) is necessarily continuous.

The translation operator \( \tau_a : \mathcal{D}(\mathbb{R}) \to \mathcal{D}(\mathbb{R}) \) is defined, as usual, by the formula

\[
(\tau_a f)(t) \equiv f(t-a), \quad t, a \in \mathbb{R}.
\]

We shall use the following notation for the Fourier Transform \( \hat{f} \) of \( f \) in \( \mathcal{D}(\mathbb{R}) \):

\[
\hat{f}(z) \equiv \int_{\mathbb{R}} e^{-2\pi i z t} f(t) dt, \quad z \in \mathbb{C}.
\]

The proof of the above theorem is based on two lemmas (given in the next section) and the following general form of the Closed-Graph Theorem proved by A. P. Robertson and W. J. Robertson in [14] and stated in [15, p. 124]:

**Closed Graph Theorem.** If \( E \) is a separated ( = Hausdorff) inductive limit of convex Baire spaces and if \( F \) is a separated inductive limit of a sequence of fully complete spaces, then any linear mapping, with a closed graph, of \( E \) into \( F \) is continuous.

We shall apply this Closed-Graph Theorem to the space \( E = F = \mathcal{D}(\mathbb{R}) \) which is known to be a separated locally convex topological vector space which is an inductive limit of a sequence of locally convex Fréchet spaces. Since Fréchet spaces are both fully complete and Baire spaces, the hypotheses of this Closed-Graph Theorem are satisfied for \( E = F = \mathcal{D}(\mathbb{R}) \).

2. **Two lemmas**

**Lemma 1.** If \( \varphi : \mathcal{D}(\mathbb{R}) \to \mathbb{C} \) is a (not assumed continuous) translation-invariant linear functional on \( \mathcal{D}(\mathbb{R}) \), then there is a complex constant \( c \) such that

\[
\varphi(f) = c \hat{f}(0) \equiv c \int_{\mathbb{R}} f(t) dt
\]
for all \( f \) in \( D(\mathbb{R}) \). That is, translation-invariant linear functionals on \( D(\mathbb{R}) \) are automatically continuous.

**Proof.** See [11], or [12, Theorem 2 page 182], or [13, Corollary of Theorem 1, p. 427].

Translation-invariance for linear functionals \( \varphi : D(\mathbb{R}) \to \mathbb{C} \) means that for all \( a \) in \( \mathbb{R} \) and for all \( f \) in \( D(\mathbb{R}) \),

\[
\varphi(\tau_a f) = \varphi(f).
\]

**Lemma 2.** Let \( \varphi \) be a (not assumed continuous) linear functional on \( D(\mathbb{R}) \) such that, for some \( z \) in \( \mathbb{C} \),

\[
\varphi(\tau_a f) = e^{-2\pi iz} \varphi(f)
\]

for all \( f \) in \( D(\mathbb{R}) \) and for all \( a \) in \( \mathbb{R} \). Then there is a complex constant \( c \) such that, for all \( f \) in \( D(\mathbb{R}) \),

\[
\varphi(f) = c \hat{f}(z) \equiv c \int_{\mathbb{R}} e^{-2\pi izt} f(t) \, dt.
\]

**Proof.** Define \( \psi : D(\mathbb{R}) \to \mathbb{C} \) by

\[
\psi(f) = \varphi_i(e^{2\pi izt} f(t))
\]

for all \( f \) in \( D(\mathbb{R}) \), where the subscript notation \( \varphi_i \) indicates the variable of the function upon which the linear functional \( \varphi \) is acting: Thus

\[
\varphi_i(g(t)) \equiv \varphi(g).
\]

Note that, for all \( f \) in \( D(\mathbb{R}) \), \( e^{2\pi izt} f(t) \) is also in \( D(\mathbb{R}) \). Then for each \( a \) in \( \mathbb{R} \),

\[
\psi(\tau_a f) = \varphi_i(e^{2\pi izt}(\tau_a f)(t)) \quad \text{by (6)}
\]

\[
= e^{2\pi iza} \varphi_i(\tau_a e^{2\pi izt} f(t)) \quad \text{by linearity of } \varphi
\]

\[
= e^{2\pi iza} e^{-2\pi iza} \varphi_i(e^{2\pi izt} f(t)) \quad \text{by (4)}
\]

\[
= \psi(f) \quad \text{by (6)}.
\]

In other words, \( \psi \) is a translation-invariant linear functional on \( D(\mathbb{R}) \). It follows from Lemma 1 that, for some constant \( c \) in \( \mathbb{C} \), and for all \( f \) in \( D(\mathbb{R}) \),

\[
\psi(f) = c \hat{f}(0).
\]

We may now compute as follows.

\[
\varphi(f) = \varphi_i(e^{2\pi izt} e^{-2\pi izt} f(t))
\]

\[
= \psi_i(e^{-2\pi izt} f(t)) \quad \text{by (6)}
\]

\[
= c[e^{-2\pi izt} f(t)]\hat{f}(0) \quad \text{by (7)}
\]

\[
= c \int_{\mathbb{R}} e^{-2\pi izt} f(t) \, dt
\]

\[
= c \hat{f}(z). \quad \text{Q.E.D.}
\]
3. PROOF OF THE THEOREM

We may write

\[(S'\tau_a f)(z) = (\tau_a S f)(z) = e^{-2\pi iza} (Sf)(z)\]

for all \(f\) in \(D(R)\) and for all \(z\) in \(C\). Therefore, the linear functional \(\varphi\) on \(D(R)\) defined by

\[\varphi(f) = (Sf)(z)\]

has the property

\[\varphi(\tau_a f) = e^{-2\pi iza} \varphi(f)\]

which is the hypothesis (2) of Lemma 2. It follows from Lemma 2 that for each \(z\) in \(C\) there is a constant \(C_z\) in \(C\) such that

\[(Sf)(z) = C_z \hat{f}(z)\]

for all \(f\) in \(D(R)\). We now apply the Closed-Graph Theorem (stated in the Introduction) to show that \(S\) is continuous:

Suppose that \(f_\alpha\) is a net converging to zero in \(D(R)\) and that \(Sf_\alpha\) converges to an element \(h\) in \(D(R)\). The linear functionals \(\varphi_z\) (one for each \(z\) in \(C\)) defined by

\[\varphi_z(f) = \hat{f}(z), \quad f \in D(R),\]

are continuous and so, for each \(z\) in \(C\),

\[\hat{h}(z) = \varphi_z(h) = \lim_{\alpha} \varphi_z(Sf_\alpha) = \lim_{\alpha} (Sf_\alpha)(z) = \lim_{\alpha} C_z \hat{f}_\alpha(z) = C_z \lim_{\alpha} \int_R e^{-2\pi izt} f_\alpha(t) \, dt = C_z \lim_{\alpha} \varphi_z(f_\alpha) = C_z \varphi_z(0) = 0.\]

Since the Fourier transform on \(D(R)\) is one-to-one, \(h = 0\). Thus the graph of \(S\) is closed. It now follows from the above-stated Closed-Graph Theorem that \(S\) is continuous. Q.E.D.

REFERENCES


