ON THE REGULARITY PROPERTIES FOR SOLUTIONS OF THE CAUCHY PROBLEM FOR THE POROUS MEDIA EQUATION

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Abstract. We consider the Cauchy problem for the equation \( \partial_t u = \Delta u^m \) in \( R^N \times (0, T) \). We assume that \( 1 < m < \frac{3N}{3N - 2} \) and the initial data \( u_0 \) is in \( C^1_0(R^N) \) and \( u_0 \geq 0 \) in \( R^N \). Then we prove that the second derivatives of \( u^m \) with respect to the space-variable are in \( L^2(R^N \times (0, T)) \).

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We consider the Cauchy problem for the porous media equation

\[
\begin{cases}
\partial_t u = \Delta u^m & \text{in } R^N \times (0, T) \\
u(x, 0) = u_0(x) & \text{on } R^N,
\end{cases}
\]

where \( m > 1 \) and \( u_0(x) \geq 0 \). The problem (1.1) has been studied by many authors. For a detailed account of (1.1) we refer to the work of Peletier [7].

We say that \( u(x, t) \) is a solution of (1.1), if

\[
\int_0^T \int_{R^N} [u(x, t)^2 + |\nabla u^m(x, t)|^2] \, dx \, dt < \infty
\]

and

\[
\int_0^T \int_{R^N} (u \partial_t \phi - \nabla_x u^m \cdot \nabla_x \phi) \, dx \, dt + \int_{R^N} u_0(x) \phi(x, 0) \, dx = 0
\]

for any continuously differentiable function \( \phi(x, t) \) with compact support in \( R^N \times [0, T) \). The existence of such a solution is due to Sabinina [8] under some condition on \( u_0 \).

We are concerned with the regularity of \( u \) in (1.1). The Hölder regularity of \( u \) was shown by Caffarelli and Friedman [6]. For \( N = 1 \) the precise
Hölder exponent with respect to the space-variable was obtained by Aronson [1]. Similar results for the time-variable were studied by di Benedetto [4], when \( N \geq 1 \). There arises a question whether the derivative \( \partial_t u \) is a function or not. Concerning this there are results such as Aronson and Bénilan [2], Bénilan [5], where the assumption on \( u_0 \) is very weak. For a function space \( A \) the assertion "\( \partial_i u \in A \)" is almost equivalent to "\( \partial_{x_i} \partial_{x_j} u^m \in A, \ 1 \leq i, j \leq N \)".

According to [5] 
\[
\partial_t u \in L^p([\delta, T] \times B_R)
\]
for any \( 1 < p < 1 + 1/m, 0 < \delta < T \) and \( R > 0 \), where \( B_R = \{x \in \mathbb{R}^N; |x| < R\} \). Further if \( N = 1 \) and \( m > 2 \) particularly, \( \partial_t u \in L^\infty(\delta, T; L^p(B_R)) \) for any \( 1 < p < 1 + 1/(m - 2) \). In this connection we note also the results in [3], [9] and [10].

Our theorem is stated in the following section.

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Let us denote by \( (, \) and \( ||| \) the inner product and the norm in \( L^2(\mathbb{R}^N) \), respectively. Let us write \( G^T = \mathbb{R}^N \times (0, T) \). Our aim is to prove

**Theorem.** Suppose \( 1 < m < 3N/(3N-2) \). Let \( u_0 \) be in \( C_0^1(\mathbb{R}^N) \), and let \( u_0 \geq 0 \) in \( \mathbb{R}^N \). Let \( u \) be a solution of (1.1). Then \( \partial_{x_i} \partial_{x_j} u^m \in L^2(G^T), \ 1 \leq i, j \leq N \).

More precisely, if \( u_0 \leq M \) in \( \mathbb{R}^N \), then for \( \alpha = 1 - 1/m \)

\[
\int_0^T \||\partial_{x_i} \partial_{x_j} u^m||^2 dt \leq C[||u_0^m||^2 + ||u_0^{m(1-\alpha/2)}||^2]
\]

\[+ M^{\alpha m}||\nabla u_0^m||^2 + (u_0^{-\alpha m}, |\nabla u_0^m|^2),\]

where \( C \) depends on \( m, N, T \) and not on \( u_0, M \).

Our method is to derive a uniform energy inequality for each solution of nondegenerate parabolic equations, which are the regular approximation of (1.1) appearing in [6].

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We prove our theorem.

For \( \eta > 0 \) let \( u^\eta(x,t) \) be the solution of

\[
\begin{aligned}
\frac{\partial}{\partial t} u^\eta &= \Delta (u^\eta)^m &\text{in } \mathbb{R}^N \times (0, \infty) \\
u^\eta(x,0) &= u_0(x) + \eta &\text{on } \mathbb{R}^N.
\end{aligned}
\]

It is known that \( \eta \leq u^\eta \leq M + \eta \) and \( u^\eta \) is classical (cf., e.g., [8]). If we set \( v^\eta = (u^\eta)^m \) and \( \psi^\eta = (u_0 + \eta)^m \), (3.1) becomes

\[
\begin{aligned}
\frac{\partial}{\partial t} v^\eta &= m\Delta v^\eta &\text{in } \mathbb{R}^N \times (0, \infty) \\
v^\eta(x,0) &= \psi^\eta(x) &\text{on } \mathbb{R}^N.
\end{aligned}
\]
where \( \alpha = 1 - 1/m \). For simplicity we denote \( \psi''(x,t) \) and \( \psi''(x) \) by \( \psi(x,t) \) and \( \psi(x) \), respectively. From the proof in Sabinina [8] we easily see that
\[
\| \psi''(\cdot,t) - \eta'' \| \leq C( \| \psi - \eta'' \| + (M + \eta'' \| \nabla \psi \|) , \quad 0 < t < T ,
\]
where \( C \) depends on \( T \) and not on \( \psi, \eta \) and \( M \).

Let \( \zeta \in C_0^2(R^N) \) and \( \zeta \geq 0 \) in \( R^N \). From (3.1') we have
\[
\frac{1}{2 - \alpha}(\zeta, \partial_t \nu^{2-a}) + m(\zeta, |\nabla_x \nu|^2) = \frac{m}{2}(\Delta \zeta, \nu^2).
\]
Hence
\[
\int_0^T (\zeta, |\nabla_x \nu|^2) dt = \frac{1}{m(2 - \alpha)} [(\zeta, \nu^{2-a}) - (\nu(t^2) - \eta''(\cdot), T^{2-a})] + \frac{1}{2} \int_0^T (\Delta \zeta, \nu^2) dt.
\]
Combining this with (3.2) we obtain
\[
\int_0^T (\zeta, |\nabla_x \nu|^2) dt \leq C[(\zeta, \nu^{2-a}) + \| \psi - \eta'' \|^2 + (M + \eta'' \| \nabla \psi \|)^2].
\]

Let \( \nu = -\zeta \partial_{x_j}^2 \) for \( 1 \leq j \leq N \). By integration by parts we have
\[
- (\Delta \nu, \nu) = (\zeta \nabla_x \partial_{x_j} \nu, \nabla_x \partial_{x_j} \nu) + (\partial_{x_j} \zeta \cdot \nabla_m \nu, \nabla_x \partial_{x_j} \nu) - (\nabla \zeta \cdot \nabla_x \nu, \partial_{x_j}^2 \nu) ,
\]
where we have assumed that \( \nu(\cdot,t) \in C^3(R^N) \). But (3.4) is valid for any \( \nu(\cdot,t) \in C^3(R^N) \) by taking an approximating sequence of \( \nu \). Similarly we have
\[
(v^{-a} \partial_t \nu, \nu) = - \alpha(v^{-a-1} \partial_{x_j} \nu \cdot \partial_t \nu, \zeta \partial_{x_j} \nu) + (v^{-a} \partial_t \nu, \zeta \partial_{x_j} \nu) + (v^{-a} \partial_t \nu, \zeta \cdot \partial_{x_j} \nu).
\]
It is easy to see that
\[
(v^{-a-1} \partial_{x_j} \nu \cdot \partial_t \nu, \zeta \partial_{x_j} \nu) = m(\zeta v^{-1} \Delta \nu, (\partial_{x_j} \nu)^2) ,
\]
and
\[
(v^{-a} \partial_t \nu, \zeta \cdot \partial_{x_j} \nu) = m(\Delta \nu, \partial_{x_j} \zeta \cdot \partial_{x_j} \nu).
\]
Combining these equalities with (3.5), we have
\[
\int_0^T (v^{-a} \partial_t \nu, \nu) dt \geq - \frac{\alpha m}{2} \int_0^T (\zeta v^{-1} \Delta \nu, (\partial_{x_j} \nu)^2) dt - \frac{1}{2} (\zeta \psi^{-a}, (\partial_{x_j} \psi)^2) + m \int_0^T (\Delta \nu, \zeta \cdot \partial_{x_j} \nu) dt .
\]
From this and (3.4), (3.1'), it follows that

\[(3.6) \int_0^T \|\zeta^{1/2} \nabla_x \partial_{x_j} v\|^2 dt \]

\[\leq \frac{1}{2m}(\zeta \psi^{-\alpha}, (\partial_{x_j} \psi)^2) + \frac{\alpha}{2} \int_0^T (\zeta v^{-1} \Delta v, (\partial_{x_j} v)^2) dt \]

\[- \int_0^T (\nabla v, \partial_{x_j} \zeta \cdot \partial_{x_j} v) dt - \int_0^T (\partial_{x_j} \zeta \cdot \nabla_x v, \nabla_x \partial_{x_j} v) dt \]

\[+ \int_0^T (\nabla \zeta \cdot \nabla_x v, \partial_{x_j} v)^2 dt.\]

Now by integration by parts

\[(\zeta v^{-1} \partial_{x_j}^2 v, (\partial_{x_j} v)^2) = - (\partial_{x_j} \zeta \cdot v^{-1}, (\partial_{x_j} v)^3) \]

\[+ (\zeta v^{-2}, (\partial_{x_j} v)^4) - 2(\zeta v^{-1} \partial_{x_j}^2 v, (\partial_{x_j} v)^2).\]

Thus we have

\[(\zeta v^{-1} \partial_{x_j}^2 v, (\partial_{x_j} v)^2) = \frac{1}{3}[(\zeta v^{-2}, (\partial_{x_j} v)^4) - (\partial_{x_j} \zeta \cdot v^{-1}, (\partial_{x_j} v)^3)],\]

so that (3.6) becomes

\[(3.7) \int_0^T \|\zeta^{1/2} \nabla_x \partial_{x_j} v\|^2 dt \]

\[\leq \frac{1}{2m}(\zeta \psi^{-\alpha}, (\partial_{x_j} \psi)^2) + \frac{\alpha}{6} \int_0^T (\zeta v^{-2}, (\partial_{x_j} v)^4) dt \]

\[- \frac{\alpha}{6} \int_0^T (\partial_{x_j} \zeta \cdot v^{-1}, (\partial_{x_j} v)^3) dt + \frac{\alpha}{2} \sum_{i \neq j} \int_0^T (\zeta v^{-1} \partial_{x_j}^2 v, (\partial_{x_j} v)^2) dt \]

\[- \int_0^T (\nabla v, \partial_{x_j} \zeta \cdot \partial_{x_j} v) dt - \int_0^T (\partial_{x_j} \zeta \cdot \nabla_x v, \nabla_x \partial_{x_j} v) dt \]

\[+ \int_0^T (\nabla \zeta \cdot \nabla_x v, \partial_{x_j} v)^2 dt.\]

Here we estimate the quantity \((\zeta v^{-2}, (\partial_{x_j} v)^4)\). By integration by parts

\[(\zeta v^{-2} \partial_{x_j} v, (\partial_{x_j} v)^3) = 2(\zeta v^{-2} \partial_{x_j} v, (\partial_{x_j} v)^3) \]

\[- 3(\zeta v^{-1} \partial_{x_j}^2 v, (\partial_{x_j} v)^2) - (\partial_{x_j} \zeta \cdot v^{-1}, (\partial_{x_j} v)^3).\]

Hence

\[(\zeta v^{-2}, (\partial_{x_j} v)^4) = 3(\zeta v^{-1} \partial_{x_j}^2 v, (\partial_{x_j} v)^2) + (\partial_{x_j} \zeta \cdot v^{-1}, (\partial_{x_j} v)^3) \]

\[\leq \frac{9}{2}(\zeta, (\partial_{x_j} v)^2) + \frac{1}{2}(\zeta v^{-2}, (\partial_{x_j} v)^4) + (\partial_{x_j} \zeta \cdot v^{-1}, (\partial_{x_j} v)^3),\]

which implies that

\[(\zeta v^{-2}, (\partial_{x_j} v)^4) \leq 9(\zeta, (\partial_{x_j} v)^2) + 2(\partial_{x_j} \zeta \cdot v^{-1}, (\partial_{x_j} v)^3).\]
Since \(2(\partial_{x_j} \zeta \cdot v^{-1}, (\partial_{x_j} v)^3) \leq \varepsilon (\zeta v^{-1}, (\partial_{x_j} v)^4) + \varepsilon^{-1} (\zeta^{-1}(\partial_{x_j} \zeta)^2, (\partial_{x_j} v)^2)\) for sufficiently small \(\varepsilon > 0\), we obtain
\[
(\zeta v^{-2}, (\partial_{x_j} v)^4) \leq \frac{9}{1-\varepsilon} (\zeta, (\partial_{x_j} v)^2) + \mathcal{C}(\varepsilon)(\zeta^{-1}(\partial_{x_j} \zeta)^2, (\partial_{x_j} v)^2).
\]
From this and Cauchy's inequality it follows that
\[
|(\partial_{x_j} \zeta \cdot v^{-1}, (\partial_{x_j} v)^3)| \leq \frac{9\varepsilon}{2(1-\varepsilon)} (\zeta, (\partial_{x_j} v)^2)
+ \mathcal{C}'(\varepsilon)(\zeta^{-1}(\partial_{x_j} \zeta)^2, (\partial_{x_j} v)^2).
\]
Now we define
\[
I_j = \frac{\alpha}{6}(\zeta v^{-2}, (\partial_{x_j} v)^4) - \frac{\alpha}{6}(\partial_{x_j} \zeta \cdot v^{-1}, (\partial_{x_j} v)^3)
+ \frac{\alpha}{2} \sum_{i \neq j} (\zeta v^{-1}(\partial_{x_i} v), (\partial_{x_j} v)^2), \quad 1 \leq j \leq N.
\]
Using (3.9) and Cauchy's inequality we have for \(\delta > 0\)
\[
I_j \leq \frac{\alpha}{4\delta} (\zeta, \left(\sum_{i \neq j} (\partial_{x_i} v)^2\right)^2)
+ \left(\frac{\alpha \delta}{4} + \frac{\alpha}{6}\right)(\zeta v^{-2}, (\partial_{x_j} v)^4)
+ \frac{3\alpha \varepsilon}{4(1-\varepsilon)} (\zeta, (\partial_{x_j} v)^2) + \mathcal{C}(\zeta^{-1}(\partial_{x_j} \zeta)^2, (\partial_{x_j} v)^2).
\]
Easily, \((\sum_{i \neq j} (\partial_{x_i} v)^2)^2 \leq (N - 1) \sum_{i \neq j} (\partial_{x_i} v)^2\). Thus by (3.8) it follows that
\[
\sum_j I_j \leq \left[\frac{\alpha}{4\delta} (N - 1)^2 + \frac{9\alpha}{1-\varepsilon} \left(\frac{\alpha \delta}{4} + \frac{\alpha}{6}\right)
+ \frac{3\alpha \varepsilon}{4(1-\varepsilon)}\right] (\zeta, \sum_j (\partial_{x_j} v)^2)
+ \mathcal{C} \sum_j (\zeta^{-1}(\partial_{x_j} \zeta)^2, (\partial_{x_j} v)^2).
\]
Here we put \(\delta = (N - 1)/3\). Then
\[
\frac{\alpha}{4\delta} (N - 1)^2 + \frac{9\alpha}{1-\varepsilon} \left(\frac{\alpha \delta}{4} + \frac{\alpha}{6}\right) = 3\alpha N/2.
\]
From our assumption on \(m\) we see that \(3\alpha N/2 < 1\).
Next we estimate the remaining terms on the right-hand side of (3.7). For sufficiently small \(\varepsilon' > 0\)
\[
\sum_j ||(\Delta v, \partial_{x_j} \zeta \cdot \partial_{x_j} v) + (\partial_{x_j} \zeta \cdot \nabla_x v, \nabla_x \partial_{x_j} v) - (\nabla \zeta \cdot \nabla_x v, \partial_{x_j}^2 v)||^2
\leq \varepsilon' \sum_j ||\varepsilon'^{1/2} \nabla_x \partial_{x_j} v||^2 + \mathcal{C}(\varepsilon')(\zeta^{-1}|\nabla \zeta|^2, |\nabla_x v|^2).
\]
Therefore combining (3.7), (3.10) with the above, we conclude that
\[
\sum_j \int_0^T ||\varepsilon'^{1/2} \nabla_x \partial_{x_j} v||^2 dt \leq \mathcal{C}((\zeta \psi^{-\alpha}, |\nabla \psi|^2)
+ \int_0^T (\zeta^{-1}|\nabla \zeta|^2, |\nabla_x v|^2) dt)\].
Let $\zeta(x)$ be in $C_0^\infty(R^N)$, and let us put $\zeta = \zeta^2$. Then from the above inequality and (3.3) it follows that for $1 \leq i, j \leq N$

(3.11)

$$\int_0^T \|\xi^{1/2} \partial_{x_i} \partial_{x_j} v^n\|^2 dt \leq C[(\xi(\psi^n)^{-\alpha}, |\nabla \psi^n|^2)

+ (|\nabla \zeta|^2, (\psi^n)^{2-\alpha}) + \|\psi^n - \eta^m\|^2 + (M + \eta)^{\alpha m}||\nabla \psi^n\|^2 + \eta^{2m}].$$

As is well known, $v^n \downarrow v (\eta \downarrow 0)$ in $G$, where $v = u^m$ and $u$ is the solution of (1.1). For each positive integer $n$ we put $\xi(x) = \xi_n(x)$, where

$$\xi_n(x) = \begin{cases} 1 & (|x| \leq n) \\
0 & (|x| > 2n) \end{cases}$$

and they are uniformly bounded in $R^N$ up to the third derivatives. Then the constant $C$ on the right-hand side of (3.11) are independent of $n$. Letting $\eta \to 0$ in (3.11), we see that $\partial_{x_i} \partial_{x_j} v^n \to \partial_{x_i} \partial_{x_j} v$ weakly in $L^2_{loc}(G)$ and $\int_0^T \xi^{1/2} \partial_{x_i} \partial_{x_j} v^n d = \to$ the right-hand side of (2.1). This completes the proof.

REFERENCES


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